FINAL EXAM

HANDED OUT: Monday, December 14th, 4:30 am (via email)
DUE: Wednesday, December 16th, by 8 am sharp.

Delivery Instructions: Please make a copy of your exam, put the original and the copy in separate envelopes, and write your name clearly on both envelopes. Slide the envelope with the original under my office door (333d NSH), and the copy under the Institute’s front door (210 NSH). Please do all this by the time and date indicated above. For those of you on travel: other types of submissions (email, fax, FedEx) are acceptable, provided that (a) the time stamp of the submission clearly indicates that the solutions were sent before the deadline, and (b) a hard copy can be generated by the deadline.

Exam rules: This is a “take-home” exam. The following rules apply:

(1) You are expressly forbidden to request and/or to receive help from anyone.
(2) You may work on this exam anywhere you wish.
(3) You are allowed to use only the following sources: your lecture notes, your textbook (Modern Quantum Mechanics by J.J. Sakurai), and your favorite text on mathematical methods for physics. As in your homework solutions, you must provide a detailed reference if you use any of the printed sources.
(4) If there is something unclear in the text below, please contact me immediately. I gladly provide clarifications to the entire class, but I cannot and will not give “hints” to individual students.

1. Quantum mechanical pictures
   (a) Demonstrate that if the commutation relation \([A, B] = iC\) is valid in any of the three (Schrödinger, Heisenberg, interaction/Dirac) pictures, then it is valid in all pictures.
   (b) Calculate the time dependence of the position operator \(x_H(t)\) of a free particle in the Heisenberg picture.

2. Particle in one dimensional potential
   (a) Show that the energy eigenstates of a simple harmonic oscillator obey the following
properties under the $x \to -x$ parity transformation:

$$\langle -x|n\rangle = (-1)^n \langle x|n\rangle$$

(b) Use the above result to obtain the energy spectrum and eigenstates of a quantum mechanical particle trapped in the following potential well: $V(x) = \infty$ if $x < 0$ and $V(x) = \frac{1}{2}m\omega^2x^2$ if $x > 0$.

3. Angular momentum operators: general properties.

(a) Show that $[\hat{\mathbf{L}}, \hat{L}_i] = 0$. Here $i \in \{x, y, z\}$.

(a) Demonstrate that, $[\hat{L}_i, \hat{\mathbf{x}}_k] = i\hbar\epsilon_{ikl}\hat{\mathbf{x}}_l$.

(b) Similarly, show that $[\hat{L}_i, \hat{p}_k] = i\hbar\epsilon_{ikl}\hat{p}_l$.

(c) Finally, show that $[\hat{L}_i, \hat{\mathbf{x}}^2] = 0$ and $[\hat{L}_i, \hat{x}^2] = 0$.

4. Angular momentum operators in spherical coordinates.

Start from the expression for the gradient operator in spherical coordinates $r, \theta, \phi$:

$$\hat{\nabla} = \hat{e}_r \partial_r + \frac{\hat{e}_\theta}{r} \partial_\theta + \frac{\hat{e}_\phi}{r \sin \theta} \partial_\phi$$

where $\hat{e}_\alpha$ is the unit vector in direction $\alpha$, and calculate the following operators (all results should be given in spherical coordinates):

(a) the angular momentum operator $\hat{\mathbf{L}}$,

(b) the angular momentum squared operator $\hat{\mathbf{L}}^2$,

(c) all cartesian components $\hat{L}_\alpha$, $\alpha \in \{x, y, z\}$, of $\hat{\mathbf{L}}$,

(d) the raising and lowering operators $\hat{L}_\pm$.

5. The rotation group.

Rotations $R$ in three dimensions form a non-commutative three-parameter (three dimensional) group. Show that rotations around a single axis form a commutative, one parameter group.

(a) Give the rotation matrix $R_z(\phi)$ that corresponds to a rotation around the $z$ axis with angle $\theta$.

(b) Calculate the generator $A_z = \lim_{\phi \to 0} \partial_\phi R_z(\phi)$.

(c) Show that $R_z(\phi) = \exp[\phi A_z]$.

(d) Show all group properties of rotations around the $z$ axis:

(d.1) existence of the identity element $1_z$,
(d.2) closure: \( R_z(\phi_1)R_z(\phi_2) \) is also a rotation around \( \vec{z} \),
(d.3) associativity: \( \left( R_z(\phi_1)R_z(\phi_2) \right)R_z(\phi_3) = R_z(\phi_1)\left( R_z(\phi_2)R_z(\phi_3) \right) \),
(d.4) commutativity: \( R_z(\phi_1)R_z(\phi_2) = R_z(\phi_2)R_z(\phi_1) \). In particular, show that \( R_z(\phi_1)R_z(\phi_2) = R_z(\phi_1 + \phi_2) \).

6. **Representations of the rotation group.**

(a) The rotation operator around the \( \vec{z} \) axis is defined by the equation

\[
\hat{D}_z(\alpha) |\phi\rangle = |\phi + \alpha\rangle.
\]

Assume that all derivatives of the ket \( |\phi\rangle \) with respect to the angle variable \( \phi \) exists. Show that \( \hat{D}_z(\alpha) = \exp(\alpha \partial_\phi) = \exp(\frac{i}{\hbar} \alpha \hat{L}_z) \) where \( \hat{L}_z = \frac{i}{\hbar} \partial_\phi \).

(b) The rotation operator with angle \( \phi \) around an arbitrary axis \( \vec{n} \) is given by

\[
\hat{D}(\vec{n}\phi) = \exp \left[ -\frac{i\vec{J} \cdot \vec{n}\phi}{\hbar} \right]
\]

Show that \( \hat{J}^2 \hat{D}(\vec{n}\phi)|jm\rangle = \hbar^2 j(j+1)\hat{D}(\vec{n}\phi)|jm\rangle \). In other words, this means that \( \hat{D}(\vec{n}\phi)|jm\rangle \) is still an eigenket of \( \hat{J}^2 \).

(Hint: the commutation rules you proved between \( \hat{J}^2 \) and \( \hat{J}_\alpha \) in the previous problem will be very useful here).

(c) The matrix elements of the rotation operator with angle \( \phi \) around the axis \( \vec{n} \), (i.e. the \( (2j + 1) \) dimensional representation of the rotation group) are \( \mathcal{D}_{m'm}^{j'j}(\vec{n}\phi) = \langle j'm'|\hat{D}(\vec{n}\phi)|jm\rangle \). Use the results obtained in the previous part of this problem to argue that \( \mathcal{D}_{m'm}^{j'j}(\vec{n}\phi) \) is diagonal in \( j \):

\[
\mathcal{D}_{m'm}^{j'j}(\vec{n}\phi) = \delta_{j,j'}\mathcal{D}_{m'm}^{j'j}(\vec{n}\phi)
\]

(d) Show that for small rotation angles \( \epsilon \)

\[
\mathcal{D}_{m'm}^{j}(\vec{n}\epsilon) = \delta_{m,m'} - \frac{i\epsilon_x + \epsilon_y}{2}\sqrt{(j - m)(j + m + 1)}\delta_{m',m+1}
\]

\[
-\frac{i\epsilon_x - \epsilon_y}{2}\sqrt{(j + m)(j - m + 1)}\delta_{m',m-1} - i\epsilon_z m\delta_{m',m}.
\]