1. (Sakurai 2.11)
Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by
$$\exp\left(-\frac{i p a}{\hbar}\right)|0\rangle$$
where $p$ is the momentum operator and $a$ is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle x \rangle$ for $t \geq 0$.

Recall that in the Heisenberg picture, the state kets/bras stay fixed, while the operators evolve in time. At time $t = 0$, Heisenberg-picture operators equal their Schrodinger-picture counterparts (this specifies their initial value), so we will write e.g. $x(0) \equiv \hat{x}$. Then
$$\langle x(t) \rangle = \langle 0 | \exp\left(\frac{i p a}{\hbar}\right) x(t) \exp\left(-\frac{i p a}{\hbar}\right) |0\rangle$$

We can express $x(t)$ in terms of Schrodinger operators by solving the Heisenberg equations of motion for $x(t)$ and $p(t)$. These are coupled first-order linear ordinary differential equations, and the method of solution is straightforward. Sakurai carries this through in section 2.3; here we will only quote the final result (Sakurai 2.3.45a):
$$x(t) = (\cos \omega t) x(0) + \left(\frac{\sin \omega t}{m \omega}\right) p(0)$$
$$= (\cos \omega t) \hat{x} + \left(\frac{\sin \omega t}{m \omega}\right) \hat{p}$$

Using this,
$$\langle x(t) \rangle = \left(0 \left| \exp\left(\frac{i p a}{\hbar}\right) \left(\cos \omega t \right) \hat{x} + \left(\frac{\sin \omega t}{m \omega}\right) \hat{p} \right| \exp\left(-\frac{i p a}{\hbar}\right) |0\rangle$$
$$= (\cos \omega t) \langle 0 | \exp\left(\frac{i p a}{\hbar}\right) \hat{x} \exp\left(-\frac{i p a}{\hbar}\right) |0\rangle + \left(\frac{\sin \omega t}{m \omega}\right) |0\rangle \hat{p} |0\rangle$$

In the first matrix element, we need to commute $\hat{x}$ with one of the exponentials. We learned how to do this in a previous homework (Sakurai problem 1.29a): we found
$$\left[ \hat{x}, \exp\left(-\frac{i p a}{\hbar}\right) \right] = i \hbar \frac{\partial}{\partial \hat{p}} \exp\left(-\frac{i p a}{\hbar}\right) = a \exp\left(-\frac{i p a}{\hbar}\right)$$

i.e.
$$\hat{x} \exp\left(-\frac{i p a}{\hbar}\right) - \exp\left(-\frac{i p a}{\hbar}\right) \hat{x} = a \exp\left(-\frac{i p a}{\hbar}\right)$$

so
$$\langle 0 | \exp\left(\frac{i p a}{\hbar}\right) \hat{x} \exp\left(-\frac{i p a}{\hbar}\right) |0\rangle = \langle 0 | \exp\left(\frac{i p a}{\hbar}\right) \left( a \exp\left(-\frac{i p a}{\hbar}\right) + \exp\left(-\frac{i p a}{\hbar}\right) \hat{x} \right) |0\rangle$$
$$= a \langle 0 | 0\rangle + \langle 0 | \hat{x} |0\rangle$$
$$= a + \langle 0 | \hat{x} |0\rangle$$
and thus
\[ x(t) = (\cos \omega t) (a + (0|\hat{x}|0)) + \left( \frac{\sin \omega t}{m\omega} \right) \langle 0|\hat{p}|0 \rangle \]

Later on in this homework, we will see that the needed matrix elements are both zero: \( \hat{x} \) and \( \hat{p} \) are non-diagonal in the SHO energy eigenbasis (#3 in this homework: Sakurai problem 2.13). Here we take this result as given; thus
\[ \langle x(t) \rangle = a \cos \omega t. \]

2. (Sakurai 2.12)

(a) Write down the wave function (in coordinate space) for the state specified in Problem 11 at \( t = 0 \). You may use
\[ \langle x'|0 \rangle = \pi^{-1/4} x_0^{-1/2} \exp \left[ -\frac{1}{2} \left( \frac{x'}{x_0} \right)^2 \right], \quad \left( x_0 \equiv \left( \frac{\hbar}{m\omega} \right)^{\frac{1}{2}} \right) \]
The state in question from the previous problem is \( |\psi \rangle \equiv \exp \left( \frac{-i\hat{p}a}{\hbar} \right) |0 \rangle \); so the desired coordinate-space wave function for this state is
\[ \psi(x) = \langle x | \exp \left( \frac{-i\hat{p}a}{\hbar} \right) |0 \rangle \]
In order to use the wave function given to us in the problem statement, we insert the identity strategically into our matrix element:
\[ \psi(x) = \int dx' \langle x | \exp \left( \frac{-i\hat{p}a}{\hbar} \right) |x' \rangle \langle x'|0 \rangle \]
The quickest way to proceed at this point is to remember, from a previous homework (Sakurai problem 1.28c), that the exponential is the spatial translation operator and so operates upon position eigenvectors in a very simple way:
\[ \exp \left( \frac{-i\hat{p}a}{\hbar} \right) |x' \rangle = |x' + a \rangle \]
(note the opposite signs attached to \( a \) on the left-hand and right-hand sides of this relation!) Then we have
\[ \psi(x) = \int dx' \langle x | x' + a \rangle \langle x'|0 \rangle = \int dx' \delta(x' + a - x) \langle x'|0 \rangle = \int dx' \delta(x' - (x - a)) \langle x'|0 \rangle = \langle x - a |0 \rangle = \pi^{-1/4} x_0^{-1/2} \exp \left[ -\frac{1}{2} \left( \frac{x - a}{x_0} \right)^2 \right] \]

(b) Obtain a simple expression for the probability that the state is found in the ground state at \( t = 0 \). Does this probability change for \( t > 0 \)?

The probability that a measurement of the energy gives the ground state energy, is given by the squared amplitude of the overlap matrix element between our state and the ground state:
\[
\text{Prob (ground state)} = |\langle 0 | \psi \rangle|^2
\]
\[
= \left| \int dx' \langle 0 | x' \rangle \langle x' | \psi \rangle \right|^2
\]
\[
= \left| \int dx' \langle x' | 0 \rangle \langle x' | \psi \rangle \right|^2
\]
\[
= \left| \int dx' \left( \pi^{-1/4} x_0^{-1/2} \exp \left[-\frac{1}{2} \left( \frac{x'}{x_0} \right)^2 \right] \right) \left( \pi^{-1/4} x_0^{-1/2} \exp \left[-\frac{1}{2} \left( \frac{x' - a}{x_0} \right)^2 \right] \right)^2 \right|^2
\]
\[
= \frac{1}{\pi x_0^2} \left| \int dx' e^{-\frac{x'^2 + ax'}{x_0^2}} \right|^2
\]

The needed integral is
\[
\int_{-\infty}^{\infty} dx \ e^{-c_1 x^2 + c_2 x} = e^{\frac{c_2^2}{4c_1}} \cdot \sqrt{\pi \ c_1}
\]

(this can be obtained from the known result \( \int_{-\infty}^{\infty} dx \ \exp (-x^2) = \sqrt{\pi} \) by completing the square in the exponent). We obtain
\[
\text{Prob (ground state)} = \frac{1}{\pi x_0^2} \exp \left(-\frac{a^2}{x_0^2} \right) \cdot \left| \exp \left( \frac{1}{4} \cdot \frac{1}{x_0^2} \right) \sqrt{\frac{\pi}{1/x_0^2}} \right|^2
\]
\[
= \exp \left(-\frac{1}{2} \frac{a^2}{x_0^2} \right).
\]

The probability at later times is most easily evaluated in the Schrödinger picture. Our state \( |\psi\rangle \) evolves via the time-evolution operator:
\[
|\psi(t)\rangle = \exp \left(-\frac{i\hat{H}t}{\hbar} \right) |\psi\rangle
\]
while the ground state base ket \( |0\rangle \) does not change (the operator \( \hat{H} \) does not change with time, so its eigenkets also stay the same). So
\[
\text{Prob (ground state at } t > 0) = \left| \langle 0 | \psi(t) \rangle \right|^2
\]
\[
= \left| \left\langle 0 \exp \left(-\frac{i\hat{H}t}{\hbar} \right) |\psi\rangle \right\rangle \right|^2
\]

Now what we can do is apply the time-evolution operator to the left, which is easy because we know its effect upon an energy eigenstate:
\[
\text{Prob (ground state at } t > 0) = \left| \exp \left(-\frac{i \left( \frac{1}{2} \omega \right) t}{\hbar} \right) \langle 0 | \psi \rangle \right|^2
\]
\[
= \left| \exp \left(-\frac{1}{2} i \omega t \right) \langle 0 | \psi \rangle \right|^2
\]

But this extra factor is just a phase factor (a complex number with magnitude 1). When we take the magnitude squared, it simply becomes 1 and drops out of the calculation. Thus we get the same probability as we did at \( t = 0 \)!
3. (Sakurai 2.13)

Consider a one-dimensional simple harmonic oscillator.

(a) Using

\[
\begin{align*}
\frac{a}{a^\dagger} &= \sqrt{\frac{\hbar}{2m\omega}} \left( x + \frac{ip}{m\omega} \right), \\
\frac{a^\dagger}{a} &= \sqrt{\frac{\hbar}{2m\omega}} \left( x - \frac{ip}{m\omega} \right),
\end{align*}
\]

evaluate \langle m|x|n \rangle, \langle m|p|n \rangle, \langle m|\{x,p\}|n \rangle, \langle m|x^2|n \rangle, \text{ and } \langle m|p^2|n \rangle.

First we solve for \(x\) and \(p\) in terms of \(a\) and \(a^\dagger\):

\[
x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad p = i\sqrt{\frac{\hbar m\omega}{2}} (a^\dagger - a)
\]

Then

\[
\langle m|x|n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle m|a + a^\dagger|n \rangle
\]

\[
= \sqrt{\frac{\hbar}{2m\omega}} \langle m| \left\{ \sqrt{n} |n - 1\rangle + \sqrt{n + 1} |n + 1\rangle \right\}
\]

\[
= \sqrt{\frac{\hbar}{2m\omega}} \left( \sqrt{n} \delta_{m,n-1} + \sqrt{n + 1} \delta_{m,n+1} \right)
\]

Similarly

\[
\langle m|p|n \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \left( \sqrt{n + 1} \delta_{m,n+1} - \sqrt{n} \delta_{m,n-1} \right)
\]

Next \(\{x,p\} = xp + px = \frac{\hbar}{2} \left( (a^\dagger + a) (a^\dagger - a) + (a^\dagger - a) (a^\dagger + a) \right) = i\hbar (a^\dagger a - a^\dagger 2)\), so

\[
\langle m|\{x,p\}|n \rangle = i\hbar \left( \sqrt{(n + 1)(n + 2)} \delta_{m,n+2} - \sqrt{n(n-1)} \delta_{m,n-2} \right)
\]

Then

\[
x^2 = \frac{\hbar}{2m\omega} (a + a^\dagger)^2 = \frac{\hbar}{2m\omega} \left( a^2 + aa^\dagger + a^\dagger a + a^\dagger 2 \right) = \frac{\hbar}{2m\omega} \left( a^\dagger 2 + 2a^\dagger a - 1 + a^2 \right) \text{ (using } [a,a^\dagger] = 1),
\]

\[
= \frac{\hbar}{2m\omega} \left( \sqrt{(n + 1)(n + 2)} \delta_{m,n+2} + (2n + 1) \delta_{m,n} + \sqrt{n(n-1)} \delta_{m,n-2} \right)
\]

Finally

\[
p^2 = -\frac{\hbar m\omega}{2} (a^\dagger - a)^2 = \frac{\hbar m\omega}{2} \left( -a^2 + aa^\dagger + a^\dagger a - a^\dagger 2 \right) = \frac{\hbar m\omega}{2} \left( -a^\dagger 2 + 2a^\dagger a - 1 - a^2 \right) \text{, so}
\]

\[
\langle m|p^2|n \rangle = \frac{\hbar m\omega}{2} \left( -\sqrt{(n + 1)(n + 2)} \delta_{m,n+2} + (2n + 1) \delta_{m,n} - \sqrt{n(n-1)} \delta_{m,n-2} \right)
\]

(b) Check that the virial theorem holds for the expectation values of the kinetic and the potential energy taken with respect to an energy eigenstate.

The virial theorem in one dimension takes the form (see e.g. Bransden & Joachain, Intro to QM, 1st ed., p.227):

\[
2 \langle T \rangle = \left\langle x \frac{\partial V}{\partial x} \right\rangle
\]

where the expectation values are evaluated in an energy eigenstate. For our potential \(V = \frac{1}{2}m\omega^2 x^2\),

\[
x \frac{\partial V}{\partial x} = x \cdot m\omega^2 x = 2V
\]

so the virial theorem reduces to \(\langle T \rangle = \langle V \rangle\). We check that this holds:

\[
\langle T \rangle \equiv \langle n|T|n \rangle = \frac{1}{2m} \langle n|p^2|n \rangle = \frac{1}{2m} \frac{\hbar m\omega}{2} (2n + 1) = \frac{\hbar \omega}{4} (2n + 1)
\]
\[ \langle V \rangle \equiv \langle n|V|n \rangle = \frac{m\omega^2}{2} \langle n|x^2|n \rangle = \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} (2n + 1) = \frac{\hbar\omega}{4} (2n + 1) \]

so the theorem is satisfied.

4. (Sakurai 2.15)
Consider a function, known as the **correlation function**, defined by

\[ C(t) = \langle x(t) x(0) \rangle , \]

where \( x(t) \) is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of a one-dimensional simple harmonic oscillator.

As in the first problem, we use Sakurai 2.3.45a for the simple harmonic oscillator:

\[ x(t) = (\cos \omega t) x(0) + \left( \frac{\sin \omega t}{m\omega} \right) p(0) \]

Then writing \( x(0) \) and \( p(0) \) as \( \hat{x} \) and \( \hat{p} \), we have

\[ C(t) = (\cos \omega t) \langle 0|\hat{x}^2|0 \rangle + \left( \frac{\sin \omega t}{m\omega} \right) \langle 0|\hat{p}\hat{x}|0 \rangle \]

Now we use the results of the previous problem to evaluate the matrix elements. To do the second one, we rewrite \( \hat{p}\hat{x} \):

\[ \hat{p}\hat{x} = \frac{1}{2} (\{\hat{x},\hat{p}\} - [\hat{x},\hat{p}]) = \frac{1}{2} (\{\hat{x},\hat{p}\} - i\hbar) \]

Then

\[ C(t) = (\cos \omega t) \cdot \frac{\hbar}{2m\omega} (1) + \left( \frac{\sin \omega t}{m\omega} \right) \cdot \frac{1}{2} (0 - i\hbar) \]

\[ = \frac{\hbar}{2m\omega} (\cos \omega t - i\sin \omega t) \]

\[ = \frac{\hbar e^{-i\omega t}}{2m\omega} . \]