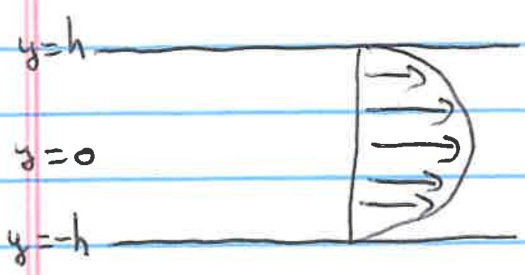
Poiseuille Flow



Boundary Conditions

$u = 0$ @ $y = \pm h$

$v = 0$ @ $y = \pm h$

Assumptions $\Rightarrow$ ① Incompressible

② Steady State

③ Fully Developed

Conservation of Mass $\Rightarrow$ $\underbrace{\cancel{\frac{\delta u}{\delta x}}}_{F.D.} + \frac{\delta v}{\delta y} = 0$ $\Rightarrow$ $v =$ constant

@ $y = \pm h$ $v = 0$ $\Rightarrow$ constant $= 0$

$\Rightarrow$ $\boxed{v = 0}$ everywhere

Conservation of Momentum

$$\rho\left(\underbrace{\cancel{\frac{\delta u}{\delta t}}}_{S.S.} + \underbrace{u\cancel{\frac{\delta u}{\delta x}}}_{F.D} + \underbrace{v\cancel{\frac{\delta u}{\delta y}}}_{v=0}\right) = -\frac{\delta p}{\delta x} + \mu\left(\underbrace{\cancel{\frac{\delta^2 u}{\delta x^2}}}_{F.D.} + \frac{\delta^2 u}{\delta y^2}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta y^2} = \frac{1}{\mu}\frac{\delta p}{\delta x}$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\delta p}{\delta x}y + C_1$$

$$u = \frac{1}{2\mu}\frac{\delta p}{\delta x}y^2 + C_1 y + C_2$$

By symmetry $C_1 = 0$

@ $y = \pm h$ $u = 0$ $\Rightarrow$ $C_2 = -\frac{1}{2\mu}\frac{\delta p}{\delta x} h^2$

$$\therefore \boxed{u = \frac{1}{2\mu}\frac{\delta p}{\delta x}(y^2 - h^2)}$$

Volume Flow Rate

$$q = \int u \, dA = \int_{-h}^{h} u \, dy \quad (\text{assuming } dA = 1 \times dy)$$

$$\Rightarrow q = \int_{-h}^{h} \frac{1}{2\mu}\frac{\delta p}{\delta x}(y^2 - h^2)\, dy$$

$$= \frac{1}{2\mu}\frac{\delta p}{\delta x}\left[\frac{y^3}{3} - h^2 y\right]_{-h}^{h}$$

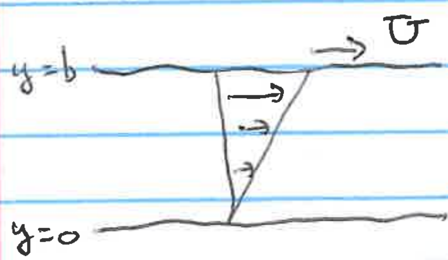
$$q = -\frac{2}{3}\frac{h^3}{\mu}\frac{\delta p}{\delta x}$$

Average Velocity $\Rightarrow$ $\bar{u} = \frac{q}{2h} = -\frac{h^2}{2\mu}\frac{\delta p}{\delta x}$

Shear Stress $\Rightarrow$ $\tau = \mu \frac{du}{dy} = \frac{\delta p}{\delta x} y$

Vorticity $\Rightarrow$ $\zeta = -\frac{\delta u}{\delta y} = \frac{-1}{2\mu}\frac{\delta p}{\delta x} y$

Couette Flow



Boundary Conditions

$u = 0 \quad @ \; y = 0$

$u = U \quad @ \; y = b$

$v = 0 \quad @ \; y = 0, b$

Assumptions $\Rightarrow$ ① Incompressible
② Steady State
③ Fully Developed

Conservation of Mass $\quad \underset{\text{F.D.}}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow v = \text{constant}$

$v = 0 \quad @ \; y = 0, b$

$\Rightarrow \text{constant} = 0$

$\therefore \; v = 0$ everywhere

Conservation of Momentum

$$\rho\left(\underset{\text{S.S.}}{\cancel{\frac{\partial u}{\partial t}}} + u\underset{\text{F.D.}}{\cancel{\frac{\partial u}{\partial x}}} + \underset{v=0}{\cancel{v\frac{\partial u}{\partial y}}}\right) = \underset{\substack{\text{Boundary}\\ \text{Driven}}}{-\cancel{\frac{\partial p}{\partial x}}} + \mu\left(\underset{\text{F.D.}}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \mu \frac{d^2 u}{dy^2} = 0$$

$$\frac{du}{dy} = C_1$$

$$u = C_1 y + C_2$$

@ $y = 0$ $\quad u = 0 \quad \Rightarrow \quad C_2 = 0$

@ $y = b$ $\quad u = U \quad \Rightarrow \quad C_1 = \frac{U}{b}$

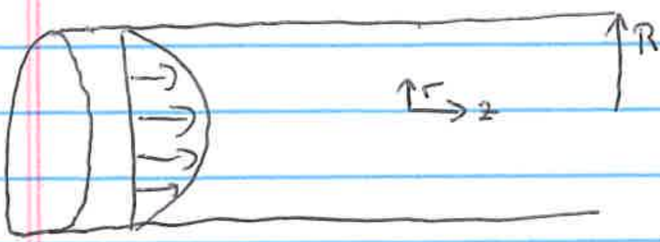
$$\therefore \quad u = U \frac{y}{b}$$

Shear Stress $\Rightarrow \quad \tau = \mu \frac{du}{dy}$

$$\tau = \mu \frac{U}{b}$$

Vorticity $\Rightarrow \quad \zeta = -\frac{du}{dy} = -\frac{U}{b}$

Hagen - Poiseuille Flow (Flow in a Pipe)



Boundary Conditions

$v_r = 0$ @ $r = R$

$v_z = 0$ @ $r = R$

Assumptions : ① Incompressible

② Steady State

③ Fully Developed

Conservation of Mass

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \underset{F.D.}{\cancel{\frac{\partial}{\partial z}(v_z)}} = 0$$

$\Rightarrow$ $r v_r = \text{constant}$

@ $r = R$ $v_r = 0$ $\Rightarrow$ constant $= 0$

$\Rightarrow$ $v_r = 0$ every where

Conservation of Momentum

$$\rho\left(\underset{S.S.}{\cancel{\frac{\partial v_z}{\partial t}}} + \underset{v_r=0}{\cancel{v_r\frac{\partial v_z}{\partial r}}} + \underset{F.D.}{\cancel{v_z\frac{\partial v_z}{\partial z}}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underset{F.D.}{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}\right]$$

$$\mu \frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right) = \frac{\delta p}{\delta z}$$

$$\Rightarrow \int \delta\left(r\frac{\delta v_z}{\delta r}\right) = \int \frac{1}{\mu}\frac{\delta p}{\delta z} r\,dr$$

$$r\frac{\delta v_z}{\delta r} = \frac{r^2}{2\mu}\frac{\delta p}{\delta z} + C_1$$

$$\delta v_z = \int\left(\frac{r}{2\mu}\frac{\delta p}{\delta z} + \frac{C_1}{r}\right)dr$$

$$v_z = \frac{r^2}{4\mu}\frac{\delta p}{\delta z} + C_1 \ln(r) + C_2$$

@ $r = 0$ v_z is finite $\Rightarrow C_1 = 0$

@ $r = R$ $v_z = 0$ $\Rightarrow C_2 = -\frac{R^2}{4\mu}\frac{\delta p}{\delta z}$

$$\therefore V_z = \frac{1}{4\mu}\frac{\delta p}{\delta z}\left[r^2 - R^2\right]$$

Volume Flow Rate

$$Q = \int v_z\,dA = \int_0^R v_z\, 2\pi r\,dr$$

$$= \frac{\pi}{2\mu}\frac{\delta p}{\delta z}\int_0^R (r^3 - R^2 r)\,dr$$

$$= \frac{\pi}{2\mu}\frac{\delta p}{\delta z}\left[\frac{r^4}{4} - \frac{R^2 r^2}{2}\right]_0^R$$

$$\boxed{Q = -\frac{\pi}{8\mu}\frac{\delta p}{\delta z} R^4}$$

Average Velocity

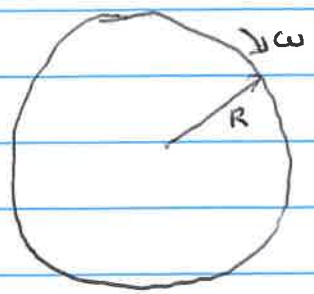
$$\bar{v} = \frac{Q}{A} = \frac{Q}{\pi R^2} = -\frac{R}{8\mu}\frac{\delta p}{\delta z}$$

$V_{max} = V_z$ @ $r = 0$

$$V_{max} = -\frac{R^2}{4\mu}\frac{\delta p}{\delta z}$$

Note $v_{max} = 2\bar{v}$

Rotating Cylinder



Boundary Condition

$V_r = 0$ @ $r = R$

$V_\theta = \omega R$ @ $r = R$

Assumptions : ① Incompressible

② Steady State

③ Fully Developed $\left(\frac{\partial}{\partial \theta} = 0\right)$

Conservation of Mass

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\overbrace{\cancel{\frac{\partial v_\theta}{\partial \theta}}}^{F.D.} = 0$$

$\Rightarrow$ $r V_r$ = constant

@ $r = R$ $v_r = 0$ $\Rightarrow$ constant $= 0$

$\therefore$ $V_r = 0$ everywhere

Momentum

$$\rho\left(\underbrace{\cancel{\frac{\partial v_\theta}{\partial t}}}_{S.S.} + \underbrace{v_r\cancel{\frac{\partial v_\theta}{\partial r}}}_{v_r=0} + \underbrace{\frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}}_{F.D.} + \underbrace{\cancel{\frac{v_r v_\theta}{r}}}_{v_r=0}\right) = \overbrace{-\frac{1}{r}\cancel{\frac{\partial p}{\partial \theta}}}^{\text{Boundary Driven}} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \underbrace{\frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}}}_{F.D.} + \underbrace{\frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}}}_{F.D.}\right]$$

$$\Rightarrow \frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_\theta}{\delta r}\right) = \frac{v_\theta}{r^2}$$

$$\frac{\delta}{\delta r}\left(r\frac{\delta v_\theta}{\delta r}\right) = \frac{v_\theta}{r}$$

$$r\frac{\delta^2 v_\theta}{\delta r^2} + \frac{\delta v_\theta}{\delta r} - \frac{v_\theta}{r} = 0$$

OR $$r^2\frac{\delta^2 v_\theta}{\delta r^2} + r\frac{\delta v_\theta}{\delta r} - v_\theta = 0$$

$$\Rightarrow \quad v_\theta = \frac{C_1}{r} + C_2 r$$

Internal Flow (Inside Cylinder $0<r<R$)

@ $r=0$ v_θ is finite $\Rightarrow C_1 = 0$

$$\therefore \; v_\theta = C_2 r$$

@ $r=R$ $v_\theta = \omega R \Rightarrow C_2 = \omega \Rightarrow \boxed{v_\theta = \omega r}$

External Flow (Outside Cylinder $R<r<\infty$)

as $r \to \infty$ $v_\theta \to 0 \Rightarrow C_2 = 0$

$$\Rightarrow v_\theta = \frac{C_1}{r}$$

@ $r=R$ $v_\theta = \omega R \Rightarrow C_1 = \omega R^2 \Rightarrow \boxed{v_\theta = \frac{\omega R^2}{r}}$