Problem 1. Fourier Transforms - Solving the Wave Equation

This problem is designed to make sure that you understand how to apply the Fourier transform to differential equations in general, which we will need later in the course.
In Physics there is an equation similar to the Diffusion equation called the Wave equation

$$\frac{\partial^2 C}{\partial t^2} = v^2 \frac{\partial^2 C}{\partial x^2}. \tag{1}$$

The main difference is that the derivative in time is also second order. Using Fourier Transforms in space solve the above equation for the following initial condition

$$C(t=0) = \delta(x) \qquad \frac{\partial C}{\partial t}|_{t=0} = 0 \tag{2}$$

Feel free to use Mathematica to help you, but also try and do it by hand so that you understand everything that is going on. Write the solution and explain what the solution is telling you. By he way v is called the wave speed.

Problem 2. Multidimensional Fourier Transforms - The Diffusion Equation in 2D

This problem is meant to push you outside your comfort zone and see if you can figure out what applying multiple Fourier transforms in multiple spatial dimensions looks like. The principles in 2d are the exact same as 1d so just try your best and do not be shy about coming to speak to me.
The diffusion equation in two dimensions for concentration $C(x, y, t)$is given by

$$\frac{\partial C(x,y,t)}{\partial t} = \frac{\partial^2 C(x,y,t)}{\partial x^2} + \frac{\partial^2 C(x,y,t)}{\partial y^2} \tag{1}$$

Consider initial condition

$$C(x, y, t=0) = \delta(x)\delta(y) \tag{2}$$

that is a delta pulse located at $x = 0$ and $y = 0$. Now define the following two Fourier transforms (one in the x direction and the other in the y direction)

$$F[f(x,y)] = \hat{f}(k,y) = \int_{-\infty}^{\infty} f(x,y)e^{ikx}dx \qquad G[f(x,y)] = \tilde{f}(x,l) = \int_{-\infty}^{\infty} f(x,y)e^{ily}dy \tag{3}$$

Apply both transforms to equation (3) and the initial condition to write the governing equation in double Fourier space for the concentration $\tilde{\hat{C}}(k, l, t)$. You should have an ordinary differential equation in time. Solve it for the given initial condition. Now define the equivalent inverse Fourier transforms

$$F^{-1}[f(k)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ikx}dk \qquad G^{-1}[f(l)] = f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(l)e^{-ily}dl \tag{4}$$

and apply them to your solution to obtain the solution in real space. What is the main influence of adding a new dimension? Without solving it explicitly what do you expect the solution in 3d to look like where the governing equation is now

$$\frac{\partial C(x,y,t)}{\partial t} = \frac{\partial^2 C(x,y,t)}{\partial x^2} + \frac{\partial^2 C(x,y,t)}{\partial y^2} + \frac{\partial^2 C(x,y,t)}{\partial z^2} \tag{5}$$

Consider initial condition

$$C(x,y,z,t=0) = \delta(x)\delta(y)\delta(z) \tag{6}$$

Problem 3. Performing Fourier Transforms numerically

If you cannot get an analytical solution for the Fourier Transform (by hand or with Mathematica), you can always try doing so numerically. To do this you need to approximate

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx} dk. \tag{1}$$

If you have taken a numerical methods class you should already know how to do this, but if not here's the overarching idea. Represent the integral as a discrete sum

$$\int_a^b h(x)dx \approx \sum_{i=1}^{N} h(x_i)\Delta x \tag{2}$$

where you break the domain $[a,b]$ into N points, separated by distance Δx (this need not be a constant, but for the simplest case let it be). The accuracy of this approximation improves with smaller Δx or if you implement a better numerical integration method (this is one of the most basic ones - look up a numerical methods book for better approaches if you so desire, or implement one of the higher order methods available in programs like Matlab).

Now write two codes, one that implements this approximation (or a better one if you're eager) for the Fourier Transform and one for the Inverse Fourier Transform. Test your code by calculating

- The Fourier Transform of $f(x) = \frac{1}{\sqrt{4\pi}} e^{\frac{-x^2}{4}}$ (i.e. diffusion equation for $D = 1$ and $t = 1$)
- The Inverse Fourier Transform of $f(k) = e^{-k^2}$ (i.e. diffusion equation for $D = 1$ and $t = 1$)

Obviously you cannot implement an integral from $-\infty$ to ∞ on a computer so you need to define a lower and upper truncation to the integral with the understanding that you may loose some accuracy, but if the domain is sufficiently large this should not be significant (you can pick this truncation by looking at where your function becomes very very small or approaches zero). Compare your results with the analytical solution that you know it should be (i.e. calculate them analytically). Keep your codes as we will return to using them, particularly in the second half of the course.