Problem 1. Fractional Dispersion - Peaks and Tails

By now from lectures you should know that the solution to the following fractional diffusion equation

$$\frac{\partial C}{\partial t} = D_+ \frac{\partial^\alpha C}{\partial x^\alpha} + D_- \frac{\partial^\alpha C}{\partial (-x)^\alpha} \qquad C(t=0) = \delta(x) \tag{1}$$

is given by

$$C(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(D_+(-ik)^\alpha + D_-(ik)^\alpha)t} e^{ikx} dk \tag{2}$$

and while we explored visually how this solution changes we did not expressly look at certain quantitative features that may be important.

(i) Recall that for the standard solution of the diffusion equation the concentration of the peak (i.e. at $x = 0$) decreases with time as $C(x = 0, t) \sim t^{-1/2}$. Can you derive from the expression in (2) what the scaling in time should be for C(x=0,t) for fractional diffusion - i.e. $C(x = 0, t) \sim t^\gamma$; what is γ? If you cannot derive it immediately, first calculate several solutions with the Matlab code provided to you in class to empirically explore what it is and then go back to trying to calculate it. In particular, does this scaling depend on α and/or D_+ and D_-?

(ii) As we saw in the plots in class the solution in (2) is characterized by heavy tails far away from the peak of the solution. Can you tell me how these scales with distance? i.e. for large x, $C \sim x^\beta$; what is β in terms of the parameters of the fractional dispersion equation? Again, if you are not sure how to do this analytically, begin by exploring it empirically with the provided Matlab codes. Again, does this scaling depend on α and/or D_+ and D_-?

(iii) Why, from a practical perspective, do you think the information from (i) or (ii) is potentially useful?

Problem 2. Fractional Advection Dispersion - Tails

If we now consider the fractional advection dispersion equation

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D_+ \frac{\partial^\alpha C}{\partial x^\alpha} + D_- \frac{\partial^\alpha C}{\partial (-x)^\alpha} \qquad C(t=0) = \delta(x) \tag{3}$$

we said the solution is given by

$$C(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(D_+(-ik)^\alpha + D_-(ik)^\alpha)t} e^{ik(x-ut)} dk \tag{4}$$

Recall from our previous discussions that one of the ways we usually measure transport in environmental flows is to measure breakthrough curves, that is the evolution of concentration at a fixed downstream point in space over time.

(i) Consider the case $D_+ = D_- = u = 1$ with the pulse initial condition in (3). Modify the Matlab codes provided to you and plot the breakthrough curves that you would obtain at a downstream distance of $x = 10$. Consider $\alpha = 1.9$,1.7 and 1.5. Be careful to make sure that your curves look the way you expect them to. Given that you are using a numerical solution blips or oddities can appear, which may require you to adjust you resolution of the k vector.

For x=10 plot your solutions up to a time of 10^4 and show your results on logrithmic axes. To best do this define your time vector in matlab with logarithmically distributed time steps - i.e. us the command time=logspace(-1,4,1000), which creates a vector of length 1000, starting at time 10^{-1} out to 10^4 with logarithmic increments.

Now focus on the tails of these breakthrough curves, that is look only at the last 100 elements of the concentration breakthrough vector. Plot log(time) against log(C) for these elements. What do you see? Is there are trend with the different values of α that emerges? How do these late time tails scale with time and α?

(ii) What happens when (a) $D_+ = 1.5$ and $D_- = 0.5$, (b) $D_+ = 2$ and $D_- = 0$, (c) $D_+ = 0.5$ and $D_- = 1.5$ and (d) $D_+ = 0$ and $D_- = 2$. Are your observations similar or different. When they are different, can you explain them using your physical reasoning and understanding of how fractional dispersion in each of these cases works? Again - be careful to always look at the full solution you obtain and make sure that no funny numerical artifacts are creeping in. If they are recognize them and explain them away rather than believing them (because in some cases you just cannot avoid them).

Problem 3. Fractional Advection Dispersion - Real Data

The following is real data obtained from a transport experiment in a river in Michigan. You can download it from the course website. It is some of the cleanest data I know, but has also been smoothed to remove noise in this case. It is a breakthrough curve measured at a distance 40 meters downstream of the injection location. You can assume the mass injected is 1 kg. From this data, can you estimate u, $D-+$, D_- and α assuming a fractional advection dispersion equation? While you're at it, also estimate what your best bet would be if you assume traditional advection dispersion, i.e. what would the best u and D be there and demonstrate why one model works better than the other. Data is collected every second over a period of 4000 seconds.

As always, rather than taking a brute force approach, try to estimate some of the parameters simply by looking at certain details of the curves and then constrain them and others.

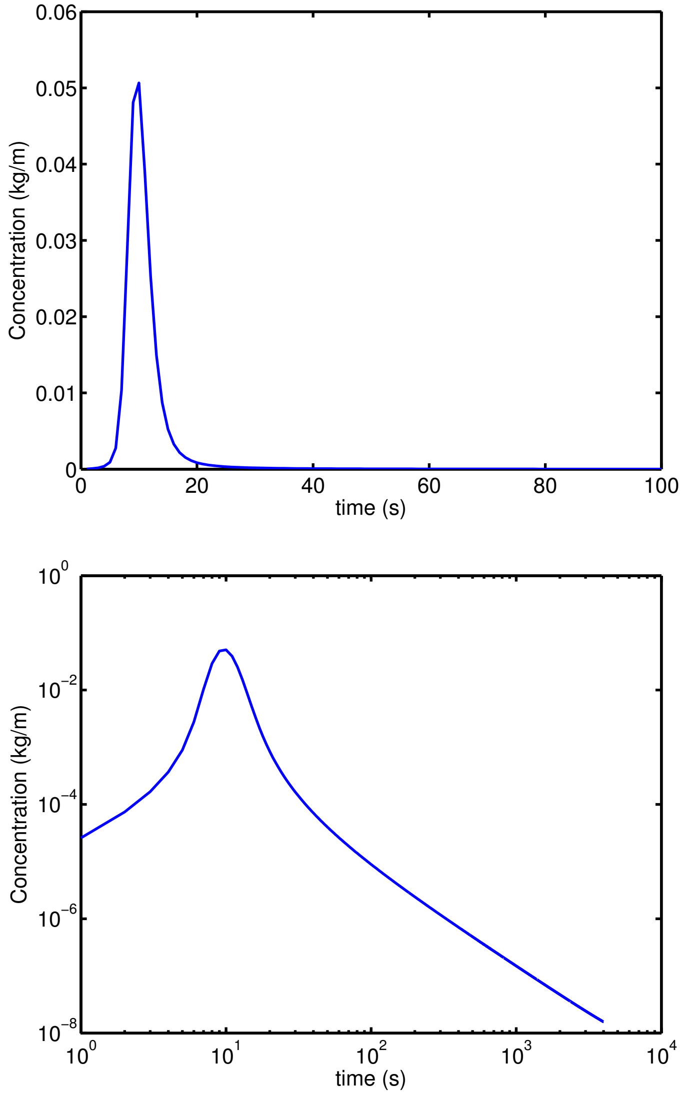

Figure 1: Breakthrough Curve for Problem 3.