Hyper-Mixing in Linear Shear Flow

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Abstract. In this technical note we study mixing in a two-dimensional linear shear flow. We derive analytical expressions for the concentration field for an arbitrary initial condition in an unbounded two dimensional shear flow. We focus on the solution for a point initial condition and study the evolution of (i) the second centered moments as a measure for the plume dispersion, (ii) the dilution index as a measure of the mixing state and (iii) the scalar dissipation rate as a measure for the rate of mixing. It has previously been shown that the solute spreading grows with the cube of time and thus is hyper-dispersive. Herein we demonstrate that the dilution index increases quadratically with time in contrast to a homogeneous medium, for which it increases linearly. Similarly, the scalar dissipation rate decays as t^{-3}, while for a homogeneous medium it decreases slower as t^{-2}. Mixing is much stronger than in an homogeneous medium and therefore we term the observed behavior hyper-mixing.

1. Introduction

Mixing is a fundamental process in many fluid flows. Understanding and predicting mixing is a critical first step to understanding and predicting chemical reactions. Mixing drives many chemical reactions by physically bringing reactants into contact [e.g. *Rezaei et al.*, 2005; *Cirpka and Valocchi*, 2007; *Tartakovsky et al.*, 2008; *de Simoni et al.*, 2005]. As such it is a topic that has attracted attention across a wide range of disciplines. In the context of geophysical flows with application to water resources it has been an important topic of research in porous media flows [e.g. *Kapoor and Kitanidis*, 1998; *Tartakovsky et al.*, 2008] as well as higher Reynolds number turbulent flows associated with surface water flows [e.g. *Ghisalberti and Nepf*, 2002] and geophysical flows in the atmosphere and the ocean [e.g. *Weiss and Provenzale*, 2008; *Rees*, 2006].

In this study we quantify the mixing properties of linear shear flow in terms of global measures of mixing. Mixing can be characterized in a variety of ways. Second centered moments of the solute distribution measure the plume extent. Entropy based measures such as the dilution index [*Kitanidis*, 1994] characterize the volume occupied by the solute and thus quantify the mixing state. Mechanical mixing measures such as the scalar dissipation rate [*Pope*, 2000] describe the degradation of concentration contrasts and quantify the mixing dynamics. These measures are commonly used to study mixing in porous media [e.g. *Rolle et al*, 2009; *Luo et al.*, 2008; *LeBorgne et al.*, 2010]

For transport in a uniform flow field, mixing processes are driven by local diffusion. However, many geophysical flows are not truly uniform and the velocity field varies in space. Spatial heterogeneity can significantly change mixing patterns observed for homogeneous media [e.g. *Kapoor and Kitanidis*, 1998; *LeBorgne et al.*, 2010]. In heterogeneous flows, two competing mechanisms drive mixing. Local shear action of the flow field (stirring), leads to the creation of concentration gradients, which are smoothed out by local dispersion and diffusion and, thus enhances mixing.

In this Technical Note, we study these mechanisms for the particular case of linear shear flow (i.e., a velocity field that varies linearly with distance normal to the direction of flow, see Fig 1). It is often deemed representative in the context of turbulent vortical flows [e.g. *Zhiang and Glimm*, 1992] and has previously been used as a simple representation for a heterogeneous velocity field in a porous medium [e.g. *Carleton and Montas*, 2009]. In fact, flow in a spatially heterogeneous medium can be approximated locally as a linear shear flow. Linear shear flow may be considered a simple subset of flows through stratified media [e.g. *Matheron and de Marsily*, 1980; *Bolster et al.*, 2011]. For horizontal miscible displacement of freshwater by saltwater, a linear shear regime can develop for diffusion dominated scenarios [e.g., *Dentz et al.*, 2006; *Bolster et al.*, 2007]. Enhanced contaminant mixing under such conditions has been observed by *Dror et al.* [2003a, b].

In the following, we present a derivation of the Green's function for transport in a linear shear flow using the method of characteristics. Other forms equivalent to this solution have been presented previously by *Okubo* [1968]; *Okubo and Karweit* [1969] and *Monin and Yaglom* [1971]. Based on this explicit analytical solution, we study the mixing dynamics caused by the interaction of shear action and local dispersion.

2. Mixing in Linear Shear Flow

We consider transport in a $d = 2$ dimensional linear shear flow far from domain boundaries. Transport is given by the advection-dispersion equation

$$\frac{\partial c(\mathbf{x},t)}{\partial t} + [q + \alpha x_2] \frac{\partial c(\mathbf{x},t)}{\partial x_1} - \nabla \cdot [\mathbf{D} \nabla c(\mathbf{x},t)] = 0. \qquad (1)$$

for initial condition $c(\mathbf{x}, t = 0) = \rho(\mathbf{x})$ with natural boundary conditions at infinity. The x_1-axis of the coordinate system is aligned with the flow direction. The dispersion tensor is diagonal with $D_{ij} = D_i \delta_{ij}$. The flow velocity is composed of the constant contribution q and the shear contribution αx_2, in which α is the shear rate.

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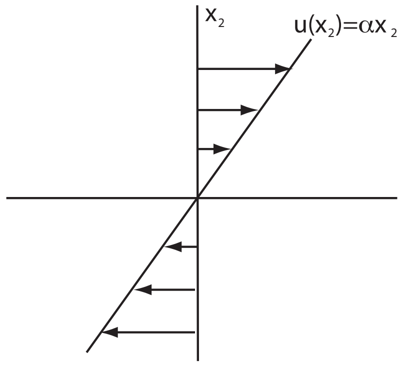

Figure 1. Schematic of the shear flow considered in this paper.

The solution $c(\mathbf{x},t)$ of (1) reads in terms of the associated Green function $g(\mathbf{x},t)$ as

$$c(\mathbf{x},t) = \int_{-\infty}^{\infty} d\mathbf{x}' \rho(\mathbf{x}') g(\mathbf{x},t|\mathbf{x}'). \tag{2}$$

The Green function $g(\mathbf{x},t|\mathbf{x}')$ satisfies (1) with initial condition $g(\mathbf{x},t=0|\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$. In Fourier space, $\tilde{g}(\mathbf{k},t|\mathbf{x}')$ satisfies

$$\frac{\partial \tilde{g}(\mathbf{k},t|\mathbf{x}')}{\partial t} - \alpha k_1 \frac{\partial \tilde{g}(\mathbf{k},t|\mathbf{x}')}{\partial k_2} - [\mathbf{k}\cdot(\mathbf{D}\mathbf{k}) - iqk_1]\,\tilde{g}(\mathbf{k},t|\mathbf{x}') = 0, \tag{3}$$

with initial condition $\tilde{g}(\mathbf{k},t=0|\mathbf{x}') = \exp(i\mathbf{k}\cdot\mathbf{x})$. This equation can be solved by integration along the characteristics $k_2(t) = k_2(0) - \alpha k_1 t$. Thus, we obtain

$$\tilde{g}(\mathbf{k},t|\mathbf{x}') = \exp[ik_1(x_1' + qt - \alpha t x_2') + ik_2 x_2'] \times \exp\left[-\frac{\mathbf{k}\cdot\boldsymbol{\kappa}(t)\mathbf{k}}{2}\right] \tag{4}$$

where $\boldsymbol{\kappa}(t)$ is the variance matrix

$$\kappa_{11}(t) = 2D_1 t + \frac{2}{3} D_2 \alpha^2 t^3, \qquad \kappa_{21}(t) = D_2 \alpha t^2$$
$$\kappa_{12}(t) = D_2 \alpha t^2, \qquad \kappa_{22}(t) = 2D_2 t. \tag{5}$$

The principal axes of the variance matrix (5) are not aligned with the axes of the coordinate system, but rotate clockwise due to the shear action of velocity field as quantified by the shear rate α. The typical time scale associated to the shear rate is denoted by $\tau_s = \alpha^{-1}$. The inverse Fourier transform of (4) and thus the Green function is given by the Gaussian

$$g(\mathbf{x},t|\mathbf{x}') = \frac{\exp\left[-\dfrac{\boldsymbol{\xi}(\mathbf{x},\mathbf{x}',t)\cdot\boldsymbol{\kappa}^{-1}(t)\boldsymbol{\xi}(\mathbf{x},\mathbf{x}',t)}{2}\right]}{2\pi\sqrt{\det[\boldsymbol{\kappa}(\boldsymbol{t})]}} \tag{6}$$

with $\boldsymbol{\xi}(\mathbf{x},\mathbf{x}',t) = \mathbf{x} - \mathbf{x}' + (\alpha t x_2' - qt)\mathbf{e}_1$.

In the following, we consider a solute plume evolving from a point-like initial distribution at $\mathbf{x} = \mathbf{0}$, $\rho(x) = \delta(\mathbf{x})$. Furthermore, we set $q = 0$. Note that a non-zero q merely translates the center of mass of the plume and does not affect spreading or mixing, which are the focus of this work. The plume extends in the direction transverse to the flow due to dispersion. The shear action then leads to an enhanced horizontal spreading and mixing of the solute. The main axes of the plume rotate in clockwise direction with increasing time. The concentration distribution is obtained from (6) by setting $\mathbf{x}' = \mathbf{0}$ and $q = 0$,

$$c(\mathbf{x},t) = \frac{\exp\left[-\frac{1}{2}\mathbf{x}^T\boldsymbol{\kappa}^{-1}(t)\mathbf{x}\right]}{2\pi\sqrt{\det[\boldsymbol{\kappa}(t)]}}. \tag{7}$$

Figure 2 illustrates the concentration distribution (7) for $D_1 = D_2 = D = 1$ at $t = 10^{-1}\tau_s$ and $t = 10\tau_s$. For $t \ll \tau_s$, the main axes of the variance tensor are aligned with $(1,1)^t$ and $(1,-1)^t$, where the superscript t denotes the transpose. With increasing time, the axes rotate and for times $t \gg \tau_s$, they are in leading order aligned with $[1, 3/(2\alpha t)]^t$ and $[-3/(2\alpha t), 1]^t$.

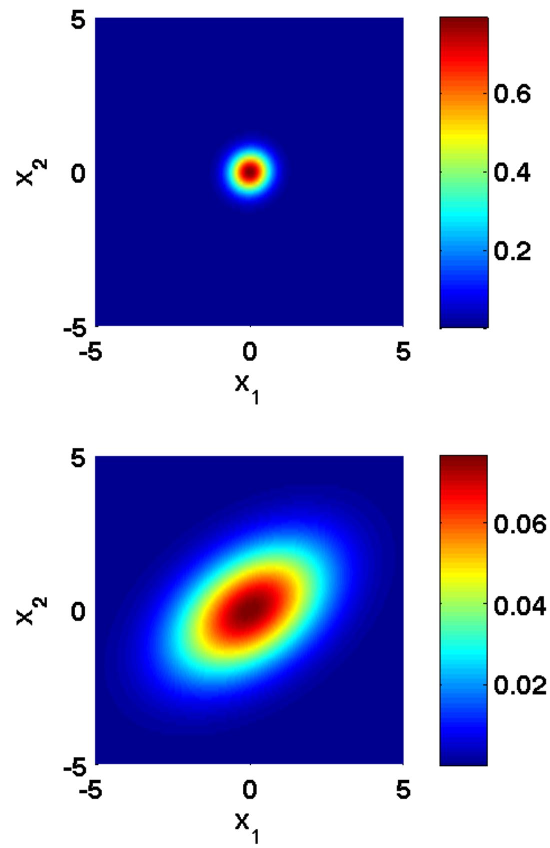

Figure 2. Concentration fields for $D_1 = 1$, $D_2 = 1$, $\alpha = 1$ at $t = 0.1\tau_s$ (top) and $10\tau_s$ (bottom). The initial condition is a point-like distribution at the origin and the mean flow is $q = 0$.

2.1. Dispersion

Dispersion in the x_1- and x_2-directions is quantified by $\kappa_{11}(t)$ and $\kappa_{22}(t)$, respectively. Plume spreading in the longitudinal direction is hyper-dispersive and for times $\alpha t \gg 1$ and scales as $\kappa_{11}(t) \sim t^3$, while spreading in the transverse direction is dispersive and scales as $\kappa_{22}(t) \sim t$, see (5).

As noted above, the main axes of the variance tensor rotate with time and are in general not aligned with the axes of the coordinate system. The plume dispersion along the main axes of the variance matrix (5) is quantified by the eigenvalues of $\boldsymbol{\kappa}(t)$, which are given by

$$\Lambda_{1/2}(t) = (D_1 + D_2)t + \frac{\alpha^2 t^2}{3} D_2 t \pm t\sqrt{(D_1 - D_2)^2 + \frac{\alpha^2 t^2}{3}\left[2D_1 D_2 + D_2^2\left(1 + \frac{\alpha^2 t^2}{3}\right)\right]}. \quad (8)$$

For illustration we show in Figure 3 the dispersion behavior along the main axis of the variance tensor in the limit of $D_1 = 0$. In this case the eigenvalues for $\alpha t \ll 1$ behave as

$$\Lambda_1(t) = 2D_2 t, \qquad \Lambda_2(t) = D_2 t(\alpha t)^2/6. \quad (9)$$

For late times, $\alpha t \gg 1$, the eigenvalues to leading order are

$$\Lambda_1(t) = 2D_2 t(\alpha t)^2/3 + 3D_2 t/2, \quad \Lambda_2(t) = D_2 t/2. \quad (10)$$

The cross-over between the early and late time regimes is marked by the shear scale $\tau_s = \alpha^{-1}$.

In this analysis we consider the approximation of an infinite domain. It is worth noting that for a vertically bounded domain, the spreading behavior is asymptotically given by Taylor-Aris dispersion, which is characterized by a constant effective dispersion coefficient in flow direction [e.g. *Taylor*, 1953; *Aris*, 1956; *Brenner and Edwards*, 1993; *Young and Jones*, 1991; *Bolster et al.*, 2009; *Porter et al.*, 2010].

Next we consider the impact of the interaction of shear and transverse dispersion on the mixing within the plume.

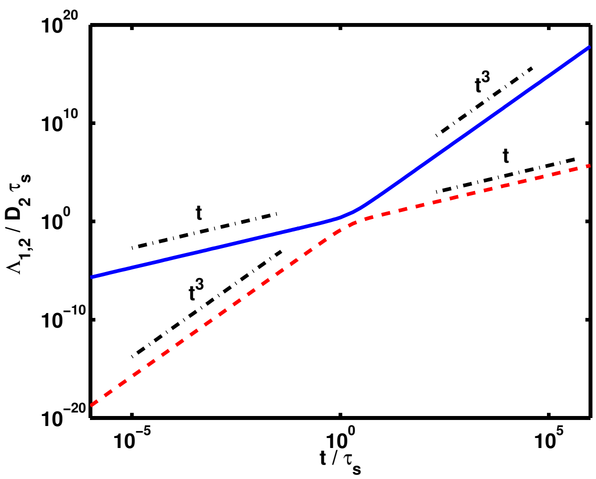

Figure 3. Spatial Variance along the main axes of the variance tensor for $D_1 = 0$ in (8). The behavior of $\Lambda_1(t)$ is shown by the solid blue line, while the behavior of $\Lambda_2(t)$ is shown by the dashed red line. The dash-dot black lines illustrate slopes of t and t^3.

2.2. Dilution Index

The dilution index [*Kitanidis*, 1994] is a measure of the volume that is occupied by a solute. Thus, it characterizes the mixing state of the system. The dilution index for an unbounded system is defined as

$$E(t) = \exp[-H(t)], \quad H(t) = \int d\mathbf{x} c(\mathbf{x}, t) \ln[c(\mathbf{x}, t)], \quad (11)$$

in which $H(t)$ is the entropy of the system under consideration. For the point-like injection considered here it can be shown that [*Kitanidis*, 1994]

$$E(t) = 2\pi e \sqrt{\det \boldsymbol{\kappa}(t)}. \quad (12)$$

Using (5), the dilution index for linear shear flow is

$$E(t) = 2\pi e \sqrt{4D_1 D_2 t^2 + \frac{1}{3} D_2^2 \alpha^2 t^4}. \quad (13)$$

For diffusion only, $\alpha = 0$, the dilution index evolves linearly with time, $E(t) \propto t$ as pointed out by *Kitanidis* [1994]. In the presence of shear, its long time behavior is $E(t) \propto t^2$, that is, the volume occupied by the solute increases quadratically with time. For $D_1 = 0$ the dilution index $E(t) \sim t^2$ scales hyper-dispersively at all times. This demonstrates that a pure shear flow causes a dramatic increase of the mixing state relative to the pure diffusion case. From *Kitanidis* [1994] we know that for a homogeneous system the dilution index scales as $E(t) \sim t^{d/2}$ where d is the number of spatial dimensions. Here we see that the dilution index in a two dimensional system with a pure shear scales even faster than for a $d = 3$ dimensional homogeneous case. The actual long time scaling of t^2 would be equivalent to a homogenous system in $d = 4$ spatial dimensions. This means, in order to obtain such a rapid rate of increase in the dilution index for a homogeneous environment one would require four spatial dimensions. It is also noteworthy that for large shear rates α, the dilution index increases linearly with α.

2.3. Scalar Dissipation Rate

Here we study the impact of linear shear action on the mixing dynamics as quantified by the scalar dissipation rate

$$\chi(t) = \int_\Omega d\mathbf{x} \nabla c(\mathbf{x}, t) \cdot \mathbf{D} \nabla c(\mathbf{x}, t). \quad (14)$$

It measures the degradation of concentration variability within the plume [e.g., *Pope*, 2000; *Kapoor and Kitanidis*, 1998]. It quantifies the basic mixing mechanisms, which are the creation of concentration contrasts by shear action and their dissipation by local dispersion. Similar expressions can be identified in equilibrium reaction rates for mixing limited reactions [e.g. *de Simoni et al.*, 2005; *Luo et al.*, 2008; *Donado et al.*, 2009; *LeBorgne et al.*, 2010]. A concentration weighted form of the dissipation rate is closely linked to the growth rate of the entropy in an advection-dispersion transport system [*Kitanidis*, 1994]. Using (7) in (14), the scalar dissipation rate is

$$\chi(t) = \frac{\sqrt{3}}{2\pi\sqrt{D_2 t^2}} \frac{\alpha^2 D_2 t^2 + 6D_1}{(12D_1 + \alpha^2 D_2 t^2)^{\frac{3}{2}}}. \quad (15)$$

For asymptotically large times we obtain the scaling $\chi(t) \sim t^{-3}$. For a homogenous velocity field it can be shown that the scalar dissipation rate depends on the dimensionality of space as expressed by the scaling $\chi(t) \sim t^{-d/2-1}$. The t^{-3} behavior observed for linear shear flow corresponds to a $d = 4$ dimensional homogeneous flow, which is consistent with the dilution index calculation. This is unsurprising since as noted above a concentration weighted form of the scalar dissipation rate can be related to the growth rate of entropy $H(t)$.

Observations for the scalar dissipation rate in $d = 2$ dimensional heterogeneous velocity fields from simulations [*LeBorgne et al.*, 2010] and theory [*Bolster et al.*, 2011], as well as from effective nonlocal models [*Bolster et al.*, 2010] scale somewhere in between values associated with $d = 2$ and $d = 3$ spatial dimensions, suggesting that the heterogeneity causes the system to behave as if it had a dimension between these two limits, but not as high as for the pure shear flow. In a real heterogeneous flow field shearing occurs at the small scale. However, the shearing can be interrupted or altered, resulting in the fact that the hyper-dispersive regime may be interrupted too. As for the case of the dilution index, setting $D_1 = 0$ results in the hyper-dispersive scaling $\chi(t) \sim t^{-3}$ from $t = 0$ onwards at all times.

3. Conclusions

Linear shear flow is a very efficient driver of mixing that greatly enhances mixing relative to a homogeneous flow. The origin of such hyper mixing is the hyper-dispersive growth of longitudinal dispersion that creates a rapidly growing interface for diffusion to act. The temporal scaling of the mixing measures are in this case directly related to the temporal scaling of dispersion. In fact, both the solutions for the dilution index and scalar dissipation rate predict that the late time scaling for these measures corresponds to an equivalent homogeneous system in $d = 4$ spatial dimensions. Previous studies in two dimensions predict that mixing in heterogeneous velocity fields may scale like a system between $d = 2$ and $d = 3$ dimensions. Linear shear flow acts as an efficient mixer, which can for example explain the enhanced mixing observed in variable density flows at low Péclet numbers [e.g. *Dror et al.*, 2003a, b; *Dentz et al.*, 2006; *Bolster et al.*, 2007].

Stretching of the plume in heterogeneous flow fields can be linked to shear regions due to correlation of the flow field in both longitudinal and transverse directions [e.g. *Le Borgne et al.*, 2008]. This phenomenon is expected to occur at small scales, at which the heterogeneous flow fields can be approximated as a linear shear flow [*Tennekes and Lumley*, 1972]. The behavior observed for linear shear cannot persist at large times because the shear rate varies spatially and thus the plume is in general exposed to different shear regimes as it travels through the heterogeneous medium. How one can link small scale hyper-mixing to large scale mixing is an area of active research.

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