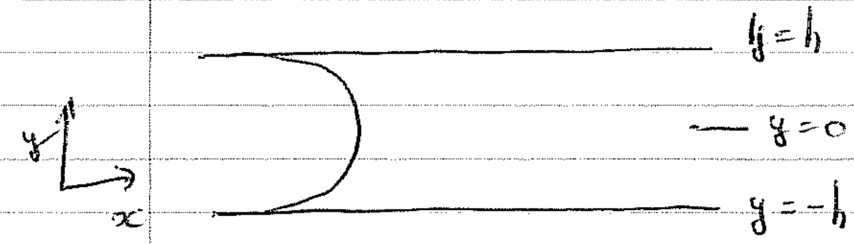
Pressure-driven flow between two flat plates



Assume : Incompressible
Steady State $\partial/\partial t = 0$
Fully Developed $\frac{\partial v}{\partial x} = 0$

$u = 0$ @ $y = \pm h$
$v = 0$ @ $y = \pm h$

Conservation of Mass $\Rightarrow$ $\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{F.D.} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow$ $v = \text{const} = 0$

X-momentum $\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{S.S.} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{F.D.} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{v=0}\right) = \underbrace{-\frac{\partial p}{\partial x}}_{P_x} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{F.D.} + \frac{\partial^2 u}{\partial y^2}\right)$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} P_x$$

Integrate once w.r.t. to y $\Rightarrow$ $\frac{\partial u}{\partial y} = \frac{1}{\mu} P_x y + C_1$

" " " " " $\Rightarrow$ $u = \frac{P_x y^2}{2\mu} + C_1 y + C_2$

System is symmetric about $y = 0$ $\Rightarrow$ $C_1 = 0$

$u = 0$ @ $y = h$ $\Rightarrow$ $C_2 = -\frac{P_x h^2}{2\mu}$

$$u = \frac{P_x}{2\mu}\left(y^2 - h^2\right)$$

$\tau = \mu \frac{du}{dy} = \frac{P_x y}{\mu}$ ... max at boundary

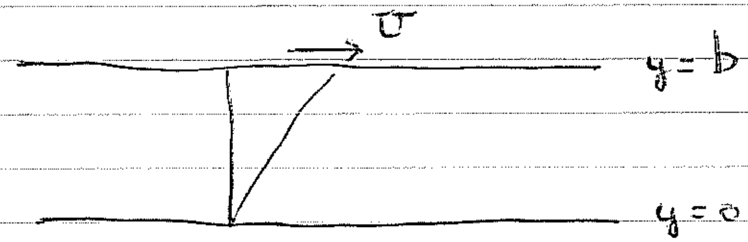
$$u_{max} = \frac{-P_x h^2}{2\mu} \quad \text{at center}$$

$$u_{mean} = \frac{1}{2h}\int_{-h}^{h} u(y)\,dy = \frac{1}{2h}\int_{-h}^{h} \frac{P_x}{2\mu}(y^2-h^2)\,dy$$

$$= \frac{-P_x h^2}{3\mu}$$

$$\frac{u_{max}}{u_{mean}} = 3/2$$

Shear (no-pressure drop) flow between two flat plates



Assume: Incompressible
Fully Developed
Steady State

B.C. $v = 0 \; @ \; y = 0, b$
$u = 0 \; @ \; y = 0$
$u = U \; @ \; y = 0$

Mass $\quad \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{F.D.} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = \text{const.} = 0$

X-momentum $\quad \rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{S.S.} + u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{F.D.} + v\underbrace{\cancel{\frac{\partial u}{\partial y}}}_{V=0}\right) = -\underbrace{\cancel{\frac{\partial p}{\partial x}}}_{\text{No pressure drop}} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{F.D.} + \frac{\partial^2 u}{\partial y^2}\right)$

$$\therefore \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial u}{\partial y} = C_1 \Rightarrow u = C_1 y + C_2$$

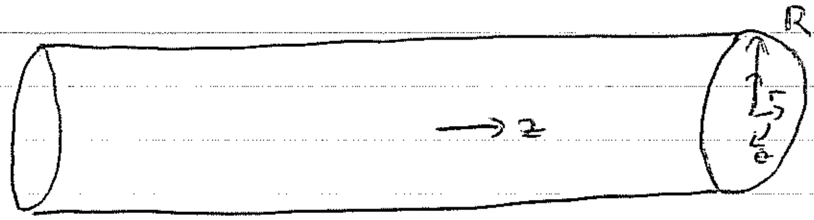
$@ \; y = 0 \quad u = 0 \Rightarrow C_2 = 0$

$@ \; y = b \quad u = U \Rightarrow C_1 = \frac{U}{b} \Rightarrow u = \frac{U}{b} y$

$$u_{mean} = \frac{1}{b}\int_0^b u\,dy = \frac{U}{b^2}\int_0^b y\,dy = \frac{U}{b^2}\frac{b^2}{2} = \frac{U}{2}$$

$$\tau = \mu \frac{du}{dy} = \underbrace{\mu \frac{U}{b}}_{\text{constant}}$$

Pressure - Driven Flow in a Circular Pipe



Assume: Incompressible
Fully Developed $\rightarrow \delta/\delta z$
Steady State
~~Axisymmetric~~ $V_\theta = 0$ $\rightsquigarrow$ only consider r, z plane

B.C. $\Rightarrow$ $V_r = 0$ @ $r = R$
$V_z = 0$ @ $r = R$

$$\text{Mass} \Rightarrow \frac{1}{r}\frac{\delta}{\delta r}(r v_r) + \underbrace{\cancel{\frac{\delta v_z}{\delta z}}}_{F.D} = 0$$

$$\therefore \quad \delta/\delta r\,(r v_r) = 0 \Rightarrow r v_r = \text{const} \Rightarrow v_r = 0$$

$$\text{Z-momentum} \Rightarrow \rho\left(\overset{S.S.}{\cancel{\frac{\delta v_z}{\delta t}}} + \overset{v_r = 0}{\cancel{v_r\frac{\delta v_z}{\delta r}}} + \overset{F.D.}{\cancel{v_z\frac{\delta v_z}{\delta z}}}\right) = \overbrace{-\frac{\delta p}{\delta z}}^{-P_z} + \mu\left[\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right) + \overset{F.D.}{\cancel{\frac{\delta^2 v_z}{\delta z^2}}}\right]$$

$$\therefore \quad \frac{1}{r}\,\delta/\delta r\left(r\frac{\delta v_z}{\delta r}\right) = \frac{P_z}{\mu}$$

$$\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right) = \frac{P_z}{\mu} r$$

Let $\ell = r\frac{\delta v_z}{\delta r}$

$$\Rightarrow \quad \frac{\delta \ell}{\delta r} = \frac{P_z}{\mu} r$$

$$d\ell = \frac{P_z}{\mu} r\,dr$$

Integrate $\Rightarrow \quad \ell = \frac{P_z}{\mu}\frac{r^2}{2} + C_1$

$\ell = r\frac{dv_z}{dr} \quad \Rightarrow \quad r\frac{dv_z}{dr} = \frac{P_z}{\mu}\frac{r^2}{2} + C_1$

$$dv_z = \frac{P_z}{\mu}\frac{r}{2} + \frac{C_1}{r}\,d$$

Integrate $\Rightarrow \quad V_z = \frac{P_z}{4\mu} r^2 + C_1 \ln(r) + C_2$

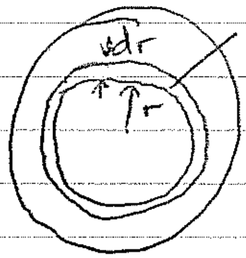
V_z is finite at $r=0 \quad \Rightarrow \quad C_1 = 0$

$v_z = 0$ @ $r = R \quad \Rightarrow \quad C_2 = -\frac{P_z}{4\mu} R^2$

$$\therefore \quad v_z = \frac{P_z}{4\mu}\left(r^2 - R^2\right)$$

$$V_{z,max} = -\frac{P_z}{4\mu} R^2 \quad @ \; r = 0$$

$dA = 2\pi r\, dr$



$$V_{z,\text{mean}} = \frac{1}{A}\int v_z \, dA$$

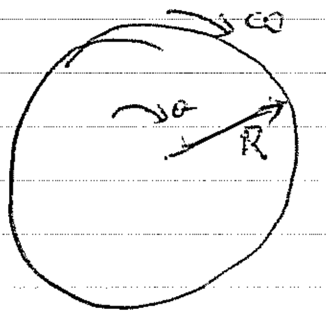
$$= \frac{1}{\pi R^2}\int_0^R \frac{P_z}{4\mu}\left(r^2 - R^2\right) 2\pi r\, dr$$

$$= \frac{2 P_z}{4\mu R^2}\int_0^R \left(r^2 - R^2\right) r\, dr$$

$$= \frac{P_z R^2}{8\mu}$$

$$\frac{V_{z\,\text{mean}}}{V_{z\,\text{max}}} = 2$$

Flow inside a rotating cylinder



$0 \le r \le R$

B.C. $v_r = 0$ @ $r = R$

$v_\theta = \omega R$ @ $r = R$

Assumptions ⇒ Incompressible
Steady State
Axisymmetric $\partial/\partial\theta = 0$

r, θ plane
$(v_z = 0)$

$$\text{Mass} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \underbrace{\cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}}_{\text{Axisym}} = 0$$

$$\therefore \frac{\partial}{\partial r}(r v_r) = 0$$

$$\Rightarrow \quad r v_r = \text{const} \Rightarrow v_r = 0$$

$$\theta\text{-momentum} \Rightarrow \rho\left(\overset{S.S}{\cancel{\frac{\partial v_\theta}{\partial t}}} + \overset{v_r=0}{\cancel{v_r\frac{\partial v_\theta}{\partial r}}} + \overset{\text{Axis.}}{\cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}} + \overset{v_r=0}{\cancel{\frac{v_r v_\theta}{r}}}\right)$$

$$= -\underbrace{\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}}}_{\text{Axis.}} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}}_{\text{Axis}} + \underbrace{\cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}}_{v_r=0}\right]$$

$$\therefore \quad \frac{1}{r} \frac{\delta}{\delta r}\left(r \frac{\delta V_\theta}{\delta \theta}\right) = \frac{V_\theta}{r^2}$$

$$\frac{\delta}{\delta r}\underbrace{\left(r \frac{\delta V_\theta}{\delta \theta}\right)}_{\text{Product Rule}} = \frac{V_\theta}{r}$$

$$\frac{\delta r}{\delta r} \frac{\delta V_\theta}{\delta \theta} + r \frac{\delta^2 V_\theta}{\delta r^2} = \frac{V_\theta}{r}$$

Rearrange $\Rightarrow$ $r^2 \frac{\delta^2 V_\theta}{\delta r^2} + r \frac{\delta V_\theta}{\delta \theta} - V_\theta = 0$

Solution $\Rightarrow$ $V_\theta = \frac{C_1}{r} + C_2 r$

Velocity finite @ $r = 0$ $\Rightarrow$ $C_1 = 0$

$V_\theta = \omega R$ @ $r = R$ $\Rightarrow$ $C_2 = \omega$

$$\therefore \quad \boxed{V_\theta = \omega r}$$