1. Quantum mechanical pictures

(a) Demonstrate that if the commutation relation $[A, B] = iC$ is valid in any of the three (Schrödinger, Heisenberg, interaction/Dirac) pictures, then it is valid in all pictures.

Consider the three statements

$$[A_S, B_S] = iC_S, \quad [A_H, B_H] = iC_H, \quad [A_I, B_I] = iC_I$$

where the subscript $S, H, I$ on the operators indicates the Schrödinger, Heisenberg, or interaction picture respectively. We need to show that any one of these statements implies the other two. One standard way of doing this, which ensures that we don't do any unnecessary work, is to show that the first statement implies the second, the second implies the third, and the third implies the first. This is the strategy we will pursue.

We need the relations between operators in the various pictures. The Heisenberg-picture operators are given in terms of the Schrödinger-picture operators as

$$A_H = e^{iHt/\hbar} A_S e^{-iHt/\hbar}$$

with analogous relations for $B$ and $C$. (We have taken the initial time $t_0$ equal to zero without any loss of generality). Next we relate the Schrödinger and interaction pictures:

$$A_I = e^{iHt/\hbar} A_S e^{-iHt/\hbar}$$

(We define the interaction picture as usual, by splitting up the full Hamiltonian $H$ into a solvable part $H_0$ plus a perturbation $V$.) Let us invert this relation to find $A_S$ in terms of $A_I$:

$$A_S = e^{-iHt/\hbar} A_I e^{iHt/\hbar}$$

Finally we relate the Heisenberg and interaction pictures: we invert (2) to solve for $A_S$, and then substitute the result into (3). The result is

$$A_I = e^{iHt/\hbar} e^{-iHt/\hbar} A_H e^{iHt/\hbar} e^{-iHt/\hbar}$$

With these preliminaries done, we can proceed to our proofs. First we assume that $[A_S, B_S] = iC_S$; then we use (2):

$$[A_H, B_H] = A_H B_H - B_H A_H$$

$$= \left( e^{iHt/\hbar} A_S e^{-iHt/\hbar} \right) \left( e^{iHt/\hbar} B_S e^{-iHt/\hbar} \right) - \left( e^{iHt/\hbar} B_S e^{-iHt/\hbar} \right) \left( e^{iHt/\hbar} A_S e^{-iHt/\hbar} \right)$$

$$= e^{iHt/\hbar} A_S B_S e^{-iHt/\hbar} - e^{iHt/\hbar} B_S A_S e^{-iHt/\hbar}$$

$$= e^{iHt/\hbar} ([A_S, B_S]) e^{-iHt/\hbar}$$

$$= e^{iHt/\hbar} (iC_S) e^{-iHt/\hbar}$$

$$= iC_H.$$
Next we assume $[A_H, B_H] = iC_H$ and use (5):

$$[A_I, B_I] = A_I B_I - B_I A_I$$
$$= \left( e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} A_H e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} \right) \left( e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} B_H e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} \right)$$
$$= e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} A_H B_H e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} - e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} B_H A_H e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t}$$
$$= e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} (A_H B_H) e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t}$$
$$= e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t} (iC_H) e^{i\frac{H_0}{\hbar}t} e^{-i\frac{H_0}{\hbar}t}$$
$$= iC_I.$$ 

Finally we assume $[A_I, B_I] = iC_I$ and use (4):

$$[A_S, B_S] = A_S B_S - B_S A_S$$
$$= \left( e^{-i\frac{H_0}{\hbar}t} A_I e^{i\frac{H_0}{\hbar}t} \right) \left( e^{-i\frac{H_0}{\hbar}t} B_I e^{-i\frac{H_0}{\hbar}t} \right) - \left( e^{-i\frac{H_0}{\hbar}t} B_I e^{i\frac{H_0}{\hbar}t} \right) \left( e^{-i\frac{H_0}{\hbar}t} A_I e^{-i\frac{H_0}{\hbar}t} \right)$$
$$= e^{-i\frac{H_0}{\hbar}t} (A_I B_I) e^{-i\frac{H_0}{\hbar}t} - e^{-i\frac{H_0}{\hbar}t} B_I A_I e^{-i\frac{H_0}{\hbar}t}$$
$$= e^{-i\frac{H_0}{\hbar}t} \left( [A_I, B_I] \right) e^{-i\frac{H_0}{\hbar}t}$$
$$= e^{-i\frac{H_0}{\hbar}t} (iC_I) e^{-i\frac{H_0}{\hbar}t}$$
$$= iC_S.$$ 

(b) Calculate the time dependence of the position operator $x_H (t)$ of a free particle in the Heisenberg picture.

We need to solve the Heisenberg equation of motion for $x_H (t)$:

$$\frac{d}{dt} x_H (t) = \frac{1}{i\hbar} [x, H]_H$$

where operators without a subscript are in the Schrodinger picture, and the Hamiltonian is $H = p^2 / 2m$ for a free particle. The needed commutator is

$$[x, H] = \left[ x, \frac{p^2}{2m} \right] = \frac{1}{2m} [x, p^2] = \frac{1}{2m} (i\hbar \cdot 2p) = i\hbar \frac{p}{m}$$

using a known result from previous homework (Sakurai problem 1.29) to evaluate the commutator. The equation of motion becomes

$$\frac{d}{dt} x_H (t) = \frac{1}{m} p_H (t)$$

and we see that we need to know the time evolution of $p_H$. Its equation of motion is

$$\frac{d}{dt} p_H (t) = \frac{1}{i\hbar} [p, H]_H = \frac{1}{i\hbar} \left[ p, \frac{p^2}{2m} \right]_H = 0$$

since $p$ commutes with any function of itself. Thus $p_H$ is just a constant in time:

$$p_H (t) = p_H (0) = p (0)$$

(The initial condition for a Heisenberg-picture operator is that it equals the Schrodinger operator at the initial time $t_0$, which we took equal to zero.) Using (8), we can trivially integrate the differential equation (7) and apply the initial condition $x_H (0) = x (0)$, to find

$$x_H (t) = x (0) + \frac{p (0)}{m} t$$
which is our desired result.

2. **Particle in one-dimensional potential**

(a) Show that the energy eigenstates of a simple harmonic oscillator obey the following property under the $x \rightarrow -x$ parity transformation:

$$\langle -x| n \rangle = (-1)^n \langle x| n \rangle$$

The parity operator $P$ takes $|x \rangle$ to $|-x \rangle$, and is defined in such a way that $P^2 = 1$ and that $P$ is Hermitian (see Sakurai for details); thus

$$\langle -x| = \langle x| P$$  \hspace{1cm} (9)

Now we take as given that the ground state of the simple harmonic oscillator has even parity:

$$P|0 \rangle = +|0 \rangle$$  \hspace{1cm} (10)

(we saw this from our explicit solution for the energy wavefunctions). The higher eigenstates are found from the ground state by

$$|n \rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0 \rangle$$  \hspace{1cm} (11)

where $a^\dagger$ is the usual raising operator:

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{ip}{m\omega} \right)$$

How does this operator behave under parity? Since position and momentum both invert:

$$PxP = -x \hspace{1cm} PpP = -p$$

then clearly $a^\dagger$ inverts as well: $Pa^\dagger P = -a^\dagger$, or equivalently

$$Pa^\dagger = -a^\dagger P$$  \hspace{1cm} (12)

That is, $P$ anti-commutes with $a^\dagger$.

Now we put together our results (9,10,11,12):

$$\langle -x| n \rangle = \frac{1}{\sqrt{n!}} \langle -x| (a^\dagger)^n|0 \rangle$$

$$= \frac{1}{\sqrt{n!}} \langle x| P(a^\dagger)^n|0 \rangle$$

$$= \frac{(-1)^n}{\sqrt{n!}} \langle x| (a^\dagger)^n P|0 \rangle$$

$$= \frac{(-1)^n}{\sqrt{n!}} \langle x| (a^\dagger)^n |0 \rangle$$

$$= (-1)^n \langle x| n \rangle$$

(b) Use the above result to obtain the energy spectrum and eigenstates of a quantum mechanical particle trapped in the following potential: $V(x) = \infty$ if $x < 0$ and $V(x) = \frac{1}{2}m\omega^2x^2$ if $x > 0$.  

3
In the region $x > 0$, the particle experiences a harmonic oscillator potential, so its wavefunction will be a linear combination of simple harmonic oscillator energy eigenfunctions. In particular, any eigenfunction for our problem must be an eigenfunction of the ordinary SHO.

However, since the potential is infinity for $x < 0$, the particle is absolutely forbidden to be there: the wavefunction is forced to be zero. And since $\psi$ must be continuous even where the potential changes by an infinite amount, it must be the case that

$$\psi(0^+) = \lim_{x \to 0^+} \psi(x) = 0$$

which acts as a boundary condition which the wavefunction in the $x > 0$ region must satisfy.

What does this condition do? For the even-parity SHO eigenfunctions, there is nothing forcing $\psi(0)$ to be zero, and in fact it can be shown from the exact form of the wavefunction (given in Appendix A.4 of Sakurai) that it will always be non-zero. Therefore, the even-parity solutions do not satisfy the boundary conditions and can be eliminated.

But, the odd-parity solutions satisfy $\psi(x) = -\psi(-x)$; in particular, letting $x = 0$, we find that $\psi(0) = 0$. These solutions do satisfy the boundary condition, and hence are allowed to be eigenfunctions of our current potential. There is a small adjustment we have to make to the normalization of these eigenfunctions: as is, they are normalized so that

$$\int_{-\infty}^{\infty} dx \, |\psi|^2 = 1$$

where the integral goes over the full range of $x$, from $-\infty$ to $\infty$. But now $\psi$ is set to zero for negative values of $x$; so the effective range of integration shrinks to $0 \leq x < \infty$, and over this range we have

$$\int_0^{\infty} dx \, |\psi|^2 = \frac{1}{2}$$

To restore normalization, we have to scale up $\psi$ by a factor of $\sqrt{2}$.

What, then, happens to the energy of the odd-parity functions, as given by

$$\int_{-\infty}^{\infty} dx \, \psi^* H \psi = E \int_{-\infty}^{\infty} dx \, \psi^* \psi = E \int_0^{\infty} dx \, |\psi|^2$$

On the one hand, $|\psi|^2$ is scaled up by a factor of 2, but on the other hand the range of integration shrinks to half of the original. These factors cancel each other, with the consequence that the total energy of the state remains the same.

So let us define $n = 2p + 1$, where $p = 0, 1, 2, \ldots$ (and thus $n$ is odd). Then the eigenfunctions for our potential are given by

$$\psi_p(x) = \begin{cases} 0, & x \leq 0 \\ \langle x | n \rangle, & x \geq 0 \end{cases}$$

where $|n\rangle$ is an eigenstate of the original simple harmonic oscillator. The energy of the state $\psi_p(x)$ is $\hbar \omega \left(n + \frac{1}{2}\right) = \hbar \omega \left((2p + 1) + \frac{1}{2}\right)$.

3. Angular momentum operators: general properties.

(a) Show that $[\hat{L}^2, \hat{L}_i] = 0$. (Here $i \in \{x, y, z\}$.)

(I will omit the hats from the operators, and show all sums explicitly rather than using the Einstein summation convention.)
First let us establish a general relation for any operators $A, B, C$, which we will use throughout this entire problem:

$$[A, BC] = ABC - BCA$$
$$= ABC - BAC + BAC - BCA$$
$$= [A, B]C + B[A, C]$$

So, since $\vec{L}^2 = \sum_k L_k L_k$, we have

$$[\vec{L}^2, L_i] = \sum_k [L_k L_k, L_i] = -\sum_k [L_i, L_k L_k]$$
$$= -\sum_k ([L_i, L_k] L_k + L_k [L_i, L_k])$$

Next we use the fundamental property satisfied by angular momenta, namely $[L_i, L_k] = \sum_l i \epsilon_{ikl} L_l$:

$$[\vec{L}^2, L_i] = -i \sum_k \sum_l \epsilon_{ikl} (L_l L_k + L_k L_l)$$

Now notice that we are summing over indices $k$ and $l$, and what we are summing is a product of one factor which is anti-symmetric in $k, l$ and a second factor which is symmetric in $k, l$. We can instantly conclude that the sums give zero, the reason being that the summands with $k \neq l$ cancel in pairs (for example, the term with $(k, l) = (1, 2)$ cancels $(k, l) = (2, 1)$, since $\epsilon_{121} = -\epsilon_{211}$ while $L_1 L_2 + L_2 L_1 = L_2 L_1 + L_1 L_2$), while the terms with $k = l$ are zero all by themselves (e.g. $\epsilon_{111} = 0$). This is another technique we will be using repeatedly in this problem.

So $[\vec{L}^2, L_i] = 0$: we are done.

(b) Demonstrate that $[L_i, x_k] = i \epsilon_{ikl} x_l$.

First we use the definition of orbital angular momentum, $\vec{L} = \vec{x} \times \vec{p}$, to write

$$L_i = \sum_j \sum_l \epsilon_{ijl} x_j p_l$$

(notice how we avoid using the index $k$ which is already being used in the problem statement. Improperly reusing an index is a frequent source of error in this type of computation.) Then

$$[L_i, x_k] = \sum_j \sum_l \epsilon_{ijl} [x_j p_l, x_k] = -\sum_j \sum_l \epsilon_{ijl} [x_k, x_j p_l]$$
$$= -\sum_j \sum_l \epsilon_{ijl} ([x_k, x_j] p_l + x_j [x_k, p_l])$$
$$= -\sum_j \sum_l \epsilon_{ijl} (0 + x_j \cdot i \hbar \delta_{kl})$$
$$= -i \hbar \sum_j \epsilon_{ijk} x_j$$
$$= i \hbar \sum_j \epsilon_{ikj} x_j$$
$$= i \hbar \sum_l \epsilon_{ikl} x_l.$$ 

(c) Similarly, show that $[L_i, p_k] = i \hbar \epsilon_{ikl} p_l$. 

5
The method of proof is very similar to that of part (b):

\[ [L_i, p_k] = \sum_j \sum_l \epsilon_{ijl} [x_j p_l, p_k] = -\sum_j \sum_l \epsilon_{ijl} [p_k, x_j p_l] \]

\[ = -\sum_j \sum_l \epsilon_{ijl} ([p_k, x_j] p_l + x_j [p_k, p_l]) \]

\[ = -\sum_j \sum_l \epsilon_{ijl} (-i\hbar \delta_{kj} p_l + 0) \]

\[ = i\hbar \sum_l \epsilon_{ikl} p_l. \]

(d) Finally, show that \([L_i, \vec{x}^2] = 0\) and that \([L_i, p^2] = 0\).

Proceeding directly:

\[ [L_i, \vec{x}^2] = \sum_k [L_i, x_k x_k] \]

\[ = \sum_k ([L_i, x_k] x_k + x_k [L_i, x_k]) \]

Use the result of part (b):

\[ [L_i, \vec{x}^2] = i\hbar \sum_k \sum_l \epsilon_{ikl} (x_l x_k + x_k x_l) \]

As in part (a), we are summing over the product of anti-symmetric and symmetric; thus \([L_i, \vec{x}^2] = 0\).

Similarly (using part (c)):

\[ [L_i, \vec{p}^2] = \sum_k [L_i, p_k p_k] \]

\[ = \sum_k ([L_i, p_k] p_k + p_k [L_i, p_k]) \]

\[ = i\hbar \sum_k \sum_l \epsilon_{ikl} (p_l p_k + p_k p_l) \]

\[ = 0. \]

4. Angular momentum operators in spherical coordinates.

Start from the expression for the gradient operator in spherical coordinates \(r, \theta, \phi\):

\( \vec{\nabla} = e^*_r \partial_r + \frac{e^*_\theta}{\sin \theta} \partial_\theta + e^*_\phi \partial_\phi \)

where \(e^*_\alpha\) is the unit vector in direction \(\alpha\), and calculate the following operators (all results should be given in spherical coordinates):

(a) the angular momentum operator \(\vec{L}\):

We use the definition \(\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (-i\hbar \vec{\nabla})\); then we write \(\vec{r} = e^*_r r\), and use the given expression for \(\vec{\nabla}\) and the fact that \(e^*_r, e^*_\theta, e^*_\phi\) form an orthogonal triad with

\( e^*_r \times e^*_\theta = e^*_\phi, \quad e^*_\theta \times e^*_\phi = e^*_r, \quad e^*_\phi \times e^*_r = e^*_\theta \)
\[ \vec{L} = (-i\hbar) (\vec{e}_r \cdot \vec{r}) \times \left( e_r \partial_r + \frac{\vec{e}_\theta}{r} \partial_\theta + \frac{\vec{e}_\phi}{r \sin \theta} \partial_\phi \right) \]
\[ = (-i\hbar) \left( e_\phi \partial_\theta - \frac{\vec{e}_\theta}{\sin \theta} \partial_\phi \right) \]

**[I will do part (c) before part (b)]**

(c) all Cartesian components \( L_\alpha \ (\alpha \in \{x, y, z\}) \) of \( \vec{L} \):

The Cartesian components are given by e.g. \( L_x = \vec{e}_x \cdot \vec{L} \), so we need to be able to express the Cartesian unit vectors in terms of spherical coordinates (and spherical unit vectors). To do this, we recall what the unit vector for a given coordinate means: it is a vector of length 1, in the direction of greatest increase for that coordinate – i.e. in the direction of the gradient of that coordinate: so

\[ \vec{e}_x = \frac{\vec{\nabla} x}{\left| \vec{\nabla} x \right|} \]

For reference, we write out the transformation between spherical and Cartesian:

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

So we calculate:

\[ \vec{\nabla} x = \left( e_r \partial_r + \frac{\vec{e}_\theta}{r} \partial_\theta + \frac{\vec{e}_\phi}{r \sin \theta} \partial_\phi \right) (r \sin \theta \cos \phi) \]
\[ = e_r \sin \theta \cos \phi + e_\theta \cos \theta \cos \phi - e_\phi \sin \phi \]

It is easy to check that this vector already has length 1; so this is also \( \vec{e}_x \). Therefore,

\[ L_x = \vec{e}_x \cdot \vec{L} \]
\[ = (-i\hbar) \left( e_r \sin \theta \cos \phi + e_\theta \cos \theta \cos \phi - e_\phi \sin \phi \right) \cdot \left( e_\phi \partial_\theta - \frac{\vec{e}_\theta}{\sin \theta} \partial_\phi \right) \]
\[ = (-i\hbar) \left( -\frac{\cos \theta}{\sin \theta} \cos \phi \partial_\phi - (\sin \phi) \partial_\theta \right) \]

We repeat this for the next component:

\[ \vec{\nabla} y = \left( e_r \partial_r + \frac{\vec{e}_\theta}{r} \partial_\theta + \frac{\vec{e}_\phi}{r \sin \theta} \partial_\phi \right) (r \sin \theta \sin \phi) \]
\[ = e_r \sin \theta \sin \phi + e_\theta \cos \theta \sin \phi + e_\phi \cos \phi \]

This vector has length 1, and so equals \( \vec{e}_y \). So

\[ L_y = \vec{e}_y \cdot \vec{L} \]
\[ = (-i\hbar) \left( e_r \sin \theta \sin \phi + e_\theta \cos \theta \sin \phi + e_\phi \cos \phi \right) \cdot \left( e_\phi \partial_\theta - \frac{\vec{e}_\theta}{\sin \theta} \partial_\phi \right) \]
\[ = (-i\hbar) \left( -\frac{\sin \theta}{\sin \theta} \sin \phi \partial_\phi + (\cos \phi) \partial_\theta \right) \]
Finally we work through $z$:

\[
\hat{\nabla} z = \left( e_r \partial_r + \frac{e_\theta}{r} \partial_\theta + \frac{e_\phi}{r \sin \theta} \partial_\phi \right) (r \cos \theta) \\
= e_r \cos \theta - e_\theta \sin \theta
\]

Again this is unit and so equals $e_z^\ast$. Then

\[
L_z = e_z^\ast \cdot \vec{L} \\
= (-i\hbar) (e_r \cos \theta - e_\theta \sin \theta) \cdot \left( e_\phi \partial_\phi - \frac{e_\theta}{\sin \theta} \partial_\phi \right) \\
= (-i\hbar) \partial_\phi.
\]

**(b) the angular momentum squared operator $\vec{L}^2$**:

We know that $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$; so now that we have expressed the individual components of $\vec{L}$ as differential operators in spherical coordinates, our strategy will be to apply $\vec{L}^2$ to a generic function $f(r, \theta, \phi)$:

\[
\vec{L}^2 f = L_x^2 f + L_y^2 f + L_z^2 f
\]

First we calculate a useful derivative:

\[
\frac{d}{d\theta} \frac{\cos \theta}{\sin \theta} = \frac{(\sin \theta)(-\cos \theta) - (\cos \theta)\cos \theta}{\sin^2 \theta} = -\frac{1}{\sin^2 \theta}
\]

With this done, we compute each term in turn:

\[
L_x^2 f = (-i\hbar)^2 \left\{ \frac{-\cos \theta}{\sin \theta} (\cos \phi) \partial_\phi + (\sin \phi) \partial_\theta \right\} \left\{ \frac{-\cos \theta}{\sin \theta} (\cos \phi) \frac{\partial f}{\partial \phi} - (\sin \phi) \frac{\partial f}{\partial \theta} \right\} \\
= (-i\hbar)^2 \left\{ \frac{-\cos \theta}{\sin^2 \theta} \frac{\cos \phi}{\partial \phi} + \frac{\cos \theta}{\sin^2 \theta} \frac{\cos \phi}{\partial \phi} \cdot \frac{\partial^2 f}{\partial \phi^2} + \frac{\cos \theta}{\sin^2 \theta} \frac{\cos \phi}{\partial \phi} \cdot \frac{\partial f}{\partial \theta} + \frac{\cos \theta}{\sin^2 \theta} \frac{\cos \phi}{\partial \phi} \cdot \frac{\partial^2 f}{\partial \phi \partial \theta} \right\}
\]

\[
L_y^2 f = (-i\hbar)^2 \left\{ \frac{-\cos \theta}{\sin \theta} (\sin \phi) \partial_\phi + (\cos \phi) \partial_\theta \right\} \left\{ \frac{-\cos \theta}{\sin \theta} (\sin \phi) \frac{\partial f}{\partial \phi} + (\cos \phi) \frac{\partial f}{\partial \theta} \right\} \\
= (-i\hbar)^2 \left\{ \frac{-\cos \theta}{\sin^2 \theta} \frac{\sin \phi}{\partial \phi} + \frac{\cos \theta}{\sin^2 \theta} \frac{\sin \phi}{\partial \phi} \cdot \frac{\partial^2 f}{\partial \phi^2} + \frac{\cos \theta}{\sin^2 \theta} \frac{\sin \phi}{\partial \phi} \cdot \frac{\partial f}{\partial \theta} + \frac{\cos \theta}{\sin^2 \theta} \frac{\sin \phi}{\partial \phi} \cdot \frac{\partial^2 f}{\partial \phi \partial \theta} \right\}
\]

\[
L_z^2 f = (-i\hbar)^2 \frac{\partial^2 f}{\partial \phi^2}
\]

Adding these, we find

\[
\vec{L}^2 f = (-i\hbar)^2 \left\{ 0 \cdot \frac{\partial f}{\partial \phi} + \left( \frac{\cos \theta}{\sin^2 \theta} (\cos \phi + \sin^2 \phi) + 1 \right) \frac{\partial^2 f}{\partial \phi^2} + \frac{\cos \theta}{\sin \theta} (\cos^2 \phi + \sin^2 \phi) \frac{\partial f}{\partial \theta} \\
+ 0 \cdot \frac{\partial^2 f}{\partial \phi \partial \theta} + (\cos^2 \phi + \sin^2 \phi) \frac{\partial^2 f}{\partial \theta^2} \right\}
\]

\[
= (-i\hbar)^2 \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{\cos \theta}{\sin ^3 \theta} \frac{\partial f}{\partial \phi} + \frac{\partial^2 f}{\partial \theta^2} \right\}
\]

\[
= (-i\hbar)^2 \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) \right\}
\]
so we conclude that

\[ \mathbf{\hat{L}}^2 = (-i\hbar)^2 \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right\}. \]

(d) the raising and lowering operators \( L_\pm \).

We calculate directly from the definition of \( L_\pm \):

\[
\begin{align*}
L_{\pm} &= L_x \pm i L_y \\
&= (-i\hbar) \left\{ \left( \frac{-\cos \theta}{\sin \theta} \cos \phi - (\sin \phi) \partial_\theta \right) \pm i \left( \frac{-\cos \theta}{\sin \theta} \sin \phi + (\cos \phi) \partial_\theta \right) \right\} \\
&= (-i\hbar) \left\{ \left( \frac{-\cos \theta}{\sin \theta} \cos \phi \pm i \sin \phi \right) \partial_\phi \pm i \left( \cos \phi \pm i \sin \phi \right) \partial_\theta \right\} \\
&= (-i\hbar e^{\pm i\phi}) \left\{ \left( \frac{-\cos \theta}{\sin \theta} \partial_\phi \pm i \partial_\theta \right) \right\}.
\end{align*}
\]

5. The rotation group.

Rotations \( R \) in three dimensions form a non-commutative three-parameter (three dimensional) group. Show that rotations around a single axis form a commutative, one-parameter group:

(a) Give the rotation matrix \( R_z (\phi) \) that corresponds to a rotation around the \( \mathbf{z} \) axis with angle \( \phi \).

If we consider active rotations (i.e. rotations of the object rather than the axes), then the rotation matrix is given by

\[
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

(Working this problem in terms of passive rotations is OK as well – just switch the off-diagonal terms of the matrix, throughout the entire problem.)

(b) Calculate the generator \( A_z = \lim_{\phi \to 0} \partial_\phi R_z (\phi) \).

The derivative of \( R_z (\phi) \) is given by

\[
\begin{bmatrix}
-\sin \phi & -\cos \phi \\
\cos \phi & -\sin \phi
\end{bmatrix}
\]

As \( \phi \to 0 \), we have \( \cos \phi \to 1 \) and \( \sin \phi \to 0 \), so the generator is

\[
A_z = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

(c) Show that \( R_z (\phi) = \exp [\phi A_z] \).

First we show that \( A_z^2 = -I \):

\[
A_z^2 = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}.
\]
So now, let \( n \) be an even number; then we can write \( n = 2p \) where \( p \) is an integer, and so

\[
A_n^p = A_{2p} = (-I)^p = (-1)^p \cdot I
\]

We can obtain a corresponding result for odd \( n = 2p + 1 \), by multiplying through by \( A_z \):

\[
A_{2p+1} = (-1)^p \cdot A_z
\]

Then we compute directly:

\[
\exp[\phi A_z] = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi A_z)^n
\]

\[
= \sum_{p=0}^{\infty} \frac{\phi^{2p}}{(2p)!} A_{2p} + \sum_{p=0}^{\infty} \frac{\phi^{2p+1}}{(2p+1)!} A_{2p+1}
\]

\[
= I \sum_{p=0}^{\infty} (-1)^p \frac{\phi^{2p}}{(2p)!} A_{2p} + A_z \sum_{p=0}^{\infty} (-1)^p \frac{\phi^{2p+1}}{(2p+1)!}
\]

\[
= I \cos \phi + A_z \sin \phi
\]

\[
= \begin{pmatrix}
\cos \phi & 0 \\
0 & \cos \phi
\end{pmatrix} + \begin{pmatrix}
0 & -\sin \phi \\
\sin \phi & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\]

\[
= R_z(\phi).
\]

(d) Show all group properties of rotations around the \( \vec{z} \) axis:

(d.1) existence of the identity element \( 1_z \):

We take the binary operation of the group to be ordinary matrix multiplication. Then as we know, the identity element for matrix multiplication is the identity matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

And if we let \( \phi = 0 \), then \( R_z(\phi = 0) \) is exactly the identity matrix. Thus the group does possess an identity element, as required. (Actually \( \phi = 2\pi n \) where \( n \) is any integer, will also work; if we are concerned with the uniqueness of the identity element, we should restrict the allowed range for the group parameter to e.g. \( 0 \leq \phi < 2\pi \).)

(d.2) closure: \( R_z(\phi_1)R_z(\phi_2) \) is also a rotation around \( \vec{z} \);

Calculating directly:

\[
R_z(\phi_1)R_z(\phi_2) = \begin{pmatrix}
\cos \phi_1 & -\sin \phi_1 \\
\sin \phi_1 & \cos \phi_1
\end{pmatrix} \begin{pmatrix}
\cos \phi_2 & -\sin \phi_2 \\
\sin \phi_2 & \cos \phi_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 & -\cos \phi_1 \sin \phi_2 - \sin \phi_1 \cos \phi_2 \\
\sin \phi_1 \cos \phi_2 + \cos \phi_1 \sin \phi_2 & -\sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos (\phi_1 + \phi_2) & -\sin (\phi_1 + \phi_2) \\
\sin (\phi_1 + \phi_2) & \cos (\phi_1 + \phi_2)
\end{pmatrix}
\]

\[
= R_z(\phi_1 + \phi_2)
\]

where we have used the standard angle addition formulas for sin and cos. This shows explicitly that \( R_z(\phi_1)R_z(\phi_2) \) is also a rotation around the \( z \)-axis, and so the group is closed with respect to the group operation.
Assume that all derivatives of the ket $|\phi\rangle$ with respect to the angle variable $\phi$ exist. Show that $\hat{D}_z(\alpha) = \exp(\alpha \partial_\phi) = \exp\left(\frac{i}{\hbar} \alpha L_z\right)$ where $L_z = \frac{\hbar}{i} \partial_\phi$.

To express $|\phi + \alpha\rangle$ as the result of some operator upon $|\phi\rangle$, we simply expand $|\phi + \alpha\rangle$ in a Taylor series around $\phi$

$$|\phi + \alpha\rangle = |\phi\rangle + \alpha \frac{d}{d\phi} |\phi\rangle + \frac{\alpha^2}{2!} \frac{d^2}{d\phi^2} |\phi\rangle + \ldots$$

(note: we are committing a (very common) abuse of notation by using $\phi$ to represent both the general variable and the particular value of the variable at which we are evaluating the ket and derivatives of the ket.) At any rate, we can then write

$$|\phi + \alpha\rangle = \left( I + \alpha \frac{d}{d\phi} + \frac{\alpha^2}{2!} \frac{d^2}{d\phi^2} + \ldots \right) |\phi\rangle = \exp(\alpha \partial_\phi) |\phi\rangle$$
so we can identify \( \hat{D}_z (\alpha) = \exp (\alpha \partial_\phi) \). Then remembering that \( L_z = -i\hbar \partial_\phi = \hbar \alpha \partial_\phi \) which we derived in problem 4c, we have also \( \hat{D}_z (\alpha) = \exp \left( \frac{\alpha}{\hbar} L_z \right) \).

(b) The rotation operator with angle \( \phi \) around an arbitrary axis \( \vec{n} \) is given by

\[
\hat{D} (\vec{n} \phi) = \exp \left[ -i \vec{J} \cdot \vec{n} \phi / \hbar \right].
\]

Show that \( \hat{J}^2 \hat{D} (\vec{n} \phi) |jm\rangle = \hbar^2 j (j + 1) \hat{D} (\vec{n} \phi) |jm\rangle \). In other words, this means that \( \hat{D} (\vec{n} \phi) |jm\rangle \) is still an eigenket of \( \hat{J}^2 \).

(Hint: the commutation rules you proved between \( \vec{J} \) and \( \hat{J}_\alpha \) in problem 3a will be very useful here).

We showed in problem 3a that \( [\hat{J}_2, \hat{J}_\alpha] = 0 \) for any given component \( \alpha \) of \( \vec{J} \); then, by linearity, \( [\hat{J}^2, \hat{J}_\vec{n}] = 0 \) for any constant vector \( \vec{n} \) as well. So \( \hat{J}^2 \) will also commute with any function of \( \hat{J} \cdot \vec{n} \); in particular with \( \hat{D} (\vec{n} \phi) \). So

\[
\hat{J}^2 \hat{D} (\vec{n} \phi) |jm\rangle = \hat{D} (\vec{n} \phi) \hat{J}^2 |jm\rangle = \hat{D} (\vec{n} \phi) \hbar^2 j (j + 1) |jm\rangle = \hbar^2 j (j + 1) \hat{D} (\vec{n} \phi) |jm\rangle.
\]

Thus \( \hat{D} (\vec{n} \phi) |jm\rangle \) is an eigenket of \( \hat{J}^2 \) with eigenvalue \( \hbar^2 j (j + 1) \), and so is itself a state with definite \( j \).

(c) The matrix elements of the rotation operator with angle \( \phi \) around the axis \( \vec{n} \), i.e. the \( (2n + 1) \)-dimensional representation of the rotation group) are \( \mathcal{D}^{j}^{j'}_{m'm} (\vec{n} \phi) = \langle j'm' | \hat{D} (\vec{n} \phi) |jm\rangle \). Use the results obtained in the previous part of this problem to argue that \( \mathcal{D}^{j}^{j'}_{m'm} \) is diagonal in \( j \):

\[
\mathcal{D}^{j}^{j'}_{m'm} (\vec{n} \phi) = \delta_{j,j'} \mathcal{D}^{j}^{j}^{j}_{m'm} (\vec{n} \phi).
\]

We saw in the previous part that \( \hat{D} (\vec{n} \phi) |jm\rangle \) has definite \( j \); then we can write this state as \( \sum_{m''} c_{j'm''} |jm''\rangle \): i.e. a linear combination of states with different \( m'' \) but the same \( j \). So then,

\[
\mathcal{D}^{j}^{j'}_{m'm} (\vec{n} \phi) = \langle j'm' | \hat{D} (\vec{n} \phi) |jm\rangle = \sum_{m''} c_{j'm''} \langle j'm' | jm''\rangle
\]

But the various states \( |jm\rangle \) are orthonormal, so for any overlap matrix element \( \langle j'm' | jm''\rangle \) to be non-zero, we must have \( j' = j \). Thus the whole sum will also be zero unless \( j' = j \), and so we are done.

(d) Show that for small rotation angles \( \epsilon \)

\[
\mathcal{D}^{j}^{j}_{m'm} (\vec{n} \epsilon) = \delta_{m,m'} - \frac{i \epsilon_x + \epsilon_y}{2} \sqrt{(j - m) (j + m + 1)} \delta_{m',m+1} + \frac{i \epsilon_x - \epsilon_y}{2} \sqrt{(j + m) (j - m + 1)} \delta_{m',m-1} - i \epsilon_z m \delta_{m',m}.
\]
We write \( \vec{n} \epsilon = \epsilon_x \vec{e}_x + \epsilon_y \vec{e}_y + \epsilon_z \vec{e}_z \); then for \( \epsilon_x, \epsilon_y, \epsilon_z \ll 1 \) we can write

\[
\hat{D} (\vec{n} \epsilon) = \exp \left[-i \left( \frac{\epsilon_x J_x + \epsilon_y J_y + \epsilon_z J_z}{\hbar} \right) \right]
\approx 1 - \frac{i}{\hbar} (\epsilon_x J_x + \epsilon_y J_y + \epsilon_z J_z)
\]

Since \( J_{\pm} = J_x \pm i J_y \), we have \( J_x = \frac{1}{2} (J_+ + J_-) \) and \( J_y = \frac{1}{2i} (J_+ - J_-) = -\frac{i}{2} (J_+ - J_-) \), so we rewrite the above as

\[
\hat{D} (\vec{n} \epsilon) = 1 - \frac{i}{\hbar} \left( \frac{\epsilon_x}{2} (J_+ + J_-) - \frac{i \epsilon_y}{2} (J_+ - J_-) + \epsilon_z J_z \right)
\]

Then

\[
D_{m'm}^{j'j} (\vec{n} \epsilon) \approx \langle j'm' \bigg| \left( 1 - \frac{1}{\hbar} \left( \frac{i \epsilon_x + \epsilon_y}{2} J_+ + \frac{i \epsilon_x - \epsilon_y}{2} J_- - i \epsilon_z J_z \right) \right) \bigg| jm \rangle
\]

so using

\[
J_+ |jm\rangle = \hbar \sqrt{(j - m)(j + m + 1)} |j, m + 1\rangle
\]
\[
J_- |jm\rangle = \hbar \sqrt{(j + m)(j - m + 1)} |j, m - 1\rangle
\]
\[
J_z |jm\rangle = \hbar m |jm\rangle
\]

we find

\[
D_{m'm}^{j'j} (\vec{n} \epsilon) = \delta_{j'j} \delta_{m'm} - \frac{i \epsilon_x + \epsilon_y}{2} \sqrt{(j - m)(j + m + 1)} \delta_{j'j} \delta_{m',m+1},
\]
\[
- \frac{i \epsilon_x - \epsilon_y}{2} \sqrt{(j + m)(j - m + 1)} \delta_{j'j} \delta_{m',m-1} - i \epsilon_z m \delta_{j'j} \delta_{m'm}
\]

\[
\equiv \delta_{j'j} D_{m'm}^{jj} (\vec{n} \epsilon)
\]

Thus, factoring out \( \delta_{j'j} \) from the right-hand side, we obtain the desired expression for \( D_{m'm}^{jj} (\vec{n} \epsilon) \).