Homework 2

① $Q_1 + Q_2 + Q_3 = Q_{out}$ (Volume of lake is fixed and does not change over time)

$45 + 65 + Q_3 = 240 \Rightarrow Q_3 = 130 \text{ m}^3/\text{hr}$

Conservation of Mass

$$\frac{d}{dt}(CV) = Q_1 C_1 + Q_2 C_2 + Q_3 C_3 - Q_{out} C_{out}$$

Assume lake is well mixed $\Rightarrow C_{out} = C$

$$\therefore \frac{dC}{dt} = \frac{QC_1 + Q_2C_2 + Q_3C_3 - Q_{out} C}{V}$$

$$= \frac{(45)(0) + 65(10) + 130(3) - 240C}{10^4}$$

$$\frac{dC}{dt} = \frac{650+390}{10^4} - \frac{240}{10^4} C$$

$$\frac{dC}{dt} = 0.104 - 0.024C$$

Solution $$\frac{dC}{0.104 - 0.024C} = dt$$

$$\Rightarrow C = 4.33 + Ae^{-0.024t}$$

At $t=0$ $C=0$ $\Rightarrow$ $0 = 4.33 + A$ $\Rightarrow$ $A = -4.33$

$$\therefore \boxed{C = 4.33\left(1 - e^{-0.024t}\right)}$$

as $t \to \infty$ $\boxed{C_\infty = 4.33 \text{ g/liters}}$

Conservation of Mass

$$\frac{d(CV)}{dt} = Q_1 C_1 + Q_2 C_2 + Q_3 C_3 - Q_{out} C$$

Now

$C_1 = C_2 = C_3 = 0$ $\Rightarrow$ $\frac{d(CV)}{dt} = 0 + 0 + 0 - Q_{out} C$

$$\therefore \frac{dC}{dt} = -\frac{240}{10^4} C$$

$$\Rightarrow C = C_0 e^{-0.024t}$$

$C = 0$ never, but say $C = \frac{C_0}{1000}$

$$\Rightarrow \frac{1}{1000} = e^{-0.024t}$$

$$t = -\frac{1}{0.024} \ln\left(\frac{1}{1000}\right) = 288 \text{ hrs}$$

↳ Answer depends on choice of $C_{acceptable}$

(2) $v = 0.1\ m/s$

$$D = 10^{-9} + 10^{-2} + 5\times10^{-3} = 1.5\times10^{-2}\ m^2/s$$

$$C_1 = \frac{15}{(4\pi Dt)^{1/2}} e^{-\frac{(x-vt)^2}{4Dt}}$$

$$C_2 = \frac{30}{(4\pi Dt)^{1/2}} e^{-\frac{(x-25-vt)^2}{4Dt}}$$

(a) $C = C_1 + C_2$ using above expressions

(b) Set $x = 100$

$$\therefore C = \frac{15}{(6\pi\times10^{-2}t)^{1/2}} e^{-\frac{(100-0.1t)^2}{6\times10^{-2}t}} + \frac{30}{(6\pi\times10^{-2}t)^{1/2}} e^{-\frac{(75-0.1t)^2}{6\times10^{-2}t}}$$

See attached plots + Matlab codes

(c) $x_{loc} = 0 + 0.1t = (0.1)(3600) = 360\ m$

$$C_{peak} = \frac{15}{(6\pi\times10^{-2}t)^{1/2}} = 0.58\ kg/m$$

$$\text{width} = (3\times10^{-2}t)^{1/2} = 10.4\ m$$

(3) From the plots

$$m_1 = vt \Rightarrow v \approx \frac{50}{1000} = \frac{1}{20} \; m/s$$

$$K_{11} = 2Dt \Rightarrow 2D \approx \frac{2000}{1000} \Rightarrow D = 1 \; m^2/s$$

Temporal Moments

$$M_0 = \frac{M}{V} \Rightarrow 20 = \frac{1}{v} \Rightarrow v = \frac{1}{20} \; m/s$$

$$M_1 = \frac{2D + vx}{v^3} \Rightarrow 5.6 \times 10^5 = \frac{2D + (\frac{1}{20})(100)}{(\frac{1}{20})^3}$$

$$\Rightarrow D = 32.5 \; m^2/s$$

~> velocities are consistent, but dispersion coefficients are not.

(4)

$$C = \frac{C_0}{2}\left\{ \mathrm{erfc}\left(\frac{x - x_0 - vt}{(4Dt)^{1/2}}\right) - \mathrm{erfc}\left(\frac{x - x_1 - vt}{(4Dt)^{1/2}}\right)\right\}$$

$C_0 = 25\ g/l$

$x_0 = 0\ m$

$x_1 = -75\ m$

$v = 0.25\ m/s$

$D = 3\times10^{-2}\ m^2/s$

$\therefore$ (a) $$C = 12.5\left\{ \mathrm{erfc}\left(\frac{x - 0.25t}{(12\times10^{-2}\,t)^{1/2}}\right) - \mathrm{erfc}\left(\frac{x+75 - 0.25t}{(12\times10^{-2}\,t)^{1/2}}\right)\right\}$$

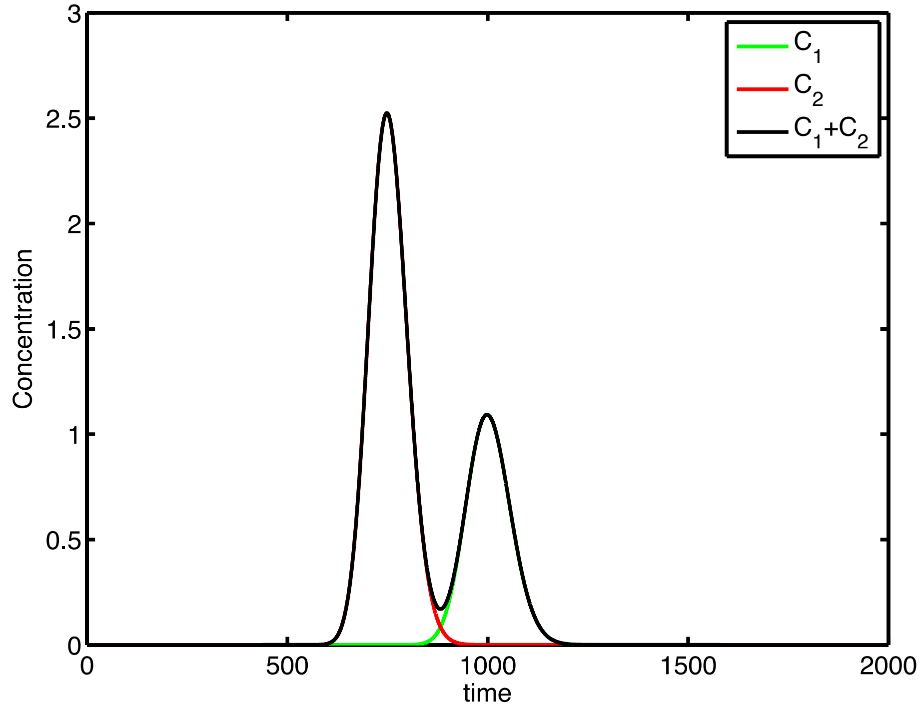
(b) Set $x = 200$ $\quad C_{max} = 25\ g/l$

$x = 2000$ $\quad C_{max} = 22.~~75~~5\ g/l$

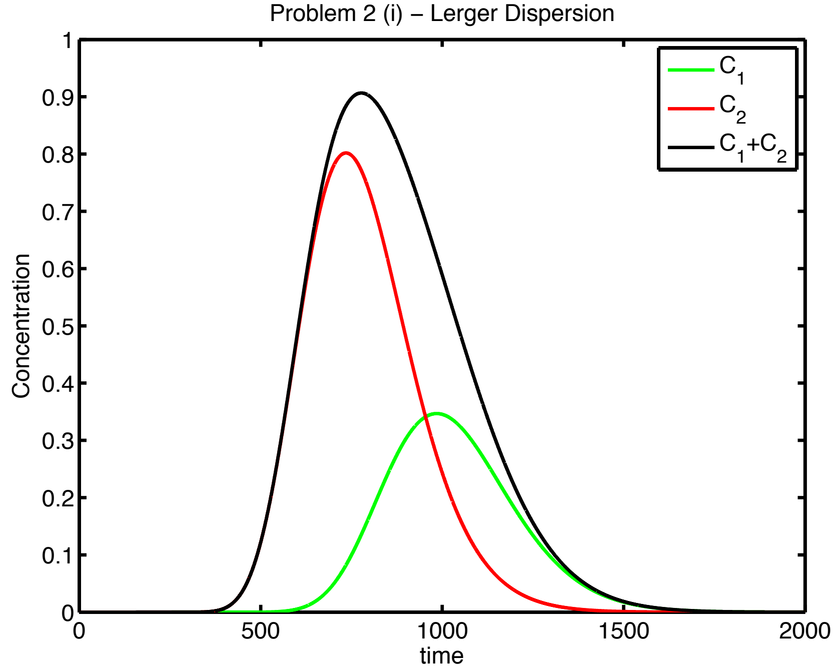
See Matlab + Plots attached

```
clear
clc
close all

%Problem 2
Dmol=1e-9; Dturb=1e-2; Dmech=5e-3;
D=Dmol+Dturb+Dmech;

M1=15; M2=30;
v=0.1;
x1=0; x2=25;

x=100;  %location where I will plot the breakthrough curve
t=linspace(1,2000,1e4); %times over which I will plot

C1=M1./sqrt(4*pi*D*t).*exp(-(x-x1-v*t).^2./(4*D*t));
C2=M2./sqrt(4*pi*D*t).*exp(-(x-x2-v*t).^2./(4*D*t));
Ctot=C1+C2;

%plot concentrations
figure(1)
plot(t,C1,'g')
hold on
plot(t,C2,'r')
plot(t,Ctot,'k')
xlabel('time')
ylabel('Concentration')
legend('C_1','C_2','C_1+C_2')
title('Problem 2 (i) - Smaller Dispersion')

%Same Problem but with a dispersion coefficient ten times as big

D=10*D;
C1=M1./sqrt(4*pi*D*t).*exp(-(x-x1-v*t).^2./(4*D*t));
C2=M2./sqrt(4*pi*D*t).*exp(-(x-x2-v*t).^2./(4*D*t));
Ctot=C1+C2;
figure(2)
plot(t,C1,'g')
hold on
plot(t,C2,'r')
plot(t,Ctot,'k')
xlabel('time')
ylabel('Concentration')
legend('C_1','C_2','C_1+C_2')
title('Problem 2 (i) - Lerger Dispersion')
```

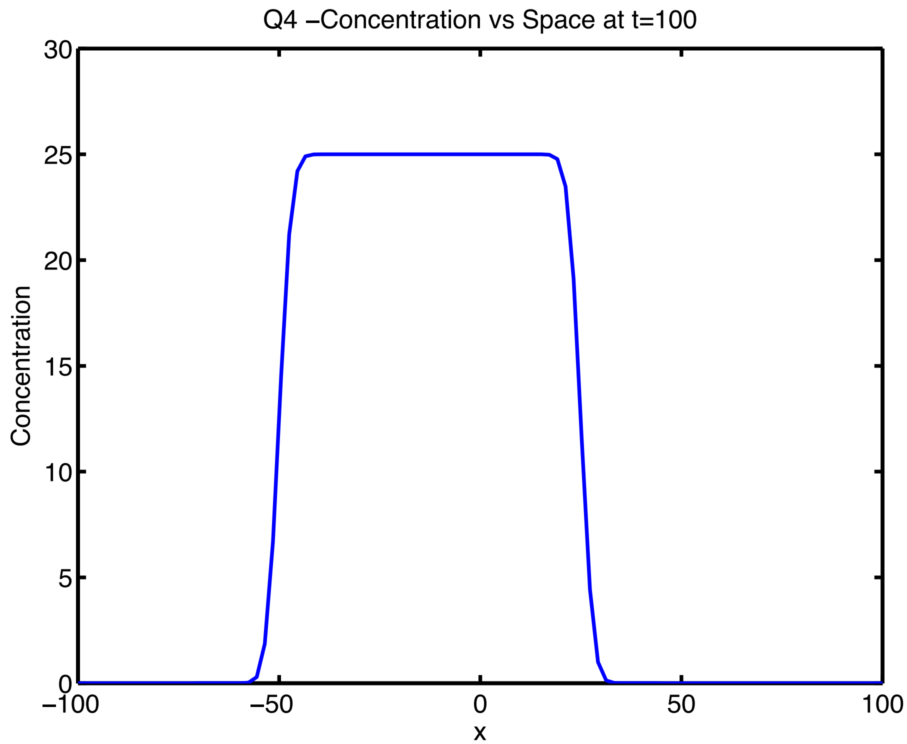
Problem 2 (i) – Smaller Dispersion



Problem 2 (i) – Lerger Dispersion



```
%Problem 4
clear

v=0.25;
D=3e-2;
C0=25;

x=linspace(-100,100);
t=100;
C=C0/2*erfc((x-v*t)./(sqrt(4*D*t)))-C0/2*erfc((x+75-v*t)./(sqrt(4*D*t)));

%plot concentration vs space at time=100 to make sure that my step looks
%right
figure(3)
plot(x,C)
axis([-100 100 0 30])
xlabel('x')
ylabel('Concentration')
title('Q4 -Concentration vs Space at t=100')


%plot breakthough curve at x=200;
x=200;
t=linspace(0,2000);
C=C0/2*erfc((x-v*t)./(sqrt(4*D*t)))-C0/2*erfc((x+75-v*t)./(sqrt(4*D*t)));
figure(4)
plot(t,C)
axis([0 2000 0 30])
xlabel('x')
ylabel('Concentration')
title('Q4 - Breakthrough Curve at x=200')

%plot breakthough curve at x=2000;
x=2000;
t=linspace(0,20000,1e4);
C=C0/2*erfc((x-v*t)./(sqrt(4*D*t)))-C0/2*erfc((x+75-v*t)./(sqrt(4*D*t)));
figure(5)
plot(t,C)
axis([0 10000 0 30])
xlabel('x')
ylabel('Concentration')
title('Q4 - Breakthrough Curve at x=2000')
```



This curve just plots concentration against space at time=100 to make sure that I have a step function of the right width moving at the right speed. Backwards edge is around -50 and leading edge around x=25 – which makes sense

Below are the plots which were asked for in the homework from which the maximum concentrations are taken