6.6 The three components of velocity in a flow field are given by

\[ u = x^2 + y^2 + z^2 \]
\[ v = -3xyz - y^2z + z^2 \]
\[ w = -3xyz + y^2z + z^2 + 4 \]

(a) Determine the volumetric dilatation rate, and interpret the results.
(b) Determine an expression for the rotation vector. Is this an irrotational flow field?

(a) Volumetric dilatation rate = \( \frac{2u'}{x^2 + y^2 + z^2} + \frac{2v'}{3xyz + y^2z + z^2} \)

Thus, for velocity components given

\[ \frac{2u'}{x^2 + y^2 + z^2} + \frac{2v'}{3xyz + y^2z + z^2} = 0 \quad (Eqs. 6.9) \]

(b) From Eqs. 6.12-6.13, and 6.14 with the velocity components given:

\[ \dot{\omega}_x = \frac{1}{2} \left( \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) = \frac{1}{2} \left( 0 - (y - z) \right) = -\frac{y - z}{2} \]
\[ \dot{\omega}_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = \frac{1}{2} \left[ 2z - (3z) \right] = \frac{z}{2} \]
\[ \dot{\omega}_z = \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = \frac{1}{2} \left[ y - (y + z) \right] = 0 \]

Thus, \( \dot{\omega} = -\frac{y - z}{2} + \frac{z}{2} - \frac{y}{2} \)

Since \( \dot{\omega} \) is not zero everywhere the flow field is not irrotational. No.
6.22. It is proposed that a two-dimensional, incompressible flow field be described by the velocity components

\[ u = Ay \]
\[ v = Bx \]

where A and B are both positive constants. (a) Will the continuity equation be satisfied? (b) Is the flow irrotational? (c) Determine the equation for the streamlines and show a sketch of the streamline that passes through the origin. Indicate the direction of flow along this streamline.

(a) To satisfy the continuity equation

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Since, for the velocity distribution given

\[ \frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0 \]

The continuity equation is satisfied. Yes.

(b) In order for the flow to be irrotational \( \omega_z = 0 \), where

\[ \omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (Eq. 6.12) \]

Since

\[ \frac{\partial v}{\partial x} = B \quad \frac{\partial u}{\partial y} = A \]

\[ \omega_z = \frac{1}{2} (B - A) \]

Thus, flow will only be irrotational if \( A = B \).

(c) Along a streamline

\[ \frac{dy}{dx} = \frac{v}{u} \]

So that for the velocity distribution given \( \frac{dy}{dx} = \frac{Bx}{Ay} \)

and therefore

\[ y \cdot dy = \frac{B}{A} \cdot dx \]

Integration yields

\[ y^2 = \frac{B}{A} \cdot x^2 + C \]

where \( C \) is a constant.

For the streamline passing through the origin \( (C = 0) \) and \( y = \pm \sqrt{\frac{B}{A}} \cdot x \) (see figure).
Two fixed, horizontal, parallel plates are spaced 0.2 in. apart. A viscous liquid \((\mu = 8 \times 10^{-3} \text{ lbs/ft}^2\text{, SG} = 0.9)\) flows between the plates with a mean velocity of 0.9 ft/s. Determine the pressure drop per unit length in the direction of flow. What is the maximum velocity in the channel?

\[
V = \frac{\frac{d}{3\mu} \frac{\Delta P}{L}}{L} \quad (\text{Eq. 6.130})
\]

Thus,

\[
\frac{\Delta P}{L} = \frac{3\mu V}{\frac{d^2}{L}} = 3 \left(8 \times 10^{-3} \frac{\text{lb} \cdot \text{s}}{\text{ft} \cdot \text{lb}}\right) \left(0.9 \frac{\text{ft}}{\text{s}}\right) \left(\frac{0.1 \text{ in.}}{12 \text{ in.}}\right)^2 = 3.11 \frac{\text{lb}}{\text{ft}^2 \text{ per ft}}
\]

\[
U_{max} = \frac{3}{d} V \quad (\text{Eq. 6.131})
\]

\[
= \frac{3}{d} \left(0.9 \frac{\text{ft}}{\text{s}}\right) = 1.35 \frac{\text{ft}}{\text{s}}
\]
6.76 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.76). The bottom plate is fixed and the upper plate moves with a constant velocity \( U \). Determine the velocity at the interface. Express your answer in terms of \( U, \mu_1, \) and \( \mu_2 \). The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the \( x \) direction. The fluid velocity and shearing stress is continuous across the interface between the two fluids. Assume laminar flow.

For the specified conditions, \( \nu = 0, \sigma = 0, \frac{\partial \rho}{\partial x} = 0, \) and \( g_z = 0, \) so that the \( x \)-component of the Navier-Stokes equations (Eq. 6.1276) for either the upper or lower layer reduces to

\[
\frac{d^2 u}{dy^2} = 0
\]

Integration of Eq. (1) yields

\[
u = A_1 y + B
\]

which gives the velocity distribution in either layer.

In the upper layer at \( y = 2h \), \( u = U \) so that

\[
B_1 = U - A_1 (2h)
\]

where the subscript \( 1 \) refers to the upper layer.

For the lower layer at \( y = 0 \), \( u = 0 \) so that

\[
B_2 = 0
\]

where the subscript \( 2 \) refers to the lower layer. Thus,

\[
u_1 = A_1 (y-2h) + U
\]

and

\[
u_2 = A_2 y
\]

At \( y = h \), \( u_1 = u_2 \) so that

\[
A_1 (h-2h) + U = A_2 h
\]

or

\[
A_2 = -A_1 + \frac{U}{h} \quad \text{[cont'd]}
\]
Since the velocity distribution is linear in each layer the shearing stress
\[ \tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) = \mu \frac{\partial u}{\partial y} \]
is constant throughout each layer. For the upper layer
\[ \tau_1 = \mu_1 A_1 \]
and for the lower layer
\[ \tau_2 = \mu_2 A_2 \]
At the interface \( \tau_1 = \tau_2 \) so that
\[ \mu_1 A_1 = \mu_2 A_2 \]
or
\[ \frac{A_1}{A_2} = \frac{\mu_2}{\mu_1} \]
Substitution of Eq. (3) into Eq. (2) yields
\[ A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{R} \]
or
\[ A_2 = \frac{U/k}{1 + \mu_2/\mu_1} \]
Thus, velocity at the interface is
\[ U_2 (y = R) = A_2 k = \frac{U}{1 + \frac{\mu_2}{\mu_1}} \]
Assume: Incompressible \( \rho = \text{constant} \)

Steady \( \frac{\partial V}{\partial t} = 0 \)

Fully developed \( \frac{\partial V_r}{\partial r} = \frac{\partial V_\theta}{\partial \theta} = 0 \)

Continuity Equation:
\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho \nu_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho \nu_\theta) = 0
\]

\( \rho \nu_r \) steady or incompressible

\[\therefore \frac{\partial}{\partial r} (r \rho \nu_r) = 0\]


\( \rho \nu_r = \text{const} \)

\( \rho \nu_r \) fully developed

\( \therefore r = \text{Rin} \quad \nu_r = 0 \quad \Rightarrow \quad \text{const} = 0 \)

\( \therefore \rho \nu_r = 0 \)

Since \( r \neq 0 \) and \( \rho \neq 0 \) \( \Rightarrow \) \( \nu_r = 0 \)

\( \theta \)-momentum:
\[
\frac{\partial}{\partial t} \left( \frac{\partial \theta}{\partial t} + V_r \frac{\partial \theta}{\partial r} + \frac{\partial \theta}{\partial \theta} + V_\theta \frac{\partial \theta}{\partial \theta} \right) = -\frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial V_\theta}{\partial \theta} \frac{\rho \theta}{r^2} \right)
\]

Steady \( \nu_r = 0 \) Fully developed \( \nu_\theta = 0 \) Point

\( \nu_r = 0 \) Clearly

\( \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{\partial \theta}{\partial \theta} - 2 \frac{\partial \theta}{\partial \theta} \right) \) Fully Developed
\[ \kappa \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_0}{\partial r} \right) - \frac{V_0}{r^2} \right] = 0 \]

\[ \frac{\partial}{\partial r} \left( r \frac{\partial V_0}{\partial r} \right) - \frac{V_0}{r} = 0 \]

\[ \frac{\partial V_0}{\partial r} + r \frac{\partial^2 V_0}{\partial r^2} - \frac{V_0}{r} = 0 \]

\[ r^2 \frac{\partial^2 V_0}{\partial r^2} + r \frac{\partial V_0}{\partial r} - V_0 = 0 \]

\[ V_0 = \frac{C_1}{r} + C_2 \cdot r \]

Boundary conditions: \( V_0 = \omega_{\text{in}} \cdot R_{\text{in}} \) @ \( r = R_{\text{in}} \) and \( V_0 = \omega_{\text{out}} \cdot R_{\text{out}} \) @ \( r = R_{\text{out}} \)

Solving for \( C_1 \) and \( C_2 \): \( \omega_{\text{out}} \cdot R_{\text{out}} = \frac{C_1}{R_{\text{out}}} + C_2 \cdot R_{\text{out}} \)

\( \omega_{\text{in}} \cdot R_{\text{in}} = \frac{C_1}{R_{\text{in}}} + C_2 \cdot R_{\text{in}} \)

\[ C_1 = R_{\text{in}} \int \omega_{\text{in}} + \frac{R_{\text{in}} \omega_{\text{in}} - R_{\text{out}} \omega_{\text{out}}}{(R_{\text{out}} - R_{\text{in}})^2} \, \frac{1}{R_{\text{out}} - R_{\text{in}}} \] \[ C_2 = \frac{\omega_{\text{out}} \cdot R_{\text{out}}^2 - \omega_{\text{in}} \cdot R_{\text{in}}^2}{(R_{\text{out}} - R_{\text{in}})^2} \]