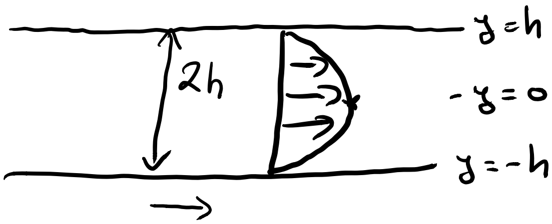
Poiseuille Flow (Pressure-driven flow between 2 plates)



Assumptions : 2 dimensional (x,y)
Incompressible ($\rho = \text{const}$)
Fully developed ($\partial/\partial x = 0$, except $\frac{\partial p}{\partial x}$)
Steady State ($\partial/\partial t = 0$)

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \cancel{\frac{\partial(\rho w)}{\partial z}} \Rightarrow \boxed{\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} = 0}$$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$\frac{\partial v}{\partial y} = 0 \Rightarrow v = \text{constant} = 0$, because $\boxed{v = 0}$ @ $y = \pm h$

$$\boxed{\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}}$$

$$\int \frac{d^2 u}{dy^2}\, dy = \int \frac{1}{\mu}\frac{dp}{dx}\, dy$$

$$\int dy \longrightarrow \frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx} y + C_1$$

$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + C_1 y + C_2$$

$u = 0$ @ $y = \pm h$ - 2 B.C. (No slip)

$$@h \Rightarrow 0 = \frac{1}{2\mu}\frac{dp}{dx}h^2 + C_1 h + C_2$$

$C_1 = 0$

AND

$$@-h \Rightarrow 0 = \frac{1}{2\mu}\frac{dp}{dx}h^2 - C_1 h + C_2$$

$$C_2 = \frac{-1}{2\mu}\frac{dp}{dx}h^2$$

$$\boxed{u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - h^2\right)}$$

$$\bar{u} = \frac{1}{2h}\int_{-h}^{h} u\,dy = -\frac{h^2}{3\mu}\frac{\delta p}{\delta x}$$

$$\boxed{\frac{\delta p}{\delta x} < 0}$$

pressure drops from high to low

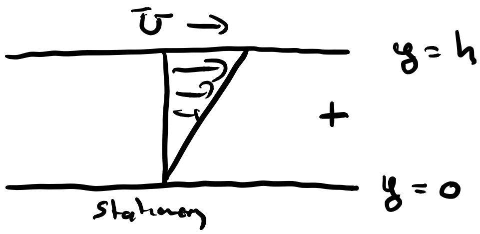
$$u_{max} = u\big|_{y=0} = \frac{1}{2\mu}\frac{\delta p}{\delta x}\left(-h^2\right)$$

$$\frac{u_{max}}{\bar{u}} = 3/2$$

$$\tau = \mu\frac{du}{dy} = \frac{\mu}{2\mu}\frac{dp}{dx}(2y) = \frac{dp}{dx}y$$

$$\omega = \frac{1}{2}\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right) = -\frac{1}{2\mu}\frac{dp}{dx}y$$

Couette Flow (no pressure drop)



① Incompressible $\Rightarrow \quad \dfrac{\delta u}{\delta x} + \dfrac{\delta v}{\delta y} + \dfrac{\delta w}{\delta z} = 0$

② 2d $\Rightarrow \quad \dfrac{\delta u}{\delta x} + \dfrac{\delta v}{\delta y} = 0$

③ Fully Developed $\cancel{\dfrac{\delta u}{\delta x}} + \dfrac{\delta v}{\delta y} = 0 \Rightarrow v = \text{constant}$

$= 0$

because $v = 0$ @ $y = 0$

④ Steady State

$$\rho\left(\underset{\text{S.S.}}{\cancel{\frac{\delta u}{\delta t}}} + u\underset{\text{F.D}}{\cancel{\frac{\delta u}{\delta x}}} + \underset{v=0}{\cancel{v\frac{\delta u}{\delta y}}} + \underset{\text{2D}}{\cancel{w\frac{\delta u}{\delta z}}}\right) = -\overset{0\ (\text{no pressure drop})}{\cancel{\frac{\delta p}{\delta x}}} + \mu\left(\underset{\text{F.D.}}{\cancel{\frac{\delta^2 u}{\delta x^2}}} + \frac{\delta^2 u}{\delta y^2} + \underset{\text{2D}}{\cancel{\frac{\delta^2 u}{\delta z^2}}}\right)$$

$$\boxed{\mu \frac{\delta^2 u}{\delta y^2} = 0}$$

$$\frac{d^2 u}{dy^2} = 0$$

$$\frac{du}{dy} = C_1$$

$$u = C_1 y + C_2$$

$$u = 0 \ @ \ y = 0 \rightarrow C_2 = 0$$

$$u = U \ @ \ y = h \rightarrow C_1 = \frac{U}{h}$$

$$u = U \frac{y}{h}$$

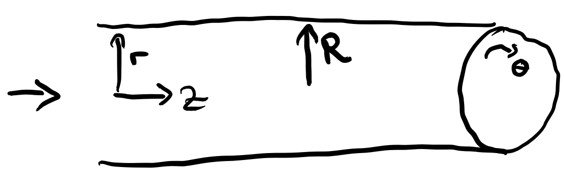
$$\bar{u} = \frac{1}{h} \int_0^h u \, dy = \frac{U}{2}$$

$$u_{max} = U$$

$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{h}$$

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = - \frac{U}{2h}$$

Hagen - Poiseuille (Pressure Driven in a Pipe)



r, z, θ

Continuity $\Rightarrow$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r \rho v_r) \right) + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Assumption : ① Incompressible ; ρ = const

② 2d $\rightsquigarrow r, z$

③ Fully developed $\rightsquigarrow \frac{\partial}{\partial z} = 0$ (except p)

④ Steady State $\rightsquigarrow$ $\delta/\delta t = 0$

Continuity $\Rightarrow$ $$\frac{1}{r}\left(\frac{\delta}{\delta r}(r v_r)\right) + \underbrace{\cancelto{0}{\frac{\delta(v_z)}{\delta z}}}_{F.D.} = 0$$

$$\frac{\delta}{\delta r}(r v_r) = 0$$

$$r v_r = \text{const} = 0$$

because $v_r = 0$ @ $r = R$

Momentum in z

$$\rho\left(\underbrace{\cancelto{0}{\frac{\delta v_z}{\delta t}}}_{S.S.} + \underbrace{\cancelto{}{v_r \frac{\delta v_z}{\delta r}}}_{v_r = 0} + \underbrace{\cancelto{0}{v_z \frac{\delta v_z}{\delta z}}}_{F.D.}\right) = -\frac{\delta p}{\delta z} + \mu\left[\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right) + \underbrace{\cancel{\frac{\delta^2 v_z}{\delta z^2}}}_{F.D.}\right]$$

$$\frac{1}{\mu}\frac{\delta p}{\delta z} = \frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right)$$

$$\frac{r}{\mu}\frac{\delta p}{\delta z} = \frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right)$$

$$\frac{r}{\mu}\frac{\delta p}{\delta z}\,dr = \delta\left(r\frac{\delta v_z}{\delta r}\right)$$

Integrate

$$\frac{r^2}{2\mu}\frac{dp}{dz} + C_1 = r\frac{\delta v_z}{\delta r}$$

$$\left(\frac{r}{2\mu}\frac{dp}{dz} + \frac{C_1}{r}\right)dr = \delta v_z$$

Integrate

$$\frac{r^2}{4\mu}\frac{dp}{dz} + C_1 \ln(r) + C_2 = v_z$$

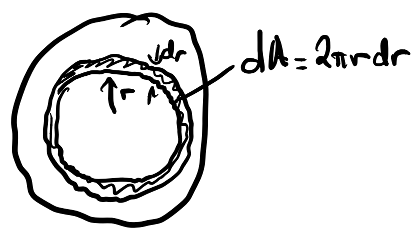
$v_z = 0$ @ $r = R$

v_z is finite @ $r = 0 \Rightarrow C_1 = 0$

$$C_2 = \frac{-R^2}{4\mu}\frac{dp}{dz}$$

$$\boxed{v_z = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - R^2\right)}$$

$$v_{z,max} = v_z\big|_{r=0} = \frac{-1}{4\mu}\frac{dp}{dz}R^2$$



$$Q = \int_A v_z \, dA$$

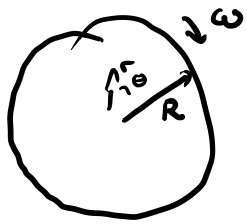
$$= \int_0^R \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - R^2\right) 2\pi r \, dr$$

$$= \frac{-\pi}{8\mu} R^4 \frac{dp}{dz}$$

$$\bar{v} = \frac{Q}{\pi R^2} = \frac{-R^2}{8\mu}\frac{dp}{dz}$$

$$v_{max} = 2\bar{v}$$

Flow Inside / Outside a rotating cylinder



r, θ, z

Inside

Assume : ① 2d r, θ

② Incompressible

③ Steady $\partial/\partial t = 0$

④ $\partial/\partial\theta = 0$ (axisymetric)

Continuity $\Rightarrow$ $$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \underbrace{\frac{1}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}}_{\partial/\partial\theta = 0} = 0$$

$$\frac{\partial}{\partial r}(r v_r) = 0$$

$$r v_r = \text{const} = 0$$

because $v_r = 0$ @ $r = R$

θ - momentum

$$\rho\left(\underbrace{\cancel{\frac{\partial v_\theta}{\partial t}}}_{S.S} + \underbrace{\cancel{v_r \frac{\partial v_\theta}{\partial r}}}_{v_r = 0} + \overbrace{\cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}}^{\text{axisym}} + \overbrace{\cancel{\frac{v_r v_\theta}{r}}}^{v_r = 0}\right)$$

$$= \underbrace{\cancel{\frac{-1}{r}\frac{dp}{d\theta}}}_{\partial \ldots = 0} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)\right.$$

v_θ

$\delta\theta = 0$ $\qquad - \frac{\quad}{r^2}$ F.D.

$$+ \frac{1}{r^2} \cancel{\frac{\delta^2 v_\theta}{\delta\theta^2}} + \frac{2}{r^2} \cancel{\frac{\delta v_r}{\delta\theta}} \Big]$$

$v_r = 0$

$$\cancel{\mu} \frac{1}{r} \frac{\delta}{\delta r}\left(r \frac{\delta v_\theta}{\delta r}\right) = \cancel{\mu} \frac{v_\theta}{r^2}$$

$$\boxed{r^2 \frac{\delta^2 v_\theta}{\delta r^2} + r \frac{\delta v_\theta}{\delta r} - v_\theta = 0}$$

Euler eqn $\Rightarrow \quad v_\theta = \frac{C_1}{r} + C_2 r$

Inside $\Rightarrow$ $\boxed{v_\theta = \omega R}$ @ $r = R$

$C_1 = 0$ because v_θ finite $(r = 0)$

$v_\theta = C_2 r$

$\Rightarrow C_2 = \omega$

$\therefore \; v_\theta = \omega r$

Outside $\Rightarrow$ $v_\theta = \frac{C_1}{r} + C_2 r$

$\boxed{v_\theta = \omega R}$ @ $r = R$

$C_2 = 0$ because v_θ is finite as $r \to \infty$

$\omega R = \frac{C_1}{R}$

$C_1 = \omega R^2$

$$\boxed{v_\theta = \frac{\omega R^2}{r}}$$

6.65 //

$$u = \frac{1}{2\mu}\frac{\delta p}{\delta x}\left(y^2 - h^2\right)$$

$$q = \int_{-h}^{h} u\,dy = -\frac{2h^3}{3\mu}\frac{\delta p}{\delta x}$$

$$h = 2.5 \text{ mm}$$
$$\mu = 0.4 \text{ Ns m}^{-2}$$
$$\frac{\delta p}{\delta x} = -900 \text{ N m}^{-3}$$

$$q = 2.34 \times 10^{-5} \text{ m}^2\text{s}^{-1}$$

$$\tau = \mu\frac{\delta u}{\delta y} = \mu\left(\frac{1}{2\mu}\frac{\delta p}{\delta x}(2y)\right) = \frac{\delta p}{\delta x}y$$
$$= (-900)(2.5\times 10^{-3})$$
$$= 2.25 \text{ N/m}^2$$

6.81 // Hagen - Poiseuille

$$\bar{V} = \frac{R^2}{8\mu}\left(\frac{-\delta p}{\delta x}\right)$$

$$R = 2 \text{ mm}$$
$$\mu = 0.002 \text{ Ns/m}^2$$

Manometer $p_1 + \gamma\Delta h - \gamma_{gs}\Delta h = p_2$

$$\left(\frac{-\delta p}{\delta x}\right) = \frac{p_1 - p_2}{\Delta l} = \frac{(\gamma_{gs} - \gamma)\Delta h}{2}$$

$$=$$

$= \quad R^2 \quad \Delta p \quad \ldots 10^{-2}$ ...

$$\bar{V} = \frac{\quad}{8\mu}\,\frac{\quad}{\Delta \ell} = 1.1 \times 10 \ \mathrm{m/s}$$