RESEARCH PAPER

PRODUCT RULE FOR VECTOR FRACTIONAL DERIVATIVES

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Abstract

This paper establishes a product rule for fractional derivatives of a real-valued function defined on a finite dimensional Euclidean vector space. The proof uses Fourier transforms.

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Editorial Note: Information about the recent authors' book [13] (by M.M. Meerschaert, A. Sikorskii), is given in this issue's Editorial Note.

1. Introduction

The purpose of this paper is to prove a product rule for vector fractional derivatives. The product rule for scalar fractional derivatives is known. The proof in Podlubny [15] uses a finite difference approach, and the proof in Osler [14] uses the Cauchy integral formula. We provide a simple proof using Fourier transforms, and then extend to vector fractional derivatives, using the same approach.

The one dimensional Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

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is defined for any *integrable* function $f(x)$, i.e., a function such that

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

If $f(x), \mathbb{D}f(x) = f'(x), \ldots, \mathbb{D}^n f(x)$ are all integrable, then it is well known (e.g., see [13, Section 1.1]) that $(ik)^n \hat{f}(k)$ is the Fourier transform of the nth derivative $\mathbb{D}^n f(x)$. Then it is natural to define the fractional derivative $\mathbb{D}^\alpha f(x)$ as the function with Fourier transform $(ik)^\alpha \hat{f}(k)$ for some $\alpha > 0$. The fractional derivative can also be defined as the limit of a difference quotient. The first derivative is

$$\mathbb{D}f(x) = f'(x) = \lim_{h \to 0} \frac{f(x) - f(x - h)}{h}$$

when the limit exists. Higher order derivatives are defined by

$$\mathbb{D}^n f(x) = \lim_{h \to 0} \frac{\Delta^n f(x)}{h^n}$$

where

$$\begin{aligned}
\Delta f(x) &= f(x) - f(x - h) \\
\Delta^2 f(x) &= \Delta\Delta f(x) = \Delta[f(x) - f(x - h)] \\
&= f(x) - 2f(x - h) + f(x - 2h) \\
\Delta^3 f(x) &= f(x) - 3f(x - h) + 3f(x - 2h) - f(x - 3h) \\
&\vdots \\
\Delta^n f(x) &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j f(x - jh)
\end{aligned}$$

using the binomial formula. The fractional difference operator

$$\Delta^\alpha f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh),$$

where

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{j!\Gamma(\alpha - j + 1)}$$

extends the usual definition, since $\Gamma(n + 1) = n!$ for positive integers n. Then the *Grünwald-Letnikov* fractional derivative can be defined as

$$\mathbb{D}^\alpha f(x) = \lim_{h \to 0} \frac{\Delta^\alpha f(x)}{h^\alpha} \tag{1.1}$$

whenever the limit exists. If $f(x)$ is bounded, and if $f(x)$ and its derivatives up to order $n > 1 + \alpha$ are integrable, then [13, Proposition 2.1] implies that

the Grünwald fractional derivative (1.1) exists, and its Fourier transform also exists, and equals $(ik)^\alpha \hat{f}(k)$.

Some alternative forms of the fractional derivative are also useful in applications. If $0 < \alpha < 1$, then an explicit computation [13, Proposition 3.10] shows that

$$(ik)^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-iky}\right) \alpha y^{-\alpha-1} dy.$$

Invert the Fourier transform $(ik)^\alpha \hat{f}(k)$, and use the fact that

$$\int e^{-ikx} f(x-y)\, dx = e^{-iky} \hat{f}(k),$$

to see that

$$\mathbb{D}^\alpha f(x) = \int_0^\infty [f(x) - f(x-y)] \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy. \qquad (1.2)$$

This is called the *generator form* of the fractional derivative. If f, f' are integrable, and vanish at infinity, then the generator form of the fractional derivative exists and has Fourier transform $(ik)^\alpha \hat{f}(k)$, see [13, Example 3.24]. For $1 < \alpha < 2$, a computation [13, Proposition 3.12] shows that

$$(ik)^\alpha = \frac{\alpha-1}{\Gamma(2-\alpha)} \int_0^\infty \left(e^{-iky} - 1 + iky\right) \alpha y^{-\alpha-1} dy.$$

Then it follows that

$$\mathbb{D}^\alpha f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \left[f(x-y) - f(x) + yf'(x)\right] y^{-1-\alpha} dy. \qquad (1.3)$$

If $1 < \alpha < 2$, and if f, f', f'' are integrable and vanish at infinity, then the generator form of the fractional derivative exists and has Fourier transform $(ik)^\alpha \hat{f}(k)$, see [13, Theorem 3.17, Example 3.25].

One important application of fractional derivatives is the fractional advection-dispersion equation (FADE)

$$\frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + a \frac{\partial^\alpha C}{\partial x^\alpha} \qquad (1.4)$$

that models the flow and transport of pollutants in underground water. Here $C = C(x,t)$ represents the concentration of pollutants at time $t > 0$ at location $x \in \mathbb{R}$, and $\partial^\alpha / \partial x^\alpha = \mathbb{D}_x^\alpha$, the fractional derivative $\mathbb{D}^\alpha$ applied to the x variable. See Benson et al. [2] for more details.

2. The one variable case

The product rule states that if $f(x)$ and $g(x)$ are two differentiable functions, then

$$\mathbb{D}[f(x)g(x)] = [fg]' = f'g + fg' = \mathbb{D}[f(x)]g(x) + f(x)\mathbb{D}[g(x)].$$

For higher order derivatives, the product rule follows by repeated application of the first derivative formula:

$$\mathbb{D}^2[f(x)g(x)] = [\mathbb{D}^2 f(x)]g(x) + 2[\mathbb{D}f(x)][\mathbb{D}g(x)] + f(x)[\mathbb{D}^2 g(x)]$$

$$\mathbb{D}^3[f(x)g(x)] = [\mathbb{D}^3 f(x)]g(x) + 3[\mathbb{D}^2 f(x)][\mathbb{D}g(x)] + 3[\mathbb{D}f(x)][\mathbb{D}^2 g(x)]$$
$$+ f(x)[\mathbb{D}^3 g(x)]$$

$$\vdots$$

$$\mathbb{D}^n[f(x)g(x)] = \sum_{j=0}^{n} \binom{n}{j} [\mathbb{D}^{n-j} f(x)][\mathbb{D}^j g(x)].$$

A fractional product rule can be found in the book of Podlubny [15]:

$$\mathbb{D}^\alpha[f(x)g(x)] = \sum_{j=0}^{\infty} \binom{\alpha}{j} [\mathbb{D}^{\alpha-j} f(x)][\mathbb{D}^j g(x)]. \tag{2.1}$$

When $j > \alpha$, the term $\mathbb{D}^{\alpha-j} f(x)$ is called a fractional integral and is also denoted by $\mathbb{I}^{j-\alpha} f(x)$. The fractional integral is defined as

$$\mathbb{I}^{j-\alpha} f(x) = \frac{1}{\Gamma(j-\alpha)} \int_{-\infty}^{x} f(u)(x-u)^{j-\alpha-1} du.$$

The proof in Podlubny [15] uses an expansion of the Grünwald formula (1.1). A different proof, using the Cauchy integral formula, is given in Osler [14]. Liouville [9] proved (2.1) using Fourier series, while Grünwald [4] and Letnikov [8] proved (2.1) directly from the definition of $\mathbb{I}^{j-\alpha} f(x)$. This paper provides an alternative proof, using Fourier transforms. Then in Section **3**, the product rule (2.1) is extended to vector fractional derivatives, using the same approach.

First note that the Fourier transform of a product is (up to a constant) a convolution:

$$\widehat{fg}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)g(x)\, dx = \frac{1}{2\pi} \hat{f}(k) * \hat{g}(k). \tag{2.2}$$

To see this, apply the Fourier inversion formula [13, Theorem 1.4]: If $\int |h(x)|\, dx < \infty$, the Fourier transform $\hat{h}(k)$ exists. If $\int |\hat{h}(k)|\, dk < \infty$

as well, then

$$h(x) = \frac{1}{2\pi} \int e^{ikx} \hat{h}(k) \, dk \qquad (2.3)$$

for all $x \in \mathbb{R}$. Now we will apply this result to the function $h(x) = f(x)g(x)$. A substitution $z = k - y$ shows that the inverse Fourier transform of

$$\frac{1}{2\pi}[\hat{f}(k) * \hat{g}(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k - y)\hat{g}(y) \, dy$$

is

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{ikx} \left(\int_{-\infty}^{\infty} \hat{f}(k - y)\hat{g}(y) \, dy\right) dk$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{i(z+y)x} \int_{-\infty}^{\infty} \hat{f}(z)\hat{g}(y) \, dy \, dz$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{izx} \hat{f}(z) \, dz \, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \hat{g}(y) \, dy$$

$$= f(x)g(x).$$

Then under certain conditions (see Theorem **2.1**), the fractional derivative $\mathbb{D}^\alpha[f(x)g(x)]$ can be defined as the function with Fourier transform $(ik)^\alpha \frac{1}{2\pi}[\hat{f}(k) * \hat{g}(k)]$.

We will prove the fractional product rule in the sense of tempered distributions, which is the natural setting for Fourier transforms. Recall that the space of rapidly decreasing functions $S(\mathbb{R}^d)$ consists of the infinitely differentiable functions $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}^d} |x^n \mathbb{D}_x^m g(x)| < \infty,$$

where $n = (n_1, \ldots, n_d)$ and $m = (m_1, \ldots, m_d)$ are vectors of non-negative integers, and

$$x^n = x_1^{n_1} \cdots x_d^{n_d}, \text{ and } \mathbb{D}_x^m = \mathbb{D}_{x_1}^{m_1} \cdots \mathbb{D}_{x_d}^{m_d}.$$

The space $S'(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ is called the space of tempered distributions. The Fourier transform, and inverse Fourier transform, are linear continuous mappings of $S'(\mathbb{R}^d)$ into itself. If $f : \mathbb{R}^d \to \mathbb{R}$ is of polynomial growth, meaning that

$$\int |f(x)|(1 + |x|)^{-p} dx < \infty$$

for some $p > 0$, then $T_f(\varphi) = \int f(x)\varphi(x) \, dx := \langle f, \varphi \rangle$ is a tempered distribution, also called a generalized function. The Fourier transform of this

generalized function is defined as $\hat{T}_f(\varphi) = \langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle = T_f(\hat{\varphi})$ for $\varphi \in S(\mathbb{R}^d)$. See Yosida [19, Ch.VI] for more details. If f is a tempered distribution, then the fractional integral $\mathbb{I}^{j-\alpha} f(x)$ always exists as the convolution with the tempered distribution $x^{j-\alpha-1} I(x > 0)/\Gamma(j - \alpha)$.

THEOREM **2.1**. *Suppose that $f(x)g(x)$ is integrable and has a continuous integrable first derivative that vanishes at infinity if $0 < \alpha < 1$, or continuous integrable first and second derivatives that vanish at infinity if $1 < \alpha < 2$. Suppose that $\mathbb{D}^{\alpha-j} f(x)$ exists and is of polynomial growth for every integer $j \geq 0$, $g \in S(\mathbb{R})$, and the series*

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} \mathbb{D}^{\alpha-j} f \mathbb{D}^j g \tag{2.4}$$

converges in $S'(\mathbb{R})$. Then (2.1) holds in the sense of generalized functions. If the series (2.4) converges pointwise, then (2.1) holds pointwise.

P r o o f. The binomial series

$$(1 + z)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} z^j \tag{2.5}$$

converges for any complex $|z| \leq 1$ and any $\alpha > 0$ (e.g., see Hille [6, p. 147]). Since this is a power series, the convergence is uniform on $|z| \leq 1$. It follows that

$$(z + w)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} z^{\alpha-j} w^j \tag{2.6}$$

for any complex numbers z, w. To see this, suppose (without any loss of generality) that $|w| \leq |z|$. Then we may write

$$(z + w)^{\alpha} = z^{\alpha}(1 + (w/z))^{\alpha}$$

$$= z^{\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (w/z)^j$$

$$= \sum_{j=0}^{\infty} \binom{\alpha}{j} z^{\alpha-j} w^j$$

which agrees with (2.6).

Under the theorem assumptions, $f(x)g(x)$ and its derivatives (of order one if $0 < \alpha < 1$, or up to order two if $1 < \alpha < 2$) are continuous and integrable and vanish at infinity. Then the fractional derivative $\mathbb{D}^{\alpha}[f(x)g(x)]$ can be defined as the function with Fourier transform $(ik)^{\alpha}(2\pi)^{-1}[\hat{f}(k) * \hat{g}(k)]$, see for example [13, Examples 3.24–3.25]. Then

the generalized function $\mathbb{D}^\alpha[f(x)g(x)]$ defines a linear functional on $S(\mathbb{R})$ such that

$$\langle \mathbb{D}^\alpha[fg], \varphi \rangle = \int_{-\infty}^{\infty} \mathbb{D}^\alpha[f(x)g(x)]\varphi(x)\,dx$$

for any $\varphi \in S(\mathbb{R})$. Its Fourier transform satisfies

$$\langle \widehat{\mathbb{D}^\alpha[fg]}, \varphi \rangle = \int_{-\infty}^{\infty} \widehat{\mathbb{D}^\alpha[fg]}(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} \mathbb{D}^\alpha[fg](x)\hat{\varphi}(x)\,dx,$$

and hence

$$\langle \widehat{\mathbb{D}^\alpha[fg]}, \varphi \rangle = \int_{-\infty}^{\infty} (ik)^\alpha \left[\frac{\hat{f}(k) * \hat{g}(k)}{2\pi} \right] \varphi(k)\,dk$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^\alpha \hat{f}(k-y)\hat{g}(y)\varphi(k)\,dy\,dk$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \{(ik-iy) + (iy)\}^\alpha \hat{f}(k-y)\hat{g}(y)\varphi(k)\,dy\,dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \binom{\alpha}{j} (ik-iy)^{\alpha-j}(iy)^j \hat{f}(k-y)\hat{g}(y)\varphi(k)\,dy\,dk\,.$$

The series

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} (ik-iy)^{\alpha-j}(iy)^j$$

converges uniformly, and can be integrated term by term with respect to $F(dy, dk) = \hat{f}(k-y)\hat{g}(y)\varphi(k)\,dy\,dk$, see [17, Theorem 7.16]. Then it follows that

$$\langle \widehat{\mathbb{D}^\alpha[fg]}, \varphi \rangle = \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (ik-iy)^{\alpha-j} \hat{f}(k-y)(iy)^j \hat{g}(y)\varphi(k)\,dy\,dk$$

$$= \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (iz)^{\alpha-j} \hat{f}(z)(i(k-z))^j \hat{g}(k-z)\varphi(k)\,dz\,dk$$

$$= \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{\mathbb{D}^{\alpha-j}f}(z)(i(k-z))^j \hat{g}(k-z)\varphi(k)\,dk\,dz.$$

The jth integral term is the action of $\widehat{\mathbb{D}^{\alpha-j}f}$ on the function

$$\int_{-\infty}^{\infty} (i(k-z))^j \hat{g}(k-z)\varphi(k)\,dk,$$

This function belongs to $S(\mathbb{R})$, since $S(\mathbb{R})$ is closed under convolutions, and since $z^j \hat{g}(z) \in S(\mathbb{R})$, in view of the fact that $g \in S(\mathbb{R})$. The Fourier transform of the generalized function $\mathbb{D}^{\alpha-j}f(x)$ is $(ik)^{\alpha-j}\hat{f}(k)$, and $(ik)^j \hat{g}(k)$ is

the pointwise Fourier transform of $\mathbb{D}^j g(x)$, since $g \in S(\mathbb{R})$. Then it follows from (2.2) that the convolution

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (ik - iy)^{\alpha-j} \hat{f}(k - y)(iy)^j \hat{g}(y) \, dy$$

is the Fourier transform of the generalized function $\mathbb{D}^{\alpha-j} f(x) \mathbb{D}^j g(x)$. Hence

$$\langle \widehat{\mathbb{D}^\alpha[fg]}, \varphi \rangle = \sum_{j=0}^{\infty} \binom{\alpha}{j} \langle \widehat{\mathbb{D}^{\alpha-j} f \mathbb{D}^j g}, \varphi \rangle,$$

and

$$\langle \mathbb{D}^\alpha[fg], \hat{\varphi} \rangle = \sum_{j=0}^{\infty} \binom{\alpha}{j} \langle \mathbb{D}^{\alpha-j} f \mathbb{D}^j g, \hat{\varphi} \rangle \tag{2.7}$$

which agrees with (2.1), provided that (2.4) holds.

Now suppose that the series (2.4) converges pointwise. For any fixed y, define $\hat{\varphi}(x) = \omega_\epsilon(x - y)$, where the approximate identity

$$\omega_\epsilon(x) = c_\epsilon \exp\{-\epsilon^2/(\epsilon^2 - x^2)\} 1_{\{|x|<\epsilon\}},$$

and the constant c_ϵ is chosen so that $\int \omega_\epsilon(x) dx = 1$. Then for any locally integrable function $h(x)$, $\int \omega_\epsilon(x - y) h(y) \, dy$ converges to $h(x)$ uniformly on compact sets as $\epsilon \to 0$. Since (2.7) holds for every $\hat{\varphi}(x) = \omega_\epsilon(x - y)$, it follows that the product rule (2.1) holds pointwise. $\qquad \square$

REMARK **2.1**. An alternative proof of Theorem **2.1** can be written using pointwise Fourier transforms, and the Fourier inversion formula. However, to apply this result, it is required that $\mathbb{D}^{\alpha-j} f(x)$ and $\mathbb{D}^j g(x)$ are integrable for every $j \geq 0$. There are few (if any) practical examples of functions $f(x)$ whose (fractional) integrals exist *and are integrable* for all $j \geq 0$. Consider for example the step function $f(x) = 1_{\{0 \leq |x| \leq 1\}}$ (or a smooth approximation of this function). For this reason, we use generalized functions, and then we only require that $\mathbb{D}^{\alpha-j} f(x)$ is of polynomial growth for every non-negative integer j. The interpretation of fractional derivatives and Fourier transforms as generalized functions is common in applications, see for example [5, 7, 18].

REMARK **2.2**. A different proof of Theorem **2.1** can be written using Laplace transforms. This only requires that $\mathbb{D}^{\alpha-j} f$ and $\mathbb{D}^j g$ are exponentially bounded, so that the Laplace transforms exist. The proof is quite similar to Theorem **2.1**, after smoothing so that f, g and all their derivatives vanish at at $x = 0$. However, for applications to functions on $\mathbb{R}^1$ or $\mathbb{R}^d$, the Fourier approach is most useful.

EXAMPLE **2.1**. For an explicit example where Theorem **2.1** applies, take $f(x) = x^p I(x > 0)$ for some $p > \alpha$, and $g(x) = e^{-bx} I(x > 0)$ for some $b > 0$. An explicit computation shows that

$$\mathbb{D}^{\alpha-j} f(x) = \frac{\Gamma(p+1)}{\Gamma(p+1+j-\alpha)} x^{p+\alpha-j} I(x > 0)$$

and of course $\mathbb{D}^j g(x) = (-b)^j e^{-bx} I(x > 0)$. Then all the conditions of Theorem **2.1** are satisfied (using a smoothed version of $g(x)$ at $x = 0$), and the series (2.4) converges pointwise, so that the product rule (2.1) holds pointwise.

EXAMPLE **2.2**. The product rule for fractional derivatives has been applied to chemical mixing in a heterogeneous porous medium [3]. In a simple chemical reaction $C_1 + C_2 \rightarrow C_3$, the local concentrations $c_i(x, t)$ satisfy a coupled system of space-fractional reaction-diffusion equations:

$$
\begin{aligned}
\frac{\partial c_i(x,t)}{\partial t} &= -v\frac{\partial c_i(x,t)}{\partial x} + a\frac{\partial^\alpha c_i(x,t)}{\partial x^\alpha} + r(x,t) \quad \text{for } i = 1, 2, \\
\frac{\partial c_3(x,t)}{\partial t} &= -r(x,t)
\end{aligned}
\tag{2.8}
$$

since the precipitant C_3 remains immobile. The reactants are in local equilibrium: $c_1 c_2 = K$. Then the local concentration difference $u = c_1 - c_2$ satisfies the conservative FADE (1.4). Note from (2.8) that the local reaction rate is

$$r = \frac{\partial c_1}{\partial t} + v\frac{\partial c_1}{\partial x} - a\frac{\partial^\alpha c_1}{\partial x^\alpha}.$$

Write $u = c_1 - K/c_1$ and solve for c_1 as a function of u. Use the chain rule to write

$$r = \frac{\partial c_1}{\partial u}\frac{\partial u}{\partial t} + v\frac{\partial c_1}{\partial u}\frac{\partial u}{\partial x} - a\mathbb{D}_x^{\alpha-1}\left[\frac{\partial c_1}{\partial u}\frac{\partial u}{\partial x}\right],$$

using the fact that $\mathbb{D}^\alpha = \mathbb{D}^{\alpha-1}\mathbb{D}^1$. Using the semi-analytic solution to the constant coefficient FADE (1.4) for u in terms of a smooth α-stable probability density function [2], an explicit solution for the local reaction rate can be obtained using the fractional product rule (2.1). To apply Theorem **2.1** with $f = \partial c_1/\partial u$ and $g = \partial u/\partial x$, approximate the stable density $u(x)$ by a *tempered stable* [1, 16], so that $g \in S(\mathbb{R})$, and then let the tempering constant go to zero.

3. Vector fractional derivatives

Suppose that $f(x) = f(x_1, \ldots, x_d)$ is a scalar-valued function of the vector argument $x = (x_1, \ldots, x_d)'$ in d dimensions. Its Fourier transform is

defined by

$$\hat{f}(k) = \int e^{-ik\cdot x} f(x)\, dx,$$

where $k = (k_1, \ldots, k_d)'$ is also a d-dimensional vector, and

$$k \cdot x = k'x = (k_1, \ldots, k_d) \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \sum_{j=1}^{d} k_j x_j$$

is the scalar product (dot product, inner product). The gradient

$$\nabla f(x) = \begin{pmatrix} \dfrac{\partial}{\partial x_1} f(x_1, \ldots, x_d) \\ \vdots \\ \dfrac{\partial}{\partial x_d} f(x_1, \ldots, x_d) \end{pmatrix}$$

is a vector-valued function whose Fourier transform is

$$(ik)\hat{f}(k) = \int e^{-ik\cdot x} \nabla f(x)\, dx,$$

where the integral is defined componentwise. The proof is a direct application of the corresponding one dimensional formula.

The product rule for two such functions f, g is

$$\nabla[f(x)g(x)] = \begin{pmatrix} \dfrac{\partial}{\partial x_1}[f(x)g(x)] \\ \vdots \\ \dfrac{\partial}{\partial x_d}[f(x)g(x)] \end{pmatrix}$$

$$= \begin{pmatrix} \left[\dfrac{\partial}{\partial x_1}f(x)\right]g(x) + f(x)\left[\dfrac{\partial}{\partial x_1}g(x)\right] \\ \vdots \\ \left[\dfrac{\partial}{\partial x_d}f(x)\right]g(x) + f(x)\left[\dfrac{\partial}{\partial x_d}g(x)\right] \end{pmatrix}$$

$$= [\nabla f(x)]g(x) + f(x)[\nabla g(x)].$$

The Laplacian

$$\Delta f(x) = \nabla \cdot \nabla f(x) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} f(x)$$

is a scalar-valued function whose Fourier transform is $(ik) \cdot (ik)\hat{f}(k) = -\|k\|^2 \hat{f}(k)$. It is also customary to write $\Delta f(x) = \nabla^2 f(x)$ by an abuse of

notation. The product rule for the Laplacian is

$$\Delta[f(x)g(x)] = \nabla \cdot \nabla[f(x)g(x)]$$
$$= \nabla \cdot \{[\nabla f(x)]g(x) + f(x)\nabla g(x)\}$$
$$= [\nabla^2 f(x)]g(x) + 2[\nabla f(x)] \cdot [\nabla g(x)] + f(x)[\nabla g(x)] \quad (3.1)$$
$$= \sum_{j=0}^{2} \binom{2}{j} \nabla^{2-j} f(x) \cdot \nabla^j g(x).$$

The scalar-valued fractional derivative $\mathbb{D}_M^\alpha f(x)$ can be defined as the function with Fourier transform

$$\int_{\|\theta\|=1} (ik \cdot \theta)^\alpha \hat{f}(k) M(d\theta),$$

where $M(d\theta)$ is an arbitrary (positive) Borel measure on the unit sphere, and $\alpha \in \mathbb{R}$. When M is uniform, $\mathbb{D}_M^\alpha f(x)$ reduces to a constant multiple of the fractional Laplacian. The fractional Laplacian $\Delta^{\alpha/2} f(x)$ can be defined as the function with Fourier transform $-\|k\|^\alpha \hat{f}(k)$. When $\alpha = 2$, this reduces to the traditional Laplacian. The vector-valued fractional gradient $\nabla_M^\alpha f(x)$ can be defined as the function with Fourier transform

$$\int_{\|\theta\|=1} \theta(ik \cdot \theta)^\alpha \hat{f}(k) M(d\theta).$$

If $\alpha = 1$ and the mixing measure M assigns unit mass to the positive coordinate vectors, then this reduces to the traditional gradient. Using these Fourier transform definitions, it is easy to check that $\mathbb{D}_M^\alpha f(x) = \nabla_M^{\alpha-1} \cdot \nabla f(x)$. See [12] for more details.

The fractional derivative $\mathbb{D}_M^\alpha f(x)$ is a weighted average of fractional directional derivatives. The directional derivative

$$\mathbb{D}_\theta^1 f(x) = \theta \cdot \nabla f(x) = \left. \frac{d}{dt} f(x + \theta t) \right|_{t=0}$$

has Fourier transform $(ik \cdot \theta)\hat{f}(k)$. The fractional directional derivative can be defined by

$$\mathbb{D}_\theta^\alpha f(x) = \left. \mathbb{D}^\alpha f(x + \theta t) \right|_{t=0},$$

where the fractional derivative $\mathbb{D}^\alpha$ is taken with respect to the variable $t \in \mathbb{R}$. Using the generator form (1.2) of the fractional derivative for $0 < \alpha < 1$, or the form (1.3) for $1 < \alpha < 2$, it is not hard to check [13, Section 6.5] that $\mathbb{D}_\theta^\alpha f(x)$ has Fourier transform $(ik \cdot \theta)^\alpha \hat{f}(k)$. Then we can also write

$$\mathbb{D}_M^\alpha f(x) = \int_{\|\theta\|=1} \mathbb{D}_\theta^\alpha f(x) \, M(d\theta). \quad (3.2)$$

For $0 < \alpha < 1$ or $1 < \alpha < 2$, this formula is valid when f and all of its partial derivatives up to order two exist, are integrable, and vanish at infinity, see Examples 6.28–6.29 in [13]. A very similar argument shows that

$$\nabla_M^\alpha f(x) = \int_{\|\theta\|=1} \theta \mathbb{D}_\theta^\alpha f(x) \, M(d\theta).$$

When $j > \alpha$, the fractional integral

$$\mathbb{I}_\theta^{j-\alpha} f(x) = \mathbb{I}^{j-\alpha} f(x + t\theta),$$

where we integrate with respect to the variable $t \in \mathbb{R}$, and

$$\mathbb{D}_M^{\alpha-j} f(x) = \int_{\|\theta\|=1} \mathbb{I}_\theta^{j-\alpha} f(x) \, M(d\theta).$$

The multi-variable FADE model for pollution migration is

$$\frac{\partial C}{\partial t} = -v \cdot \nabla C + a \mathbb{D}_M^\alpha C, \tag{3.3}$$

where $1 < \alpha < 2$, $C = C(x,t)$ is the pollution concentration at location $x \in \mathbb{R}^d$ at time $t > 0$, $v \in \mathbb{R}^d$ is the velocity vector that gives the average velocity of the pollution plume, and $a > 0$ is a constant. Note that we can also write

$$\frac{\partial C}{\partial t} = -v \cdot \nabla C + a \nabla_M^{\alpha-1} \cdot \nabla C,$$

in which the dispersive term is the fractional divergence of the concentration gradient [12]. The fractional partial differential equation (3.3) governs the long-time limit of a random walk with jumps $X = R\Theta$, where we assume $\mathbb{P}(R > r) \sim Cr^{-\alpha}$ and $\mathbb{P}(\Theta \in B) = M(B)$ for any finite Borel subset B of the unit sphere. See [13] for more information.

The product rule for the fractional derivative $\mathbb{D}_M^\alpha f(x)$ and the fractional gradient $\nabla_M^\alpha f(x)$ can now be stated and proven using Fourier transforms. The next theorem is the main result of this paper.

THEOREM **3.1**.   *Suppose that $f(x)g(x)$ is integrable and has continuous integrable first order partial derivatives that vanish at infinity if $0 < \alpha < 1$, or continuous integrable partial derivatives up to order two that vanish at infinity if $1 < \alpha < 2$. Suppose that, for a fixed θ on the unit sphere, $\mathbb{D}_\theta^{\alpha-j} f(x)$ exists and is of polynomial growth for every integer $j \geq 0$, $g \in S(\mathbb{R}^d)$, and the series*

$$\sum_{j=0}^\infty \binom{\alpha}{j} \mathbb{D}_\theta^{\alpha-j} f \mathbb{D}_\theta^j g \tag{3.4}$$

converges in $S'(\mathbb{R}^d)$. Then

$$\mathbb{D}_\theta^\alpha[f(x)g(x)] = \sum_{j=0}^\infty \binom{\alpha}{j} \mathbb{D}_\theta^{\alpha-j}f(x)\mathbb{D}_\theta^j g(x). \tag{3.5}$$

holds in the sense of generalized functions. If the series (3.4) converges pointwise, then (3.5) also holds pointwise. If (3.4) converges uniformly in θ on the unit sphere, then

$$\mathbb{D}_M^\alpha[f(x)g(x)] = \sum_{j=0}^\infty \binom{\alpha}{j} \int_{\|\theta\|=1} \mathbb{D}_\theta^{\alpha-j}f(x)\mathbb{D}_\theta^j g(x)\, M(d\theta),$$
$$\nabla_M^\alpha[f(x)g(x)] = \sum_{j=0}^\infty \binom{\alpha}{j} \int_{\|\theta\|=1} \theta\mathbb{D}_\theta^{\alpha-j}f(x)\mathbb{D}_\theta^j g(x)\, M(d\theta). \tag{3.6}$$

P r o o f. The proof is similar to Theorem **2.1**, using Fourier transforms. The Fourier inversion theorem for real-valued functions on $\mathbb{R}^d$ is stated in [11, Theorem 1.3.7]: If $\int |h(x)|\, dx < \infty$, then the Fourier transform $\hat{h}(k)$ exists. Then if $\int |\hat{h}(k)|\, dk < \infty$, we have

$$h(x) = (2\pi)^{-d} \int e^{ik\cdot x}\hat{h}(k)\, dk \tag{3.7}$$

for all $x \in \mathbb{R}^d$. Now a substitution $z = k - y$ shows that the inverse Fourier transform of $(2\pi)^{-d}[\hat{f}(k) * \hat{g}(k)]$ is

$$(2\pi)^{-2d} \int e^{ik\cdot x} \left(\int \hat{f}(k-y)\hat{g}(y)\, dy \right)\, dk$$
$$= (2\pi)^{-2d} \int e^{i(z+y)\cdot x} \int \hat{f}(z)\hat{g}(y)\, dy\, dz$$
$$= (2\pi)^{-d} \int e^{iz\cdot x}\hat{f}(z)\, dz\, (2\pi)^{-d} \int e^{iy\cdot x}\hat{g}(y)\, dy$$
$$= f(x)g(x).$$

Use the fractional binomial formula (2.6) to write

$$(ik\cdot\theta)^\alpha = \{(i[k-y]\cdot\theta) + (iy\cdot\theta)\}^\alpha = \sum_{j=0}^\infty \binom{\alpha}{j} (i[k-y]\cdot\theta)^{\alpha-j}(iy\cdot\theta)^j$$

for any $\alpha > 0$. Since $f(x)g(x)$ and its derivatives (of order one if $0 < \alpha < 1$, or up to order two if $1 < \alpha < 2$) are continuous and integrable and vanish at infinity, the fractional directional derivative $\mathbb{D}_\theta^\alpha[f(x)g(x)]$ can be defined as the function with Fourier transform $(ik\cdot\theta)^\alpha(2\pi)^{-d}[\hat{f}(k) * \hat{g}(k)]$, see [13, Examples 6.28–6.29]. For a function $\varphi \in S(\mathbb{R}^d)$, write

$$\langle \widehat{\mathbb{D}_\theta^\alpha [fg]}, \varphi \rangle$$

$$= \int (ik \cdot \theta)^\alpha (2\pi)^{-d} [\hat{f}(k) * \hat{g}(k)] \varphi(k) \, dk$$

$$= \int \int \sum_{j=0}^\infty \binom{\alpha}{j} (i[k-y] \cdot \theta)^{\alpha-j} (iy \cdot \theta)^j (2\pi)^{-d} \hat{f}(k-y)\hat{g}(y)\varphi(k) \, dy \, dk$$

$$= \sum_{j=0}^\infty \binom{\alpha}{j} (2\pi)^{-d} \int \int (i[k-y] \cdot \theta)^{\alpha-j} \hat{f}(k-y) \ (iy \cdot \theta)^j \hat{g}(y)\varphi(k) \, dy \, dk.$$

Continuing as in the proof of Theorem **2.1**, using the Fourier inversion formula (3.7), we have

$$\langle \mathbb{D}_\theta^\alpha [fg], \hat{\varphi} \rangle = \sum_{j=0}^\infty \binom{\alpha}{j} \langle \mathbb{D}_\theta^{\alpha-j} f \mathbb{D}_\theta^j g, \hat{\varphi} \rangle. \qquad (3.8)$$

As in the one-variable case, if the series 3.4 converges in the ordinary sense, then for a fixed y we take $\hat{\varphi}(x) = \omega_\epsilon(x - y)$, where

$$\omega_\epsilon(x) = c_\epsilon \exp\{-\epsilon^2/(\epsilon^2 - |x|^2)\}1_{\{|x|<\epsilon\}},$$

and constant c_ϵ is chosen so that $\int \omega_\epsilon(x)dx = 1$. Then for a locally integrable function $h(x)$, $\int \omega_\epsilon(x-y)h(y)dy$ converges to $h(x)$ uniformly on compact sets as $\epsilon \to 0$. Since (3.8) holds for every $\hat{\varphi}(x) = \omega_\epsilon(x-y)$, it follows that the product rule (3.5) holds pointwise. Finally, if the series in (3.4) converge uniformly in θ on the unit sphere, we can apply (3.2) to show that (3.6) holds. $\qquad \square$

REMARK **3.1**. In the special case $\alpha = 2$, if the mixing measure $M(d\theta)$ assigns unit mass to the positive coordinate vectors, then the fractional product rule (3.6) reduces to the product rule (3.1) for the Laplacian.

EXAMPLE **3.1**. An application to chemical mixing in one dimension was explained in Section **2**. In higher dimensions, the local reaction rate

$$r = \frac{\partial c_1}{\partial u}\frac{\partial u}{\partial t} + v\frac{\partial c_1}{\partial u}\frac{\partial u}{\partial x} - a\nabla_M^{\alpha-1}\left[\frac{\partial c_1}{\partial u}\nabla u\right],$$

and the fractional product rule in Theorem **3.1** can be applied, along with the semi-analytic solution to the multi-variable FADE (3.3) for u in terms of a multi-variable α-stable probability density function [10], to solve for the local reaction rate $r(x,t)$. To apply Theorem **3.1** with $f = \partial c_1/\partial u$ and $g = \nabla u$, approximate the stable density $u(x)$ by a *tempered stable* as in Remark **2.2**.

Acknowledgements

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