Chapter 6
Differential Analysis of Flow

CE30460 - Fluid Mechanics
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Conservation of Momentum

- Newton’s Second Law

\[
\sum F_{\text{contents of the control volume}} = \frac{\partial}{\partial t} \int_{cv} \mathbf{v} \rho \, dV + \sum V_{\text{out}} \rho_{\text{out}} A_{\text{out}} V_{\text{out}} - \sum V_{\text{in}} \rho_{\text{in}} A_{\text{in}} V_{\text{in}}
\]

\[
\delta F = \delta m \mathbf{a}
\]

- What forces act on a differential volume?
Forces on Differential Element

- Weight (Vector)
  \[ \delta F_b = \delta m \, g \]

- Surface forces (normal and shear)

\[
\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}
\]
\[
\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}
\]
\[
\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}
\]
Surface Forces in x direction on a fluid element

(same in other directions)
\[ \delta F_{sx} = \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \delta y \delta z \]

\[ \delta F_{sy} = \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \delta x \delta y \delta z \]

\[ \delta F_{sz} = \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \delta x \delta y \delta z \]
\[ \delta F_x = \delta m a_x \]
\[ \delta F_y = \delta m a_y \]
\[ \delta F_z = \delta m a_z \]

\[ \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \]

\[ \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \]

\[ \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \]
How do we treat stresses?

\[
\begin{align*}
\sigma_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} \\
\sigma_{yy} &= -p + 2\mu \frac{\partial v}{\partial y} \\
\sigma_{zz} &= -p + 2\mu \frac{\partial w}{\partial z} \\
\tau_{xy} &= \tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\tau_{yz} &= \tau_{zy} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\tau_{zx} &= \tau_{xz} = \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)
\end{align*}
\]
And finally....

- The Navier Stokes Equations

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\]
\[ \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta v_r}{r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \]

\[ = -\frac{\partial p}{\partial r} + \rho g_r + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] \]

\[ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_\theta}{r} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \]

\[ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] \]

\[ \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta v_z}{r} + v_z \frac{\partial v_z}{\partial z} \right) \]

\[ = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \]
So....

- What can we do with these beast equations??

- Let’s look for some simple solutions.....
  - Poiseuille Flow
  - Hagen-Poiseuille Flow
  - Couette Flow
  - Taylor-Couette Flow
  - Mixed Poiseille-Couette Flow
Poiseuille Flow

- 2-d flow between stationary plates driven by pressure drop

- Let’s make some assumptions to solve this flow (you guys think about it in groups)
Poiseuille Flow

- Let’s calculate the following things:
  - Velocity Profile
  - Mean Velocity
  - Maximum Velocity
  - Vorticity
  - Shear Stress
Couette Flow

- 2-d flow between moving plates with zero pressure drop

- Again, let’s make some assumptions to solve this flow (you guys think about it in groups)
Couette Flow

- Again, let’s calculate the following things:
  - Velocity Profile
  - Mean Velocity
  - Maximum Velocity
  - Vorticity
  - Shear Stress
Hagen-Poiseuille

- Pressure driven flow in a pipe
Hagen-Poiseuille Flow

- Guess what? Let’s calculate the following things:
  - Velocity Profile
  - Mean Velocity
  - Maximum Velocity
  - Vorticity
  - Shear Stress
6.65 Oil ($\mu = 0.4$ N·s/m²) flows between two fixed horizontal infinite parallel plates with a spacing of 5 mm. The flow is laminar and steady with a pressure gradient of $-900$ (N/m²) per unit meter. Determine the volume flowrate per unit width and the shear stress on the upper plate.
6.81 A liquid (viscosity = 0.002 N·s/m²; density = 1000 kg/m³) is forced through the circular tube shown in Fig. P6.81. A differential manometer is connected to the tube as shown to measure the pressure drop along the tube. When the differential reading, $\Delta h$, is 9 mm, what is the mean velocity in the tube?
6.73  A viscous fluid (specific weight = 80 lb/ft³; viscosity = 0.03 lb · s/ft²) is contained between two infinite, horizontal parallel plates as shown in Fig. P6.73. The fluid moves between the plates under the action of a pressure gradient, and the upper plate moves with a velocity $U$ while the bottom plate is fixed. A U-tube manometer connected between two points along the bottom indicates a differential reading of 0.1 in. If the upper plate moves with a velocity of 0.02 ft/s, at what distance from the bottom plate does the maximum velocity in the gap between the two plates occur? Assume laminar flow.
6.59  The stream function for a certain incompressible, two-dimensional flow field is

\[ \psi = 3r^3 \sin 2\theta + 2\theta \]

where \( \psi \) is in \( \text{ft}^2/\text{s} \) when \( r \) is in feet and \( \theta \) in radians. Determine the shearing stress, \( \tau_{r\theta} \), at the point \( r = 2 \text{ ft}, \theta = \pi/3 \text{ radians} \) if the fluid is water.
Flow Between Concentric Cylinders

- Taylor-Couette Flow
- Assume out cylinder rotates at angular velocity $\omega_{in}$
- Assume out cylinder rotates at angular velocity $\omega_{out}$