Physics 70007, Fall 2009
Solutions to Homework #3

October 30, 2009

1. (Sakurai 1.28)

(a) Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket

$$[x,\ F(p_x)]_{\text{classical}}$$

The classical Poisson bracket is defined in Sakurai (1.6.48):

$$[A(q,p),B(q,p)]_{\text{classical}} \equiv \sum_s \left(\frac{\partial A}{\partial q_s}\frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s}\frac{\partial B}{\partial q_s} \right)$$

so

$$\begin{aligned}
[x,\ F(p_x)]_{\text{classical}} &= \frac{\partial x}{\partial x}\frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F(p_x)}{\partial x} \\
&= 1 \cdot \frac{\partial F}{\partial p_x} - 0 \cdot 0 \\
&= \frac{\partial F}{\partial p_x}
\end{aligned}$$

(b) Let x and p_x be the corresponding quantum mechanical operators this time. Evaluate the commutator

$$\left[x,\ \exp\left(\frac{ip_x a}{\hbar}\right)\right]$$

First we expand the exponential in a power series, and use the fact that the commutator is linear in its arguments:

$$\left[x,\ \exp\left(\frac{ip_x a}{\hbar}\right)\right] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n [x,\ p_x^n]$$

Now we show by induction that $[x,\ p_x^n] = i\hbar \cdot np_x^{n-1}$ for $n \geq 1$:

- Firstly for $n = 1$, the statement becomes $[x,\ p_x] = i\hbar$, which we accept as true (this is one of our basic axioms);
- we then assume that the statement holds for n and prove that it holds for $n+1$:

$$\begin{aligned}
\left[x,\ p_x^{n+1}\right] &= xp_x^{n+1} - p_x^{n+1}x \\
&= xp_x^n p_x - p_x^n p_x x \\
&= \left([x,\ p_x^n] + p_x^n x\right) p_x - p_x^n p_x x \\
&= \left(i\hbar n p_x^{n-1} + p_x^n x\right) p_x - p_x^n p_x x \\
&= i\hbar n p_x^n + p_x^n [x,\ p_x] \\
&= i\hbar (n+1) p_x^n \qquad \text{as desired.}
\end{aligned}$$

Using this result, we have

$$\begin{aligned}
\left[x,\ \exp\left(\frac{ip_x a}{\hbar}\right)\right] &= \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n i\hbar n p_x^{n-1} \qquad \text{(the n=0 term gives zero)}\\
&= \left(\frac{ia}{\hbar}\right)(i\hbar)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{ia}{\hbar}\right)^{n-1} p_x^{n-1}\\
&= -a\exp\left(\frac{ip_x a}{\hbar}\right) \qquad \text{(shifting the n index)}
\end{aligned}$$

From part (a), we find that the corresponding classical Poisson bracket would be $\frac{\partial}{\partial p_x}\exp\left(\frac{ip_x a}{\hbar}\right) = \frac{-a}{i\hbar}\exp\left(\frac{ip_x a}{\hbar}\right)$. So we see that the correspondence rule between classical Poisson brackets and quantum-mechanical commutators (Sakurai 1.6.47), wherein we take the classical result, promote x and p to operators, and multiply by $i\hbar$ to obtain the quantum commutator, is satisfied.

(c) Using the result obtained in (b), prove that

$$\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle, \qquad (x|x'\rangle = x'|x'\rangle)$$

is an eigenstate of the coordinate operator x. What is the corresponding eigenvalue?

Let $|\psi\rangle \equiv \exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle$; then

$$\begin{aligned}
x|\psi\rangle &= \hat{x}\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle\\
&= \left(\exp\left(\frac{ip_x a}{\hbar}\right)x + \left[x,\ \exp\left(\frac{ip_x a}{\hbar}\right)\right]\right)|x'\rangle\\
&= \left(\exp\left(\frac{ip_x a}{\hbar}\right)x' - a\exp\left(\frac{ip_x a}{\hbar}\right)\right)|x'\rangle\\
&= (x'-a)|\psi\rangle
\end{aligned}$$

So $|\psi\rangle$ is an eigenstate of x with eigenvalue $x'-a$. Apparently, the exponential operator has performed a spatial translation of the position eigenstate.

2. (Sakurai 1.29)

(a) On page 247, Gottfried (1966) states that

$$[x_i, G(\vec{p})] = i\hbar\frac{\partial G}{\partial p_i}, \qquad [p_i, F(\vec{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$

can be "easily derived" from the fundamental commutation relations, for all functions F and G that can be expressed as power series in their arguments. Verify this statement.

First we note that although G is a function of all the components of $\vec{p}$, the only component which does not commute with x_i is p_i; so for this computation, we can consider G to be a function of p_i only (the other components of $\vec{p}$ will act like constants as far as commutation with x_i is concerned.) We then expand G as a power series in p_i, apply the linearity of the commutator, and use the result we derived in part (b) of the previous problem:

$$\begin{aligned}
[x_i, G(\vec{p})] &= \sum_{n=0}^{\infty} g_n\,[x_i,\ p_i^n]\\
&= \sum_{n=0}^{\infty} g_n\cdot i\hbar n p_i^{n-1} \qquad \text{(the n=0 term is zero)}\\
&= i\hbar\frac{\partial}{\partial p_i}\sum_{n=0}^{\infty} g_n p_i^n\\
&= i\hbar\frac{\partial G}{\partial p_i}
\end{aligned}$$

The second relation can be derived in a very similar way (the needed lemma is $[p_i,\ x_i^n] = -i\hbar n x_i^{n-1}$ whose proof is very similar to the one already performed.)

(b) Evaluate $[x^2,\ p^2]$. Compare your result with the classical Poisson bracket $[x^2,\ p^2]_{\text{classical}}$.

Our strategy to evaluate $[x^2,\ p^2] = xxpp - ppxx$ is to move the x operators in the first term one by one, through the p operators:

$$\begin{aligned}
[x^2,\ p^2] &= xxpp - ppxx \\
&= x\left(p^2 x + [x,\ p^2]\right) - p^2 x^2 \\
&= xp^2 x + 2i\hbar xp - p^2 x^2 \qquad \text{(from part a)} \\
&= \left(p^2 x + [x,\ p^2]\right) x + 2i\hbar xp - p^2 x^2 \\
&= p^2 x^2 + 2i\hbar px + 2i\hbar xp - p^2 x^2 \\
&= 2i\hbar\,(xp + px)
\end{aligned}$$

In contrast, the classical Poisson bracket is

$$\begin{aligned}
[x^2,\ p^2]_{\text{classical}} &= \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x} \\
&= (2x)(2p) - 0 \cdot 0 \\
&= 4xp
\end{aligned}$$

At first glance, it might seem as though the classical-quantum correspondence rule is failing us here. In fact this is not the case – there is an ambiguity in the rule, due to the fact that the classical x and p commute (as they are just ordinary c-numbers) while the corresponding quantum *operators* do not. So when we are presented with a classical quantity like xp, we can quantize it as is, giving us the operator $\hat{x}\hat{p}$, or we could commute the classical quantities first to get px and then quantize, giving us a different operator $\hat{p}\hat{x}$! Both choices appear equally valid, yet they cannot possibly both be correct. The prescription that appears to work is the *symmetric prescription*, which respects both choices equally by writing xp as $1/2\,(xp + px)$ before quantizing; at any rate, it makes classical-quantum correspondence work for this problem. The bottom line, however, is that it is often far from clear how to quantize a classical system once we move beyond the simplest cases.

3. (Sakurai 1.30)

The translation operator for a finite (spatial) displacement is given by

$$\mathcal{T}(\vec{l}) = \exp\left(\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right)$$

where $\vec{p}$ is the momentum operator.

(a) Evaluate $\left[x_i,\ \mathcal{T}(\vec{l})\right]$.

First note that we can write the translation operator as

$$\begin{aligned}
\mathcal{T}(\vec{l}) &= \exp\left(\frac{-i\,(p_x l_x + p_y l_y + p_z l_z)}{\hbar}\right) \\
&= \exp\left(\frac{-ip_x l_x}{\hbar}\right)\exp\left(\frac{-ip_y l_y}{\hbar}\right)\exp\left(\frac{-ip_z l_z}{\hbar}\right)
\end{aligned}$$

since the various components of momentum commute with each other. Then, since only p_i out of all the components of momentum does not commute with x_i, we can treat the exponentials not involving p_i as constants and factor them out of the commutator:

$$\begin{aligned}
\left[x_i,\ \mathcal{T}(\vec{l})\right] &= \left[x_i, \prod_j \exp\left(\frac{-ip_j l_j}{\hbar}\right)\right] \\
&= \left(\prod_{j\neq i} \exp\left(\frac{-ip_j l_j}{\hbar}\right)\right)\left[x_i, \exp\left(\frac{-ip_i l_i}{\hbar}\right)\right]
\end{aligned}$$

The remaining commutator is the same one we evaluated in problem 1, part b:

$$\begin{aligned}\left[x_i,\ \mathcal{T}(\vec{l})\right] &= \left(\prod_{j\neq i}\exp\left(\frac{-ip_jl_j}{\hbar}\right)\right)\left(l_i\exp\left(\frac{-ip_il_i}{\hbar}\right)\right) \\ &= l_i\,\mathcal{T}(\vec{l}).\end{aligned}$$

(b) Using (a) (or otherwise), demonstrate how the expectation value $\langle\vec{x}\rangle$ changes under translation.

We consider some state $|\psi\rangle$ versus its translated counterpart $|\psi'\rangle \equiv \mathcal{T}(\vec{l})\,|\psi\rangle$, and look at the expectation value of some component of position:

$$\begin{aligned}\langle\psi'|\,x_i\,|\psi'\rangle &= \langle\psi|\,\mathcal{T}(\vec{l})^\dagger x_i\mathcal{T}(\vec{l})\,|\psi\rangle \\ &= \langle\psi|\,\mathcal{T}(\vec{l})^\dagger\mathcal{T}(\vec{l})x_i\,|\psi\rangle + \langle\psi|\,\mathcal{T}(\vec{l})^\dagger\left[x_i,\ \mathcal{T}(\vec{l})\right]|\psi\rangle \\ &= \langle\psi|\,\mathcal{T}(\vec{l})^\dagger\mathcal{T}(\vec{l})x_i\,|\psi\rangle + l_i\,\langle\psi|\,\mathcal{T}(\vec{l})^\dagger\mathcal{T}(\vec{l})\,|\psi\rangle \\ &= \langle\psi|\,x_i\,|\psi\rangle + l_i\end{aligned}$$

where we have used the fact that $\mathcal{T}(\vec{l})$ is unitary, since the components of momentum are hermitian. This result holds for each component i, so we can combine our results into the vector equation

$$\langle\vec{x}\rangle' = \langle\vec{x}\rangle + \vec{l}$$

which shows that the expectation value of position is translated, as we might have suspected.

4. (Sakurai 2.5)

Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x).$$

By calculating $[[H,x],x]$, prove

$$\sum_{a'}|\langle a''|\,x\,|a'\rangle|^2\,(E_{a'} - E_{a''}) = \frac{\hbar^2}{2m}$$

where $|a'\rangle$ is a [normalized] energy eigenket with eigenvalue $E_{a'}$ [and the sum goes over the complete set of energy eigenstates].

First we note that

$$\langle a''|\,x\,|a'\rangle^* = \langle a'|\,x^\dagger\,|a''\rangle = \langle a'|\,x\,|a''\rangle$$

since x is hermitian. Then calling the desired sum S, we have

$$\begin{aligned}S &= \sum_{a'}|\langle a''|\,x\,|a'\rangle|^2\,(E_{a'} - E_{a''}) \\ &= \sum_{a'}\langle a''|\,x\,|a'\rangle\,\langle a'|\,x\,|a''\rangle\,(E_{a'} - E_{a''}) \\ &= \sum_{a'}E_{a'}\,\langle a''|\,x\,|a'\rangle\,\langle a'|\,x\,|a''\rangle - \sum_{a'}E_{a''}\,\langle a''|\,x\,|a'\rangle\,\langle a'|\,x\,|a''\rangle \\ &= \sum_{a'}\langle a''|\,xE_{a'}\,|a'\rangle\,\langle a'|\,x\,|a''\rangle - E_{a''}\sum_{a'}\langle a''|\,x\,|a'\rangle\,\langle a'|\,x\,|a''\rangle \\ &= \sum_{a'}\langle a''|\,xH\,|a'\rangle\,\langle a'|\,x\,|a''\rangle - E_{a''}\sum_{a'}\langle a''|\,x\,|a'\rangle\,\langle a'|\,x\,|a''\rangle \\ &= \langle a''|\,xH\sum_{a'}(|a'\rangle\,\langle a'|)\,x\,|a''\rangle - E_{a''}\,\langle a''|\,x\sum_{a'}(|a'\rangle\,\langle a'|)\,x\,|a''\rangle \\ &= \langle a''|\,xHx\,|a''\rangle - E_{a''}\,\langle a''|\,x^2\,|a''\rangle\end{aligned}$$

Now, looking at the second term, we can proceed in two different ways: we can move $E_{a''}$ into the matrix element, either to the left or to the right of the x^2. Either way, it ends up next to an a'' eigenstate and so can be turned into the Hamiltonian operator H; so in the former case, we obtain

$$\begin{aligned} S &= \langle a''| \, xHx - Hxx \, |a''\rangle = \langle a''| \, [x, H] \, x \, |a''\rangle \\ &= - \langle a''| \, [H, x] \, x \, |a''\rangle \end{aligned}$$

while in the latter case we get

$$S = \langle a''| \, xHx - xxH \, |a''\rangle = \langle a''| \, x \, [H, x] \, |a''\rangle$$

Adding these two expressions, we find

$$2S = \langle a''| \, [x, [H, x]] \, |a''\rangle = - \langle a''| \, [[H, x] \, , x] \, |a''\rangle$$

or

$$S = -\frac{1}{2} \langle a''| \, [[H, x] \, , x] \, |a''\rangle$$

The nested commutator is easy to evaluate:

$$[H, x] = \left[\frac{p^2}{2m} + V(x) \, , x \right] = \frac{1}{2m} \left[p^2, x \right] = \frac{-1}{2m} (2i\hbar p) = \frac{-i\hbar}{m} p$$

and so

$$[[H, x] \, , x] = \frac{-i\hbar}{m} [p, x] = \frac{(-i\hbar)^2}{m} = -\frac{\hbar^2}{m}$$

So at last,

$$S = \left(-\frac{1}{2} \right) \left(-\frac{\hbar^2}{m} \right) \langle a''|a''\rangle = \frac{\hbar^2}{2m}.$$