Slowing of vortex rings by development of Kelvin waves

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We study experimentally the slowing of viscous vortex rings. In particular we do so using the concept of drag coefficient, which is a bulk coefficient which aims to capture the various mechanisms of slowing that can occur. At early times of flight the ring slows at a certain rate. After some time, instabilities (which we refer to as Kelvin waves) begin to form on the ring and there is a transient increase in the measured drag coefficient. After this brief transient the rings enter another regime with constant drag coefficient, which is larger than the early time one. In particular, our data illustrate that there appears to be a direct link between the number of waves that form on the ring and the relative increase in drag coefficient.

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I. INTRODUCTION

Vortex rings are considered by many to be one of the most fundamental and intriguing phenomena in fluid dynamics. As Saffman states: "One particular motion exemplifies the whole range of problems of vortex motion and is also a commonly observed phenomenon, namely, the vortex ring. Their formation is a problem of vortex sheet dynamics, the steady state a problem of existence, their duration a problem of stability and if there are several we have a problem of vortex interactions." [1]. In this paper we focus on the slowing of vortex rings, a complex topic that has been of great interest dating as far back as Reynolds, who discussed it in a paper back in 1876 [2]. We ourselves picture the vortex ring as the "hydrogen atom "of instability and turbulence.

Let us begin with a brief review of results for thin vortex rings of circulation Γ moving in a viscous fluid of kinematic viscosity ν. We focus our discussion to rings where the core radius a is much smaller than the radius of the ring R. Such a vortex ring will propagate forward under its own self induced velocity V [3],

$$V = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a} - \beta\right] \tag{1}$$

where β is a constant that depends on the core structure (see Table 1 in [3]).

There are a number of reasons why vortex rings might slow down. One common suggestion is that the core of the ring grows by diffusion ($a=\sqrt{4\nu t}$) thus reducing the velocity of the ring given by Eq. (1) [4,5]. Another slowing mechanism is that the bubble loses impulse as the radius of the ring grows due to viscous decay of circulation [6]. A further theory suggests that the vortex bubble (the ambient fluid carried along with the ring during its motion) grows by viscous entrainment along with the shedding of vorticity into a wake behind the ring [6,7]. A combination of all these effects is likely. All these studies focus on the initial stable flight of the ring, which can be relatively short lived.

Many researchers have observed that during flight bending waves grow on the perimeter of the ring [8]. The instability that causes the growth of these rings has been studied extensively [9–11]. It is widely believed that this instability, at least for a straight vortex tube case, results from subjection to a straining field in a plane perpendicular to the tube axis [12,13]). These waves can extract energy from the flow and thus cause the rings to slow. Kiknadze and Mamaladze in an analytical study [14], and Barenghi, Hanninen and Tsubota in a numerical study [15], showed that an inviscid ring with Kelvin waves will translate more slowly than one without waves. This occurs due to changes in local radii of curvature induced by the waves. They even go so far as to predict that given a sufficiently large amplitude and number of Kelvin waves a perturbed vortex ring may in fact move backward. Note that this slowing is the result of calculations based on the Biot-Savart law. They are not described by a drag coefficient such as we discuss later in this paper.

Most previous studies have looked at rings with a thick core structure [16,17] or rings that are highly turbulent [18–20]. The rings that we consider here are thin rings with small core radii. We refer to the waves as "Kelvin waves" recognizing that there are much more complicated waves that can develop [7]. In a previous study [3] it was shown that the thickness of these cores does not vary much during the propagation of the ring, and so we note that any slowing of the ring cannot be attributed to a growing core structure for these thin-cored rings. As such, we take an alternate approach and consider "drag" on the bubble of the ring. By drag we aim to include all effects that slow the ring such as entrainment into the vortex bubble, shedding of vorticity into the wake of the ring and other such phenomena that are difficult to capture individually. This is not a far leap as similar concepts have been used in the study of quantized ring vortices in superfluid helium [21]. We aim to show that the appearance of Kelvin waves on the ring does not just cause a reduced flight speed as shown by [14], but also that these waves increase the effective drag on a ring.

II. MODEL

While we currently do not have the tools necessary to predict the drag on a vortex ring, we can certainly measure a

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drag coefficient that quantifies it, e.g., [3]. Following the principles outlined therein and as is typical in drag studies, we start by defining the drag coefficient on the vortex ring as

$$C_d = \frac{D_f}{\frac{1}{2}\rho V^2 A}, \tag{2}$$

where D_f is the drag force on the ring, V is the velocity of the ring and A is the cross section area of the ring exposed perpendicular to the flow direction (i.e., $A=2a2\pi R$). The drag force causes a loss in momentum/impulse, $P=\rho\Gamma\pi R^2$, of the ring

$$D_f = -\frac{dP}{dt}, \tag{3}$$

which results in the following differential equation for the velocity of the ring:

$$\frac{dV}{dt} = \frac{\Lambda a}{2\pi R^2} C_d V^2, \tag{4}$$

where $\Lambda = \ln(8R/a) - \beta$ from Eq. (1). Solving Eq. (4) we obtain

$$V = \frac{V_0}{1 + V_0 c t}, \tag{5}$$

where V_0 is the velocity of the ring at formation and $c = \Lambda a C_d / 2\pi R^2$. This result is in agreement with the prediction that for long times the velocity of the ring decays as t^{-1} [6]. This linear decay has been observed in other studies such as [22]. From Eq. (5) the distance traveled by the ring at time t is given by

$$X = \frac{1}{c}\ln(V_0 c t + 1). \tag{6}$$

III. EXPERIMENTS

All experiments used for this paper were performed in a glass tank of dimension $75\times 30\times 30$ cm and all the data were taken photographically. The vortex rings studied in this work were all generated by impulsively accelerating fluid through a cylindrical tube. The experimental apparatus used to generate vortex rings is identical to the "new" gun described in detail in [3].

We used the Baker [23] pH technique for visualization. A thorough discussion of the Baker technique is contained in Mazo *et al.* [24]. The drag coefficients reported in Fig. 12 of [3] show that a drag coefficient can be defined for vortex rings propagating in a viscous fluid, and that coefficient is indeed a function of Reynolds number. In this study rings produced by the "old" gun produced rings with more initial instabilities than the new gun and the scatter in the results for the new gun is substantially smaller than for the earlier design. It was this observation that prompted the present investigation.

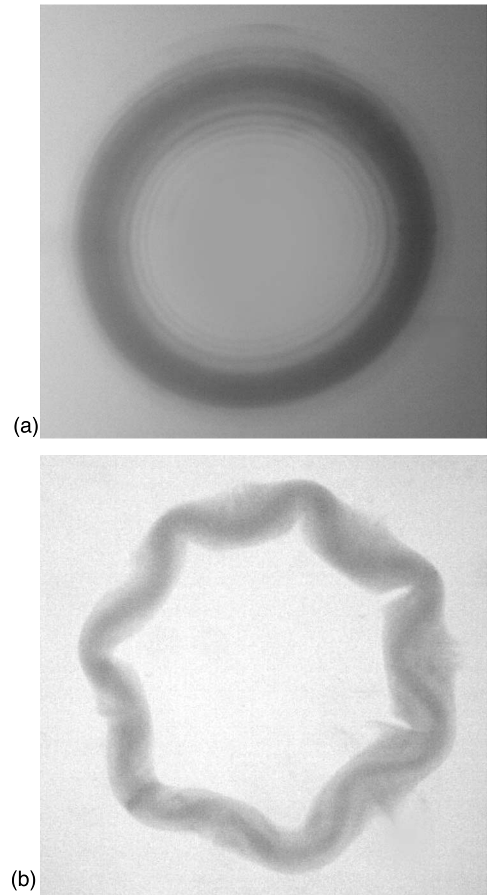

FIG. 1. Photograph of a vortex ring generated by a stroke L =3.75 cm, time T=0.60 s. Photo (a) is taken soon after the piston is actuated, and (b) after some time. The bending waves on thin core rings are called Kelvin waves. More complicated waves can be seen which are discussed in [7].

IV. RESULTS

During the initial stage of flight the vortex rings appear smooth and there are no visible Kelvin waves. During this stage the rings propagate forward smoothly slowing slightly. After some time the instability forms and there is a transition time where the Kelvin waves form such as those depicted in Fig. 1. Note that in this image seven clearly defined waves exist. After this intermediate formation time the ring continues to propagate smoothly, but decelerates at a more rapid pace. Our experiments are conducted over a range of Reynolds number $40 < \mathrm{Re} < 400$, where we define the Reynolds number as

$$\mathrm{Re} = V_p a/\nu = 2L/T^{1/2}\nu^{1/2}, \tag{7}$$

where Vp is the speed of the piston of the vortex ring gun, a is the radius of the core of the ring, ν is the viscosity of water, L is the length of the stroke of the piston and T is the stroke time. From [3] we know that the core radius is given by $a = \sqrt{4\nu T}$.

Using a high resolution digital video camera mounted perpendicular to the path, either above or at the side of our experimental tank, we recorded the path of flight of a ring.

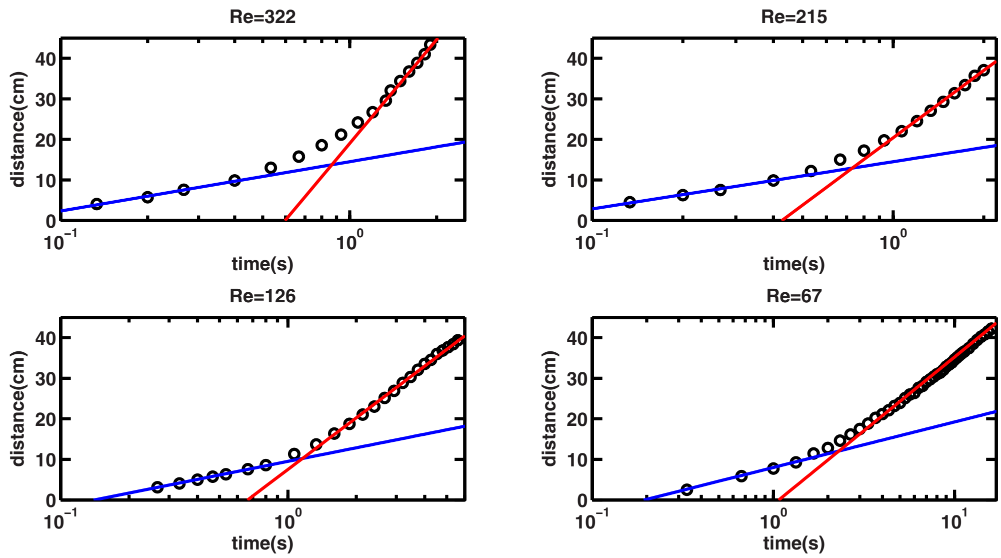

FIG. 2. (Color online) Plots of position vs time for various vortex rings.

Simultaneously another camera was used to take frontal pictures of the rings (e.g., Fig. 1) in order to observe the formation and number of Kelvin waves. The videos of the trajectories were then analyzed using MATLAB image processing tools to measure the position of the rings at various times.

A. Decay curves

A sample set of distance vs time curves for different rings is shown in Fig. 2. These curves range over the span of Reynolds numbers achievable with our current setup. In all cases two distinct decay coefficients are visible at early and late stages of flight with a transition zone in between. The slopes of the solid lines are a measure of the decay coefficient defined in Eq. (2). By early and late we, respectively, mean before and after the onset of Kelvin waves on the vortex ring. In all cases the early decay coefficient is less than that of the late time one as illustrated by the steeper slope of the lines at later times. In order to compare these early and late time decay coefficients we define their ratio as $R_c = C_{late}/C_{early}$ where C_{late} and C_{early} are the measured decay coefficients at late and early times, respectively. A plot of this ratio against Reynolds number is shown in Fig. 3(a). First, it is worth noting that R_c increases with Reynolds number, which is perhaps not surprising. Second, it is also worth noting that R_c appears to increase in discrete jumps (plateaus) rather than continuously as one might at first expect.

B. Wave numbers

Now that we have shown that there is an increase in drag on the vortex rings after the onset of Kelvin waves we also wanted to look at the waves themselves. In particular we counted the number of waves formed on each ring, which as one would expect from the theory of [10] was also found to increase with Reynolds number. A plot showing the number of waves that form on the ring against Reynolds number is shown in Fig. 3(b). Note the remarkable resemblance of this figure to Fig. 3(a). It appears that the number of Kelvin waves on a ring directly affects the relative increase in drag experienced by that ring. This would explain the discrete increases in Rc as the number of waves on a given ring must be a finite integer.

V. CONCLUSION

We have studied the slowing of viscous vortex rings using the notion of drag. We note that there are two distinct decay coefficients—one that occurs at early times before the onset

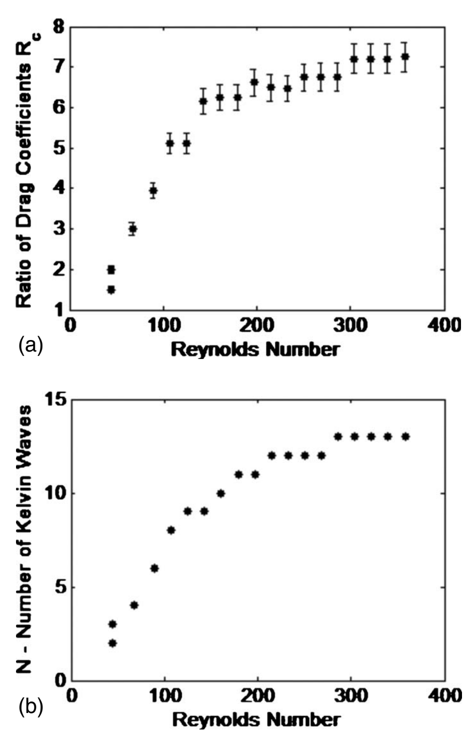

FIG. 3. (a) Ratio of late to early decay coefficients against Reynolds numbers (b) Number of waves on the ring against Reynolds number.

of Kelvin waves on the core of the ring and one that occurs post-Kelvin wave onset. In all cases the late decay rate is greater than the early one. This increase in drag can be quite dramatic—for some cases we observed an eightfold increase. As one would expect from previous work in this field the number of waves on a vortex ring increases with Reynolds number. We illustrate that a larger number of waves leads to an increase in the ratio of late to early drag. In [15] it is predicted that rings can attain negative velocities if the wave amplitude and number of waves is sufficiently large (see Fig. 2 in their paper). However, for the numbers of waves observed here the wave amplitudes must be larger than 15–20 % of the ring radius, which is difficult to achieve. Additionally, the theory in [15] is for inviscid rings and we anticipate that because the rings must reverse through a zero velocity that viscous effects will not be negligible and make such observations in water impossible. We have, however, succeeded in making vortex rings that stop and reverse direction twice, but not by means of a vortex gun. Instead we generated rings by suddenly stopping a sphere that had been in constant velocity motion. We plan to discuss this extraordinary phenomenon in a future paper.

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