THE FUNDAMENTAL GROUP AND COVERING SPACES

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Abstract. These notes, from a first course in algebraic topology, introduce the fundamental group and the fundamental groupoid of a topological space and use them to classify covering spaces.

Contents

1. Homotopy theory of paths and loops

Definition 1.1. A *path* in a topological space X from $x_0 \in X$ to $x_1 \in X$ is a map $u\colon I \to X$ of the unit interval into X with $u(0) = x_0$ and $u(1) = x_1$.

Two paths, u_0 and u_1, from x_0 to x_1 are *path homotopic*, and we write simply $u_0 \simeq u_1$, if $u_0 \simeq u_1$ rel ∂I, ie if u_0 and u_1 are are homotopic relative to the end-points $\{0,1\}$ of the unit interval I.

- The *constant path* at x_0 is the path $x_0(s) = x_0$ for all $s \in I$
- The *inverse path* to u is the path from x_1 to x_0 given by $\overline{u}(s) = u(1-s)$

If v is a path from $v(0) = u(1)$ then the *product path* path $u \cdot v$ given by

$$(u \cdot v)(s) = \begin{cases} u(2s) & 0 \le s \le \frac{1}{2} \\ v(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

where we first run along u with double speed and then along v with double speed is a path from $u(0)$ to $v(1)$.

In greater detail, $u_0 \simeq u_1$ if there exists a homotopy $h\colon I \times I \to X$ such that $h(s,0) = u_0(s)$, $h(s,1) = u_1(s)$ and $h(0,t) = x_0$, $h(1,t) = x_1$ for all $s,t \in I$. All paths in a homotopy class have the same start point and the same end point. Note the following rules for products of paths

- $x_0 \cdot u \simeq u \simeq u \cdot x_1$
- $u \cdot \overline{u} \simeq x_0$, $\overline{u} \cdot u \simeq x_1$

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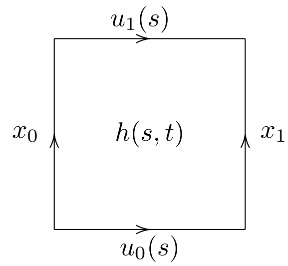
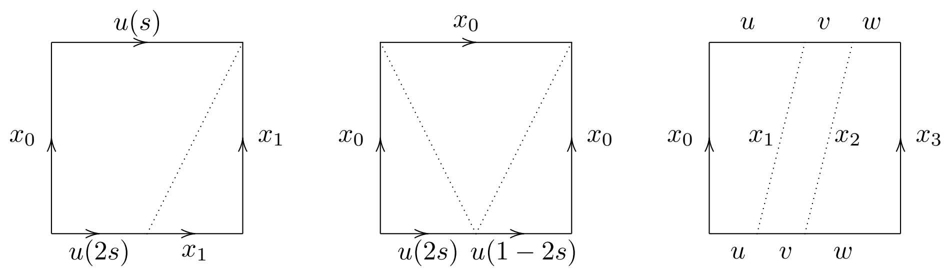


FIGURE 1. A path homototopy between two paths

- $(u \cdot v) \cdot w \simeq u \cdot (v \cdot w)$
- $u_0 \simeq u_1,\ v_0 \simeq v_1 \Longrightarrow u_0 \cdot v_0 \simeq u_1 \cdot v_1$

These drawings are meant to suggest proofs for the first three statements:

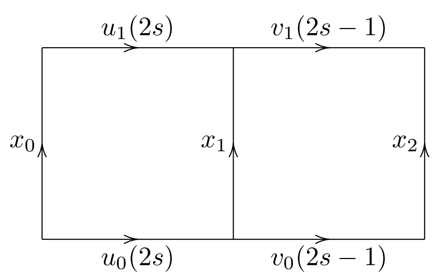


In the first case, we first run along u with double speed and then stand still at the end point x_1 for half the time. The homotopy consists in slowing down on the path u and spending less time just standing still at x_1.

In the second case, we first run along u all the way to x_1 with double speed and then back again along u also with double speed. The homotopy consists in running out along u, standing still for an increasing length of time (namely for $\frac{1}{2}(1-t) \leq s \leq \frac{1}{2}(1+t)$), and running back along u.

In the third case we first run along u with speed 4, then along v with speed 4, then along w with speed 2, and we must show that can deformed into the case where we run along u with speed 2, along v with speed 4, along w with speed 4. This can be achieved by slowing down on u, keeping the same speed along v, and speeding up on w.

The fourth of the above rules is proved by this picture,



We write $\pi(X)(x_0, x_1)$ for the set of al homotopy classes of paths in X from x_0 to x_1 and we write $[u] \in \pi(X)(x_0, x_1)$ for the homotopy class containing the path u. The fourth rule implies that the product operation on paths induces a product operation on homotopy classes of paths

$$\pi(X)(x_0, x_1) \times \pi(X)(x_1, x_2) \xrightarrow{\cdot} \pi(X)(x_0, x_2) \colon ([u], [v]) \to [u] \cdot [v] = [u \cdot v] \tag{1.2}$$

and the other three rules translate to similar rules

- $[x_0] \cdot [u] = [u] = [u] \cdot [x_1]$ (neutral elements)
- $[u] \cdot [\overline{u}] = [x_0]$, $[\overline{u}] \cdot [u] = [x_1]$ (inverse elements)
- $([u] \cdot [v]) \cdot [w] = [u] \cdot ([v] \cdot [w])$ (associativity)

for this product operation.

We next look at the functorial properties of this construction. Suppose that $f\colon X \to Y$ is a map of spaces. If u is a path in X from x_0 to x_1, then the image fu is a path in Y from $f(x_0)$ to $f(x_1)$. Since homotopic paths have homotopic images there is an induced map

$$\pi(X)(x_0, x_1) \xrightarrow{\pi(f)} \pi(Y)(f(x_0), f(x_1))\colon [u] \to [fu]$$

on the set of homotopy classes of paths. Observe that this map

- $\pi(f)$ does not change if we change f by a homotopy relative to $\{x_0, x_1\}$,
- $\pi(f)$ respects the path product operation in the sense that $\pi(f)([u] \cdot [v]) = \pi(f)([u]) \cdot \pi(f)([v])$ when $u(1) = v(0)$,
- $\pi(\mathrm{id}_X) = \mathrm{id}_{\pi(X)(x_1,x_2)}$, $\pi(g \circ f) = \pi(g) \circ \pi(f)$ for maps $g\colon Y \to Z$

We now summarize our findings.

Proposition 1.3. *For any space X, $\pi(X)$ is a groupoid, and for any map $f\colon X \to Y$ between spaces the induced map $\pi(f)\colon \pi(X) \to \pi(Y)$ is a groupoid homomorphism. In fact, π is a functor from the category of topological spaces to the category of groupoids.*

Definition 1.4. $\pi(X)$ is called the fundamental groupid of X. The *fundamental group* based at $x_0 \in X$ is the group $\pi_1(X, x_0) = \pi(X)(x_0, x_0)$ of homotopy classes of loops in X based at x_0.

The path product (1.2) specializes to a product operation

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$$

and to transitive free group actions

$$\pi_1(X, x_0) \times \pi(X)(x_0, x_1) \to \pi(X)(x_0, x_1) \leftarrow \pi(X)(x_0, x_1) \times \pi_1(X, x_1) \tag{1.5}$$

so that $\pi_1(X, x_0)$ is indeed a group and $\pi(X)(x_0, x_1)$ is an affine group from the left and from the right.

For fundamental groups, in particular, any based map $f\colon (X, x_0) \to (Y, y_0)$ induces a group homomorphism $\pi(f) = f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, given by $\pi_1(f) = f_*([u]) = [fu]$, that only depends on the based homotopy class of the based map f.

Proposition 1.6. *The fundamental group is a functor $\pi_1\colon \mathbf{hoTop}_* \to \mathbf{Grp}$ from the homotopy category of based topological spaces into the category of groups.*

This means that $\pi_1(\mathrm{id}_{(X,x_0)}) = \mathrm{id}_{\pi_1(X,x_0)}$ and $\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$. It follows immediately that if $f\colon X \to Y$ is a homotopy equivalence of *based* spaces then the induced map $\pi_1(f) = f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of groups. (See Section 5 for more information about categories and functors.)

Corollary 1.7. *Let X be a space, A a subspace, and $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ the group homomorphism induced by the inclusion map $i\colon A \to X$.*

1. *If A is a retract of X then i_* has a left inverse (so it is a monomorphism).*
2. *If A is a deformation retract of X then i_* has an inverse (so it is an isomorphism).*

Proof. (1) Let $r\colon X \to A$ be a map such that $ri = 1_A$. Then $r_* i_*$ is the identity isomorphism of $\pi_1(A, a_0)$. (2) Let $r\colon X \to A$ be a map such that $ri = 1_A$ and $ir \simeq 1_X \mathrm{rel}\, A$. Then $r_* i_*$ is the identity isomorphism of $\pi_1(A, a_0)$ and $i_* r_*$ is the identity isomorphism of $\pi_1(X, a_0)$ so i_* and r_* are each others' inverses. □

Corollary 1.8. *Let X and Y be spaces. There is an isomorphism*

$$(p_X)_* \times (p_Y)_*\colon \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

induced by the projection maps $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$.

Proof. The loops in $X \times Y$ have the form $u \times v$ where u and v are loops in X and Y, respectively (General Topology, 2.63). The above homomorphism has the form $[u \times v] \to [u] \times [v]$. The inverse homomorphism is $[u] \times [v] \to [u \times v]$. Note that this is well-defined. □

We can now compute our first fundamental group.

Example 1.9. $\pi_1(\mathbf{R}^n, 0)$ is the trivial group with just one element because $\mathbf{R}^n$ contains the subspace $\{0\}$ consisting of one point as a deformation retract. Any space that deformation retract onto on of its points has trivial fundamental group. Is it true that any contractible space has trivial fundamental group?

Our tools to compute π_1 in more interesting cases are covering space theory and van Kampen's theorem.

1.10. **Change of base point and unbased homotopies.** What happens if we change the base point? In case, the new base point lies in another path-component of X, there is no relation at all between the fundamental groups. But if the two base points lie in the same path-component, the fundamental groups are isomorphic.

Lemma 1.11. *If u is a path from x_0 to x_1 then conjugation with $[u]$*

$$\pi_1(X,x_1) \to \pi_1(X,x_0)\colon [v] \to [u]\cdot[v]\cdot[\overline{u}]$$

is a group isomorphism.

Proof. This is immediate from the rules for products of paths and a special case of (1.5). □

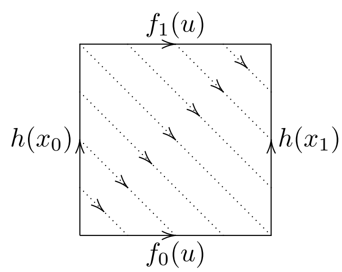
We already noted that if two maps are homotopic *relative to the base point* then they induce the same group homomorphism between the fundamental groups. We shall now investigate how the fundamental group behaves with respect to free maps and free homotopies, ie maps and homotopies that do not preserve the base point.

Lemma 1.12. *Suppose that $f_0 \simeq f_1\colon X \to Y$ are homotopic maps and $h\colon X\times I \to Y$ a homotopy. For any point $x \in X$, let $h(x) \in \pi(Y)(f_0(x), f_1(x))$ be the path homotopy class of $t \to h(x,t)$. For any $u \in \pi(X)(x_0,x_1)$ there is a commutative diagram*

$$\begin{array}{ccc} f_0(x_0) & \xrightarrow{h(x_0)} & f_1(x_0) \\ {\scriptstyle f_0(u)}\downarrow & & \downarrow{\scriptstyle f_1(u)} \\ f_0(x_1) & \xrightarrow[h(x_1)]{} & f_1(x_1) \end{array}$$

in $\pi(Y)$.

Proof. Let u be any path from x_0 to x_1 in X. If we push the left and upper edge of the homotopy $I\times I \to Y\colon (s,t) \to h(u(s),t)$ into the lower and right edge



we obtain a path homotopy $h(x_0)\cdot f_1(u) \simeq f_0(u)\cdot h(x_1)$. □

Corollary 1.13. *In the situation of Lemma 1.12, the diagram*

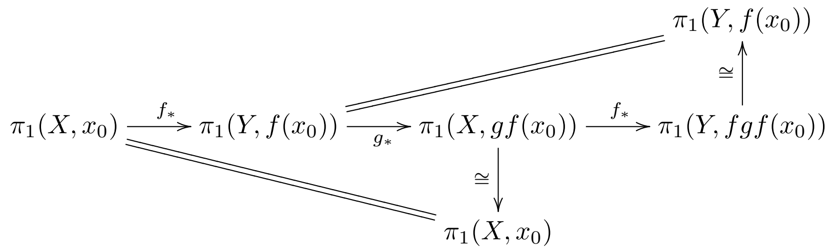
$$\begin{array}{ccc} & & \pi_1(Y, f_1(x_0)) \\ & \nearrow^{(f_1)_*} & \\ \pi_1(X,x_0) & & \cong\ \downarrow{\scriptstyle [h(x_0)]-[\overline{h(x_0)}]} \\ & \searrow_{(f_0)_*} & \\ & & \pi_1(Y, f_0(x_0)) \end{array}$$

commutes.

Proof. For any *loop* u based at x_0, $f_0(u)h(x_0) = h(x_0)f_1(u)$ or $f_0(u) = h(x_0)f_1(u)\overline{h(x_0)}$. □

Corollary 1.14. (1) *If $f\colon X\to Y$ is a homotopy equivalence (possibly unbased) then the induced homomorphism $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is a group isomorphism.*
(2) *If $f\colon X \to Y$ is nullhomotopic (possibly unbased) then $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is the trivial homomorphism.*

Proof. (1) Let g be a homotopy inverse to f so that $gf \simeq 1_X$ and $fg \simeq 1_Y$. By Lemma 1.12 there is a commutative diagram



which shows that g_* is both injective and surjective, ie g_* is bijective. Then also f_* is bijective.
(2) If f homotopic to a constant map c then f_* followed by an isomorphism equals c_* which is trivial. Thuse also f_* is trivial. □

We can now answer a question from Example 1.9 and say that any contractible space has trivial fundamental group.

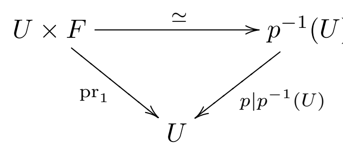
Definition 1.15. A space is *simply connected* if there is a unique path homotopy class between any two of its points.

The space X is simply connected if $\pi(X)(x_1, x_2) = *$ for all $x_1, x_2 \in X$, or, equivalently, X is path connected and $\pi_1(X, x) = *$ at all points or at one point of X.

2. Covering spaces

A covering map over X is a map that locally looks like the projection map $X \times F \to X$ for some discrete space F.

Definition 2.1. A covering map is a continuous surjective map $p\colon Y \to X$ with the property that for any point $x \in X$ there is a neighborhood U (an *evenly covered* neighborhood), a discrete set F, and a homeomorphism $U \times F \to p^{-1}(U)$ such that the diagram



commutes.

Some covering spaces, but not all (7.22), arise from left group actions. Consider a *left* action $G \times Y \to Y$ of a group G on a space Y. Let $p_G\colon Y \to G\backslash Y$ be the quotient map of Y onto the orbit space $G\backslash Y$. The quotient map p_G is open because open subsets $U \subset Y$ have open saturations $GU = \bigcup_{g \in G} gU = p_G^{-1} p_G(U)$ (General Topology 2.82). The open sets in $G\backslash Y$ correspond bijectively to saturated open sets in Y.

We now single out the left actions $G \times Y \to Y$ for which the quotient map $p_G\colon Y \to G\backslash Y$ of Y onto its orbit space is a covering map.

Definition 2.2. [5, (*) p. 72] A covering space action is a group action $G \times Y \to Y$ where any point $y \in Y$ has a neighborhood U such that the translated neighborhoods gU, $g \in G$, are disjoint. (In other words, the action map $G \times U \to GU$ is a homeomorphism.)

Example 2.3. The actions
- $\mathbf{Z} \times \mathbf{R} \to \mathbf{R}\colon (n, t) \mapsto n + t$
- $\mathbf{Z}/2 \times S^n \to S^n\colon (\pm 1, x) \mapsto \pm x$
- $\mathbf{Z}/m \times S^{2n+1} \to S^{2n+1}\colon (\zeta, x) \mapsto \zeta x$, where $\zeta \in \mathbf{C}$ is an mth root of unity, $\zeta^m = 1$,
- $\{\pm 1, \pm i, \pm j, \pm k\} \times S^3 \to S^3$, quaternion multiplication [5, Example 1.43],

are covering space actions and the orbit spaces are $\mathbf{Z}\backslash\mathbf{R} = S^1$ (the circle), $\mathbf{Z}/2\backslash S^n = \mathbf{R}P^n$ (real projective space), and $\mathbf{Z}/m\backslash S^{2n+1} = L^{2n+1}(m)$ (lense space). The action $\mathbf{Z} \times S^1 \to S^1\colon (n, z) \mapsto e^{\pi i \sqrt{2} n} z$ is *not* a covering space action for the orbits are dense.

Example 2.4. The maps $p_n\colon S^1 \to S^1$, $n \in \mathbf{Z}$, and $p_\infty\colon \mathbf{R} \to S^1$ given by $p_n(z) = z^n$, and $p_\infty(s) = e^{2\pi s} = (\cos(2\pi s), \sin(2\pi s))$ are covering maps of the circle with fibre $p_n^{-1}(1) = \mathbf{Z}/n\mathbf{Z}$ and $p_\infty^{-1}(1) = \mathbf{Z}$. There are many covering maps of $S^1 \vee S^1$. The map $S^n \to C_2\backslash S^n = \mathbf{R}P^n$, $n \geq 1$, is a covering map of real projective n-space. The map $S^{2n+1} \to C_m\backslash S^{2n+1} = L^{2n+1}(m)$ is a covering map of the lens space. $M_g \to N_{g+1}$ a double covering map of the unorientable surface of genus $g+1$ with $F = \mathbf{Z}/2\mathbf{Z}$. Can you find a covering map of M_g? Can you find a covering map of $\mathbf{R}$?

Theorem 2.5 (Unique HLP for covering maps)**.** [5, 1.30] *Let $p\colon Y \to X$ be a covering map, B be any space, and $h\colon B \times I \to X$ a homotopy into the base space. If one end of the homotopy lifts to a map $B \times \{0\} \to Y$ then the whole homotopy admits a unique lift $B \times I \to Y$ such that the diagram*

$$\begin{array}{ccc} B \times \{0\} & \xrightarrow{\widetilde{h}_0} & Y \\ \big\downarrow & \overset{\widetilde{h}}{\nearrow} & \big\downarrow p \\ B \times I & \xrightarrow[h]{} & X \end{array}$$

commutes.

Proof. We consider first the case where B is a point. The statement is then that in the situation

$$\begin{array}{ccc} \{0\} & \xrightarrow{y_0} & Y \\ \big\downarrow & \overset{\widetilde{u}}{\nearrow} & \big\downarrow p \\ I & \xrightarrow[u]{} & X \end{array}$$

there is a unique map $\widetilde{u}\colon I \to Y$ such that $p\widetilde{u} = u$ and $\widetilde{u}(0) = y_0$. For uniqueness of lifts from I see Theorem 2.12.(1). We need to prove existence. The Lebesgue lemma (General Topology, 2.158) applied to the compact space I says that there is a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of I such that u maps each of the closed subintervals $[t_{i-1}, t_i]$ into an evenly covered neighborhood in X. Suppose that we have lifted u to $\widetilde{u}$ defined on $[0, t_{i-1}]$. Let U be an evenly covered neighborhood of $u(t_{i-1})$. Suppose that the lift $\widetilde{u}(t_{i-1})$ belongs to $U \times \{\ell\}$ for some $\ell \in F$. Continue the given $\widetilde{u}$ with $(p|(U \times \{\ell\}))^{-1} \circ u|[t_{i-1}, t_i]$. After finitely many steps we have the unique lift on I.

We now turn to the general situation. Uniqueness is clear for we have just seen that lifts are uniquely determined on the vertical slices $\{b\} \times I \subset B \times I$ for any point b of B. Existence is also clear except that continuity of the lift is not clear.

We now prove that the lift is continuous. Let b be any point of X. By compactness, there is a neighborhood N_b of b and a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of I such that h maps each of the sets $N_b \times [t_{i-1}, t_i]$ into an evenly covered neighborhood of X. Suppose that $h(N_b \times [0, t_1])$ is contained in the evenly covered neighborhood $U \subset X$ and let $\widetilde{U} \subset p^{-1}(U) \subset Y$ be a neighborhood such that $p|\widetilde{U}\colon \widetilde{U} \to U$ is a homeomorphism and $\widetilde{h}_0(b, 0) \in \widetilde{U}$. We can not be sure that $\widetilde{h}_0(N_b \times \{0\}) \subset \widetilde{U}$; only if N_b is connected. Replace N_b by $N_b \cap \widetilde{h}_0^{-1}(\widetilde{U})$. Then $\widetilde{h}_0(N_b \times \{0\}) \subset \widetilde{U}$. Then $(p|\widetilde{U})^{-1} \circ h|N_b \times [0, t_1]$ is a lift of $h|N_b \times [0, t_1]$ extending $\widetilde{h}_0$. After finitely many steps we have a lift defined on $N_b \times I$ (where N_b is possibly smaller than the N_b we started with). Do this for every point b of B. These maps must agree on their overlap by uniqueness. So they define a lift $B \times I \to Y$. This lift is continuous since it is continuous on each of the open tubes $N_b \times I$. □

We emphasize the special case where B is a point. Let $y_0 \in Y$ be a point in Y and $x_0 = p(y_0) \in X$ its image in X.

Corollary 2.6 (Unique path lifting)**.** *Let x_0 and x_1 be two points in X and let y_0 be a point in the fibre $p^{-1}(x_0) \subset Y$ over x_0. For any path $u\colon I \to X$ from x_0 to x_1, the exists a unique path $\tilde{u}\colon I \to Y$ in Y starting at $\tilde{u}(0) = y_0$. Moreover, homotopic paths have homotopic lifts: If $v\colon I \to X$ is a path in X that is path homotopic to u then the lifts $\tilde{u}$ and $\tilde{v}$ are also path homotopic.*

Proof. First, in Theorem 2.5, take B to be point. Next, take B to be I and use the HLP to see that homotopic paths have homotopic lifts. □

Corollary 2.7. *Let $p\colon Y \to X$ be a covering map and let $y_0, y_1, y_2 \in Y$, $x_0 = py_0, x_1 = py_1, x_2 = py_2$.*

(1) *By recording end points of lifts we obtain maps*
$$p^{-1}(x_1) \times \pi(X)(x_1, x_2) \dot{\to} p^{-1}(x_2), \qquad p^{-1}(x_0) \times \pi_1(X, x_0) \dot{\to} p^{-1}(x_0)$$
given by $y \cdot [u] = \widetilde{u}_y(1)$ *where* $\widetilde{u}_y$ *is the lift of* u *starting at* y. *Multiplication by a path* u *from* x_1 *to* x_2 *slides the fibre over* x_1 *bijectively into the fibre over* x_2.

(2) *The covering map* $p\colon X \to Y$ *induces injective maps*
$$\pi(Y)(y_1, y_2) \xrightarrow{p_*} \pi(X)(x_1, x_2), \qquad \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(X, x_0)$$
The subset $p_*\pi(Y)(y_1, y_2) \subset \pi(X)(x_1, x_2)$ *consists of all paths from* x_1 *to* x_2 *that lift to paths from* y_1 *to* y_2. *The subbgroup* $p_*\pi_1(Y, y_0) \leq \pi_1(X, x_0)$ *consists of all loops at* x_0 *that lloft to loops at* y_0.

Definition 2.8. The *monodromy functor* of the covering map $p\colon X \to Y$ is a functor
$$F(p)\colon \pi(X) \to \mathbf{Set}$$
of the fundamental groupoid of the base space into the category **Set** of sets. This functor takes a point in $x \in X$ to the fibre $F(p)(x) = p^{-1}(x)$ over that point and it takes a path homotopy class $u \in \pi(X)(x_0, x_1)$ to $F(p)(x_0) = p^{-1}(x_0) \to p^{-1}(x_1) = F(p)(x_1)\colon y \to y \cdot u$. (The notation here is such that $F(p)(uv) = F(p)(v) \circ F(p)(u)$ for paths $u \in \pi(X)(x_0, x_1)$, $v \in \pi(X)(x_1, x_2)$.)

In particular, the fibre $F(p)(x) = p^{-1}(x)$ over any point $x \in X$ is a right $\pi_1(X, x)$-set.

Corollary 2.9 (The fundamental groupoid of a covering space)**.** *The fundamental groupoid of* Y,
$$\pi(Y) = \pi(X) \rtimes F(p)$$
is the Grothendieck construction of the fiber functor (2.8). *In other words, the map* $\pi(p)\colon \pi(Y)(y_0, y_1) \to \pi(X)(x_0, x_1)$ *is injective and the image is the set of path homotopy classes from* x_0 *to* x_1 *that take* y_0 *to* y_1. *In particular, the homomorphism* $p_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ *is injective and its image is the set of loops at* x_0 *that lift to loops at* y_0.

Proof. We consider the functor $F(p)$ as taking values in discrete categories. The objects of $\pi(X) \rtimes F(p)$ are pairs (x, y) where $x \in X$ and $y \in F(p)(x) \subset Y$. A morphism $(x_1, y_1) \to (x_2, y_2)$ is a pair (u, v) where u is a morphism in $\pi(X)$ from x_1 to x_2 and v is a morphism in $F(p)(x_2)$ from $F(p)(u)(x_1) = x_1 \cdot u$ to y_2. As $F(p)(x_2)$ have no morphisms but identities, the set of morphisms $(x_1, y_1) \to (x_2, y_2)$ is the set of $u \in \pi(X)(x_1, x_2)$ such that $y_1 \cdot u = y_2$. This is precisely $\pi(Y)(y_1, y_2)$. □

Definition 2.10. For a space X, let $\pi_0(X)$ be the set of path components of X.

Lemma 2.11. *Let* $p\colon X \to Y$ *be a covering map.*

(1) *Suppose that* X *is path connected. The inclusion* $p^{-1}(x_0) \subset Y$ *induces a bijection* $p^{-1}(x_0)/\pi_1(X, x_0) \to \pi_0(Y)$. *In particular,*
$$Y \textit{ is path connected} \iff \pi_1(X, x_0) \textit{ acts transitively on the fibre } p^{-1}(x_0)$$

(2) *Suppose that* X *and* Y *are path connected. The maps*
$$\begin{array}{cc} \pi_1(Y, y_1)\backslash\pi(X)(x_1, x_2) \leftrightarrow p^{-1}(x_2) & \pi_1(Y, y_0)\backslash\pi_1(X, x_0) \leftrightarrow p^{-1}(x_0) \\ \pi_1(Y, y_1)u \to y_1 \cdot u & \pi_1(Y, y_0)u \to y_0 \cdot u \\ [pu_y] \leftarrow y & [pu_y] \leftarrow y \end{array}$$
are bijections. Here, u_y *is any path in* Y *from* y_1 *or* y_0 *to* y. *In particular,* $|\pi_1(X, x_0)\colon \pi_1(Y, y_0)| = |p^{-1}(x_0)|$.

Proof. The map $p^{-1}(x_0) \to \pi_0(Y)$, induced by the inclusion of the fibre into the total space, is onto because X is path connected so that any point in the total space is connected by a path to a point in the fibre. Two points in the fibre are in the same path component of Y if and only if are in the same $\pi_1(X, x_0)$-orbit.

If Y is path connected, then $\pi_1(X, x_0)$ acts transitively on the fibre $p^{-1}(x_0)$ with isotropy subgroup $\pi_1(Y, y_0)$ at y_0. □

Theorem 2.12 (Lifting Theorem)**.** *Let $p\colon Y \to X$ be a covering map and $f\colon B \to X$ a map into the base space. Choose base points such that $f(b_0) = x_0 = p(y_0)$ and consider the lifting problem*

$$\begin{array}{ccc} & & (Y, y_0) \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow p \\ (B, b_0) & \xrightarrow[f]{} & (X, x_0) \end{array}$$

(1) *If B is connected, then there exists at most one lift $\tilde{f}\colon (B, b_0) \to (Y, y_0)$ of f over p.*
(2) *If B is path connected and locally path connected then*

$$\textit{There is a map } \tilde{f}\colon (B, b_0) \to (Y, y_0) \textit{ such that } f = p\tilde{f} \iff f_*\pi_1(B, b_0) \subset p_*\pi_1(Y, y_0)$$

Proof. (1) Suppose that $\widetilde{f}_1$ and $\widetilde{f}_2$ are lifts of the same map $f\colon B \to X$. We claim that the sets $\{b \in B \mid \widetilde{f}_1(b) = \widetilde{f}_2(b)\}$ and $\{b \in B \mid \widetilde{f}_1(b) \neq \widetilde{f}_2(b)\}$ are open.

Let b be any point of B where the two lifts agree. Let $U \subset X$ be an evenly neighborhood of $f(b)$. Choose $\widetilde{U} \subset p^{-1}(U) = U \times F$ so that the restriction of p to $\widetilde{U}$ is a homoeomorphism and $\widetilde{f}_1(b) = \widetilde{f}_2(b)$ belongs to $\widetilde{U}$. Then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on the neighborhood $\widetilde{f}_1^{-1}(\widetilde{U}) \cap \widetilde{f}_2^{-1}(\widetilde{U})$ of b.

Let b be any point of B where the two lifts do not agree. Let $U \subset X$ be an evenly neighborhood of $f(b)$. Choose disjoint open sets $\widetilde{U}_1, \widetilde{U}_2 \subset p^{-1}(U) = U \times F$ so that the restrictions of p to $\widetilde{U}_1$ and $\widetilde{U}_2$ are homoemorphisms and $\widetilde{f}_1(b)$ belongs to $\widetilde{U}_1$ and $\widetilde{f}_2(b)$ to $\widetilde{U}_2$. Then $\widetilde{f}_1$ and $\widetilde{f}_2$ do not agree on the neighborhood $\widetilde{f}_1^{-1}(\widetilde{U}_1) \cap \widetilde{f}_2^{-1}(\widetilde{U}_2)$ of b.
(2) It is clear that if the lift exists, then the condition is satisfied. Conversely, suppose that the condition holds. For any point b in B, define a lift $\tilde{f}$ by

$$\widetilde{f}(b) = y_0 \cdot [fu_b]$$

where u_b is any path from b_0 to b. (Here we use that B is path connected.) If v_b is any other path from b_0 to b then $y_0 \cdot [fu_b] = y_0 \cdot [fv_b]$ because $y_0 \cdot [fu_b \cdot f\overline{v_b}] = y_0$ as the loop $[fu_b \cdot f\overline{v_b}] \in \pi_1(Y, y_0)$ fixes the point y_0 by Lemma 2.11.

We need to see that $\tilde{f}$ is continuous. Note that any point $b \in B$ has a path connected neighborhood that is mapped into an evenly covered neighborhood of $f(b)$ in X. It is evident what $\tilde{f}$ does on this neighborhood of b. □

A map $f\colon B \to S^1 \subset \mathbf{C} - \{0\}$ into the circle has an nth root if and only if the induced homomorphism $f_*\colon \pi_1(B) \to \mathbf{Z}$ is divisible by n.

3. The fundamental group of the circle, spheres, and lense spaces

For each $n \in \mathbf{Z}$, let ω_n be the loop $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns)$, $s \in I$, on the circle.

Theorem 3.1. *The map $\Phi\colon \mathbf{Z} \to \pi_1(S^1, 1)\colon n \to [\omega_n]$ is a group isomorphism.*

Proof. Let $p\colon \mathbf{R} \to S^1$ be the covering map $p(t) = (\cos(2\pi t), \sin(2\pi t))$, $t \in \mathbf{R}$. Remember that the total space $\mathbf{R}$ is simply connected as we saw in Example 1.9. The fibre over 1 is $p^{-1}(1) = \mathbf{Z}$. Let $u_n(t) = nt$ be the obvious path from 0 to $n \in \mathbf{Z}$. By Lemma 2.11 the map

$$\mathbf{Z} \to \pi_1(S^1, 1)\colon n \to [pu_n] = [\omega_n]$$

is bijective.

We need to verify that Φ is a group homomorphism. Let m and n be integers. Then $u_m \cdot (m + u_n)$ is a path from 0 to $m + n$ so it can be used instead of u_{m+n} when computing $\Phi(m + n)$. We find that

$$\Phi(m+n) = [p(u_m \cdot (m+u_n))] = [p(u_m) \cdot p(m+u_n)] = [p(u_m)][p(m+u_n)] = [p(u_m)][p(u_n)] = \Phi(m)\Phi(n)$$

because $p(m + u_n) = pu_n$ as p has period 1. □

Theorem 3.2. *The n-sphere S^n is simply connected when $n > 1$.*

Proof. Let N be the North and S the South Pole (or any other two distinct points on S^n). The problem is that there are paths in S^n that visit every point of S^n. But, in fact, any loop based at N is *homotopic* to a loop that avoids S (Problem and Solution). This means that $\pi_1(S^n - \{S\}, N) \to \pi_1(S^n, N)$ is surjective. The result follows as $S^n - \{S\}$ is homeomorphic to the simply connected space $\mathbf{R}^n$. □

Corollary 3.3. *The fundamental group of real projective n-space $\mathbf{R}P^n$ is $\pi_1(\mathbf{R}P^n) = C_2$ for $n > 1$. The fundamental group of the lense space $L^{2n+1}(m)$ is $\pi_1(L^{2n+1}(m)) = C_m$ for $n > 0$.*

Proof. We proceed as in Theorem 3.1. Consider the case of the the covering map $p\colon S^{2n+1} \to L^{2n+1}(m)$ over the lense space $L^{2n+1}(m)$. Let $N = (1, 0, \dots, 0) \in S^{2n+1} \subset \mathbf{C}^{n+1}$. The cyclic group $C_m = \langle \zeta \rangle$ of mth roots of unity is generated by $\zeta = e^{2\pi i/m}$. The map $\zeta^j \to \zeta^j N$, $j \in \mathbf{Z}$, is a bijection $C_m \to p^{-1}pN$ between the set C_m and the fibre over pN. As S^{2n+1} is simply connected there is a bijection

$$\Phi\colon p^{-1}pN = C_m \to \pi_1(L^{2n+1}(m), pN)\colon \zeta^j \to [p\omega_j]$$

where ω_j is the path in S^{2n+1} from N to $\zeta^j N$ given by $\omega_j(s) = (e^{2\pi i s j/m}, 0, \dots, 0)$. Since $\omega_{i+j} \simeq \omega_i \cdot (\zeta^i \omega_j)$, it follows just as in Theorem 3.1 that Φ is a group homomorphism.

For the projective spaces, use the paths $\omega_j(s) = (\cos(2\pi js), \sin(2\pi js), 0, \dots, 0)$ from N to $(-1)^j N$, to see that

$$\Phi\colon p^{-1}pN = C_2 \to \pi_1(\mathbf{R}P^n, pN)\colon (\pm 1)^j \to [p\omega_j]$$

is a bijection. □

3.4. **Applications of $\pi_1(S^1)$.** Here are some standard applications of Theorem 3.1.

Corollary 3.5. *The nth power homomorphism $p_n\colon (S^1, 1) \to (S^1, 1)\colon z \to z^n$ induces the nth power homomorphism $\pi_1(S^1, 1) \to \pi_1(S^1, 1)\colon [\omega] \to [\omega]^n$.*

Proof. $(p_n)_*\Phi(1) = (p_n)_*[\omega_1] = [p_n\omega_1] = [\omega_1^n] = [\omega_n] = \Phi(n) = \Phi(1)^n$. □

Theorem 3.6 (Brouwer's fixed point theorem)**.** (1) *The circle S^1 is not a retract of the disc D^2.*
(2) *Any map self-map of the disc D^2 has a fixed point.*

Proof. (1) Let $i\colon S^1 \to D^2$ be the inclusion map. The induced map $i_*\colon \mathbf{Z} = \pi_1(S^1) \to \pi_1(D^2) = 0$ is not injective so S^1 can not be a retract by 1.7.
(2) With the help of a fixed-point free self map of D^2 one can construct a retraction of D^2 onto S^1. But they don't exist. □

Theorem 3.7 (The fundamental theorem of algebra)**.** *Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a normed complex polynomial of degree n. If $n > 0$, then p has a root.*

Proof. Any normed polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is nonzero when $|z|$ is large: When $|z| > 1 + |a_{n-1}| + \cdots + |a_0|$, then $p(z) \neq 0$ because

$$|a_{n-1}z^{n-1} + \cdots + a_0| \le |a_{n-1}||z|^{n-1} + \cdots + |a_0| < |a_{n-1}||z|^{n-1} + \cdots + |a_0||z|^{n-1} = (|a_{n-1}| + \cdots + |a_0|)|z|^{n-1} < |z|^n$$

Therefore any normed polynomial $p(z)$ defines a map $S^1(R) \to \mathbf{C} - \{0\}$ where $S^1(R)$ is the circle of radius R and $R > 1 + |a_{n-1}| + \cdots + |a_0|$. In fact, all the normed polynomials $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$, $t \in I$, take $S^1(R)$ into $\mathbf{C} - \{0\}$ so that we have a homotopy

$$S^1(R) \times I \to \mathbf{C} - \{0\}\colon (z, t) \to z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$$

between $p_1(z) = p(z)|S^1(R)$ and $p_0(z) = z^n$.

If $p(z)$ has no roots at all, the map $p|S^1(R)$ factors through the complex plane $\mathbf{C}$ and is therefore nullhomotopic (as $\mathbf{C}$ is contractible) and so is the homotopic map $S^1(R) \to \mathbf{C} - \{0\}\colon z \to z^n$ and the composite map

$$S^1 \xrightarrow{z \to Rz} S^1(R) \xrightarrow{z \to z^n} \mathbf{C} - \{0\} \xrightarrow{z \to z/|z|} S^1$$

But this is simply the map $S^1 \to S^1\colon z \to z^n$ which we know induces multiplication by n (3.5). However, a nullhomotopic map induces multiplication by 0 (1.14). So $n = 0$. □

A map $f\colon S^1 \to S^1$ is *odd* if $f(-x) = -f(x)$ for all $x \in S^1$. Any rotation (or reflection) of the circle is odd (because it is linear).

Lemma 3.8. *Let $f\colon S^1 \to S^1$ be an odd map. Compose f with a rotation R so that $Rf(1) = 1$. The induced map $(Rf)_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$ is multiplication by an odd integer. In particular, f is not nullhomotopic.*

Proof. We must compute $(Rf)_*[\omega_1]$. The HLP gives a lift

$$\begin{array}{ccccc} \{0\} & & \xrightarrow{\quad 0 \quad} & & \mathbf{R} \\ \big\downarrow & & \overset{\widetilde{\omega}}{\nearrow} & & \big\downarrow \\ I & \xrightarrow[\omega_1]{} & S^1 & \xrightarrow[Rf]{} & S^1 \end{array}$$

and we have $(Rf)_*[\omega_1] = [p\widetilde{\omega}]$. When $0 \leq s \leq 1/2$, $\omega_1(s+1/2) = -\omega_1(s)$ and also $Rf\omega_1(s+1/2) = -Rf\omega_1(s)$ as Rf is odd. The lift, $\widetilde{\omega}$ of $Rf\omega_1$, then satisfies the equation

$$\widetilde{\omega}(s+1/2) = \widetilde{\omega}(s) + q/2$$

for some *odd* integer q. By continuity and connectedness of the interval $[0,1/2]$, q does not depend on s. Now $\widetilde{\omega}(1) = \widetilde{\omega}(1/2) + q/2 = \widetilde{\omega}(0) + q/2 + q/2 = q$ and therefore $(Rf)_*[\omega_1] = [p\widetilde{\omega}] = [\omega_q] = [\omega_1]^q$. We conclude that $(Rf)_*$ is multiplication by the odd integer q. Since a nullhomopotic map induces the trivial group homomorphism (1.14), f is not nullhomotopic. □

Theorem 3.9 (Borsuk–Ulam theorem for $n = 2$)**.** *Let $f\colon S^2 \to \mathbf{R}^2$ be any continuous map. Then there exists a point $x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose that $f\colon S^2 \to \mathbf{R}^2$ is a map such that $f(x) \neq f(-x)$ for all $x \in S^2$. The composite map

$$S^1 \xhookrightarrow{\quad \text{incl} \quad} S^2 \xrightarrow{x \to \frac{f(x)-f(-x)}{|f(x)-f(-x)|}} S^1$$

is odd so it is not nullhomotopic. But the first map $S^1 \hookrightarrow S^2$ is nullhomotopic because it factors through the contractible space $D^2_+ = \{(x_1,x_2,x_3) \in S^2 \mid x_3 \geq 0\}$. This is a contradiction. □

This implies that ther are no injective maps of $S^2 \to \mathbf{R}^2$; in particular S^2 does not embed in $\mathbf{R}^2$.

Proposition 3.10 (Borsuk–Ulam theorem for $n = 1$)**.** *Let $f\colon S^1 \to \mathbf{R}$ be any continuous map. Then there exists a point $x \in S^1$ such that $f(x) = f(-x)$.*

Proof. Look at the map $g(x) = f(x) - f(-x)$. If g is identically 0, $f(x) = f(-x)$ for all $x \in S^1$. Otherwise, g is an odd function, $g(-x) = -g(x)$, and g has both positive and negative values. By connectedness, g must assume the value 0 at some point. □

This implies that there are no injective maps $S^1 \to \mathbf{R}$; in particular S^1 does not embed in $\mathbf{R}$.

4. The van Kampen theorem

Let G_j, $j \in J$, be a set of groups indexed by the set J. The *coproduct* (or *free product*) of these groups is a group $\coprod_{i\in J} G_j$ with group homomorphisms $\varphi_j\colon G_j \to \coprod_{j\in J} G_j$ such that

$$\operatorname{Hom}(\coprod_{j\in J} G_j, H) = \prod_{j\in J} \operatorname{Hom}(G_j, H)\colon \varphi \to (\varphi\circ\varphi_j)_{j\in J} \tag{4.1}$$

is a bijection for any group H. The group $\coprod_{j\in J} G_j$ contains each group G_j as a subgroup and these subgroups do not commute with each other. If the groups have presentations $G_j = \langle L_j \mid R_j\rangle$ then $\coprod_{j\in J}\langle L_j \mid R_j\rangle = \langle \cup_{j\in J} L_j \mid \cup_{j\in J} R_j\rangle$ as this group has the universal property. See [9, 6.2] for the construction of the free product.

The characteristic property (4.1) applied to $H = \prod_{j\in J} G_j$ shows that there is a group homomorphism $\coprod G_j \to \prod G_j$ from the free product to the direct product whose restriction to each G_j is the inclusion into the product.

Example 4.2. [9, Example II–III p 171] $\mathbf{Z}/2 \amalg \mathbf{Z}/2 = \mathbf{Z} \rtimes \mathbf{Z}/2$ and $\mathbf{Z}/2 \amalg \mathbf{Z}/3 = \mathrm{PSL}(2,\mathbf{Z})$. We can prove the first assertion:

$$\mathbf{Z}/2 \amalg \mathbf{Z}/2 = \langle a,b \mid a^2,b^2\rangle = \langle a,b,c \mid a^2,b^2,c=ab\rangle = \langle a,b \mid a^2, acac, c\rangle = \langle a,b \mid a^2, c, aca = c^{-1}\rangle$$

but the second one is more difficult.

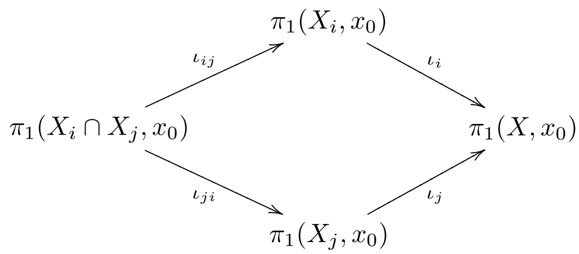
Suppose that the space $X = \bigcup_{j\in J} X_j$ is the union of open and path connected subspaces X_j and that x_0 is a point in $\bigcap_{j\in J} X_j$. The inclusion of the subspace X_j into X induces a group homomorphism $\iota_j\colon \pi_1(X_j, x_0) \to \pi_1(X, x_0)$. The coproduct $\coprod_{j\in J} \pi_1(X_j, x_0)$ is a group equipped with group homomorphisms $\varphi_j\colon \pi_1(X_j, x_0) \to \coprod_{j\in J} \pi_1(X_j, x_0)$. Let

$$\Phi\colon \amalg_{j\in J} \pi_1(X_j, x_0) \to \pi_1(\textstyle\bigcup_{j\in J} X_j, x_0) = \pi_1(X, x_0)$$

be the group homomorphism determined by $\Phi \circ \varphi_j = \iota_j$.

Is Φ surjective? In general, no. The circle, for instanec, is the union of two contractible open subspaces, so Φ is not onto in that case. But, if any loop in X is homotopic to a product of loops in one of the subspaces X_j, then Φ is surjective.

Is Φ injective? It will, in general, not be injective, because the individual groups $\pi_1(X_i)$ in the free product do not intersect but the subspaces do intersect. Any loop in X that is a loop in $X_i \cap X_j$ will in the free product count as a loop both in $\pi_1(X_i)$ and in $\pi_1(X_j)$. We always have commutative diagrams of the form



where ι_{ij} are inclusion maps. This means that $\Phi(\iota_{ij} g) = \Phi(\iota_{ji} g)$ for any $g \in \pi_1(X_i \cap X_j, x_0)$ so that

$$\forall i, j \in J \forall g \in \pi_1(X_i \cap X_j)\colon \iota_{ij}(g)\iota_{ji}(g)^{-1} \in \ker \Phi \tag{4.3}$$

Let $N \leq \coprod_{j\in J} \pi_1(X_j, x_0)$ be the smallest normal subgroup containing all the elements of (4.3). The kernel of Φ must contain N but, of course, the kernel could be bigger. The surprising fact is that often it isn't.

Theorem 4.4 (Van Kampen's theorem)**.** *Suppose that $X = \bigcup_{j\in J} X_j$ is the union of open and path connected subspaces X_j and that x_0 is a point in $\bigcap_{j\in J} X_j$.*

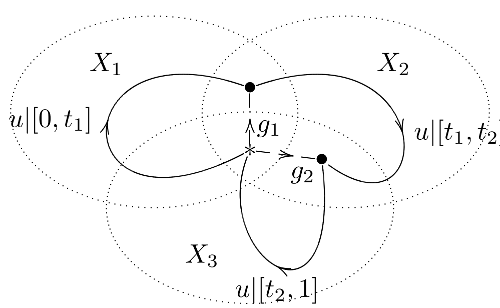
(1) *If the intersection of any two of the open subspaces is path connected then Φ is surjective.*
(2) *If the intersection of any three of the open subspaces is path connected then the kernel of Φ is N.*

Corollary 4.5. *If the intersection of any three of the open subspaces is path connected then Φ determines an isomorphism*

$$\overline{\Phi}\colon \coprod_{j\in J} \pi_1(X_j, x_0)/N \cong \pi_1(X, x_0)$$

Proof of Theorem 4.4. (1) We need to show that any loop $u \in \pi_1(X)$ in X is a product $u_1 \cdots u_m$ of loops $u_i \in \pi_1(X_{j_i})$ in one of the subspaces. Let $u\colon I \to X$ be a loop in X.

Thanks to the Lebesgue lemma (General Topology, 2.158) we can find a subdivision $0 = t_0 < t_1 < \cdots t_m = 1$ of the unit interval so that $u_i = u|[t_{i-1}, t_i]$ is a path in (say) X_i. As $u(t_i) \in X_i \cap X_{i+1}$, and also the base point $x_0 \in X_i \cap X_{i+1}$, and $X_i \cap X_{i+1}$ is path connected, there is path g_i in $X_i \cap X_{i+1}$ from the basepoint x_0 to $u(t_{i-1})$. The situation looks like this:



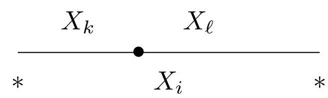
Now $u \simeq u|[0,t_1] \cdot u|[t_1,t_2] \cdots u|[t_{m-1},1] \simeq (u|[0,t_1] \cdot \overline{g_1}) \cdot (g_1 \cdot u|[t_1,t_2] \cdot \overline{g_2}) \cdots (g_m \cdot u|[t_{m-1},1])$ is a product of loops where each factor is inside one of the subspaces.

(2) Let $N \triangleleft \amalg\pi_1(X_i)$ be the smallest normal subgroup containing all the elements (4.3). Let $u_i \in \pi_1(X_{j_i})$. For simplicity, let's call X_{j_i} for X_i. Consider the product

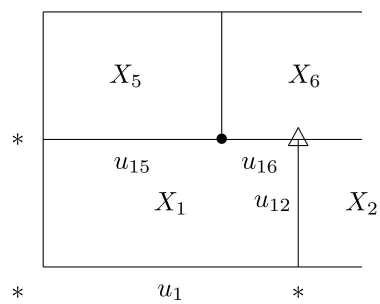
$$\underbrace{u_1}_{\pi_1(X_1)}\underbrace{u_2}_{\pi_1(X_2)}\cdots\underbrace{u_m}_{\pi_1(X_m)} \in \amalg_{j\in J}\pi_1(X_j)$$

and suppose that $\Phi(u_1\cdots u_m)$ is the unit element of $\pi_1(X)$. We want to show that $u_1\cdots u_m$ lies in the normal subgroup N or that $u_1\cdots u_m$ is the identity in the quotient group $\amalg\pi_1(X_j)/N$.

Since $u_1\cdots u_m$ is homotopic to the constant loop in X there is homotopy $I\times I \to X = \bigcup X_j$ from the loop $u_1\cdots u_m$ in X to the constant loop. Divide the unit square $I\times I$ into smaller rectangles such that each rectangle is mapped into one of the subspaces X_j. We may assume that the subdivision of $I\times\{0\}$ is a further subdivision of the subdivision at i/m coming from the product $u_1\cdots u_m$. It could be that one new vertex is (or more new vertices are) inserted between $(i-1)/m$ and i/m.



Connect the image of the new vertex $\bullet$ with a path g inside $X_i\cap X_k\cap X_\ell$ to the base point. Now u_i is homotopic in X_i to the product $(u_i|[(i-1)/m,\bullet]\cdot\overline{g})\cdot(g\cdot u_i|[\bullet,i/m])$ of two loops in X_i. This means that we may as well assume that no new subdivision points have been introduced at the bottom line $I\times\{0\}$. Now perturb slightly the small rectangles, but not the ones in the bottom and top row, so that also the corner of each rectangle lies in at most three rectangles. The lower left corner may look like this:



The loop u_1 in X_1 is homotopic to the product of paths $u_{15}u_{16}u_{12}$ by a homotopy as in the proof of 1.12. Connect the image of the point $\bullet$ to the base point by a path g_{156} inside $X_1\cap X_5\cap X_6$ and connect the image of the point $\triangle$ to the base point by a path g_{126} inside $X_1\cap X_2\cap X_6$. Then u_1 is homotopic in X_1 to the product of loops $(u_{15}\overline{g_{156}})\cdot(g_{156}u_{16}\overline{g_{126}})\cdot(g_{126}u_{12})$ in X_1. The first of these loops is a loop in $X_1\cap X_5$, the second is a loop in $X_1\cap X_6$, and the third is a loop in $X_1\cap X_2$. In $\amalg\pi_1(X_j)$ and modulo the normal subgroup N we have that

$$\underbrace{u_1}_{X_1}\underbrace{u_2}_{X_2}\cdots = \underbrace{u_{15}\overline{g_{156}}}_{X_1}\cdot\underbrace{g_{156}u_{16}\overline{g_{126}}}_{X_1}\cdot\underbrace{g_{126}u_{12}}_{X_1}\underbrace{u_2}_{X_2}\cdots = \underbrace{u_{15}\overline{g_{156}}}_{X_5}\cdot\underbrace{g_{156}u_{16}\overline{g_{126}}}_{X_6}\cdot\underbrace{g_{126}u_{12}\cdot u_2}_{X_2}\cdots$$

After finitely many steps we conclude that modulo N the product $u_1\cdots u_m$ equals a product of constant loops, the identity element. □

Corollary 4.6. *Let X_j be a set of path connected spaces. Then*

$$\coprod_{j\in J}\pi_1(X_j)\cong\pi_1(\bigvee_{j\in J}X_j)$$

provided that each base point $x_j\in X_j$ is the deformation retract of an open neighborhood $U_j\subset X_j$.

Proof. Van Kampen's theorem does not apply directly to the subspaces X_j of $\bigvee X_j$ because they are not open. Instead, let $X'_j = X_j\cup\bigvee_{i\in J}U_i$. The subspaces X'_j are open and path connected and the intersection of at least two of them is the contractible space $\bigvee_{i\in J}U_i$. Moreover, X_j is a deformation retract of X_j. □

For instance, punctured compact surfaces have free fundamental groups.

Corollary 4.7 (van Kampen with two subspaces)**.** *Suppose that $X = X_1 \cup X_2$ where X_1, X_2, and $X_1 \cap X_2 \neq \emptyset$ are open and path connected. Then*

$$\pi_1(X_1 \cup X_2, x_0) \cong \pi_1(X_1, x_0) \amalg_{\pi_1(X_1 \cap X_2, x_0)} \pi_1(X_2, x_0) \cong$$

for any basepoint $x_0 \in X_1 \cap X_2$.

This means that when $X_1 \cap X_2$ is path connected the fundamental group functor takes a push out of spaces to a push out, amalgamated product, of groups

$$\begin{array}{ccc} X_1 \cap X_2 & \xrightarrow{i_1} & X_2 \\ {\scriptstyle i_2}\downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array} \qquad \xrightarrow{\pi_1} \qquad \begin{array}{ccc} \pi_1(X_1 \cap X_2) & \xrightarrow{(i_1)_*} & \pi_1(X_2) \\ {\scriptstyle (i_2)_*}\downarrow & & \downarrow \\ \pi_1(X_2) & \longrightarrow & \pi_1(X) \end{array}$$

As a very special case, we see that a space, that is the union of two open simply connected subspaces with path connected intersection, is simply connected. This proves, again (Theorem 3.2), that S^n is simply connected when $n > 1$.

We can use this simple variant of van Kampen to analyze the effect on the fundamental group of attacing cells.

Corollary 4.8 (The fundamental group of a cellular extension)**.** *Let X be a path connected space. Then*

$$\pi_1(X \cup_{\amalg f_\alpha} \coprod D^n_\alpha) = \begin{cases} \pi_1(X)/\langle \gamma_\alpha f_\alpha \overline{\gamma_\alpha} \rangle & n = 2 \\ \pi_1(X) & n > 2 \end{cases}$$

where γ_α is a path from the base point of X to the image of the base point of $S^1_\alpha \subset D^2_\alpha$.

Proof. Let Y be X with the n-cells attached. Attach strips, fences connecting the base point of X with the base points of the attached cells, to Y and call the results Z. This does not change the fundamental group as Y is a deformation retract of Z (Corollary 1.7). Let A be Z with the top half of each cell removed and let $B = Z - X$. Then $Z = A \cup B$ and $A \cap B$ are path connected (the fences are there to make A and B path connected) so that

$$\pi_1(Z) = \pi_1(A) \amalg_{\pi_1(A \cap B)} \pi_1(B)$$

by the van Kampen theorem in the simple form of Corollary 4.7. Now B is contractible, hence simply connected (Corollary 1.14), so $\pi_1(Y) = \pi_1(Z)$ is the quotient of $\pi_1(A)$ by the smallest normal subgroup containing the image of $\pi_1(A \cap B) \to \pi_1(A)$. But $A \cap B$ is homotopy equivalent to a wedge $\bigvee_\alpha S^{n-1}_\alpha$ of $(n-1)$-spheres. In particular, $A \cap B$ is simply connected when $n > 2$ (Corollary 4.6, Theorem 3.2) so that $\pi_1(Y) = \pi_1(Z) = \pi_1(A) = \pi_1(X)$. When $n = 2$, $\pi_1(A \cap B)$ is a free group and the image of it in $\pi_1(A) = \pi_1(X)$ is generated by the path homotopy classes of the loops $\gamma_\alpha f_\alpha \overline{\gamma_\alpha}$. □

Corollary 4.9. *Let X be a CW-complex with skeleta X^k, $k \geq 0$. Then*

$$\pi_0(X^1) = \pi_0(X), \qquad \pi_1(X^2) = \pi_1(X)$$

Corollary 4.10. *The fundamental groups of the compact surfaces of positive genus g are*

$$\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] \rangle, \qquad \pi_1(N_g) = \langle a_1, \ldots, a_g \mid \prod a_i^2 \rangle,$$

The compact orientable surfaces M_g, $g \geq 0$, are distinct, $\pi_1(M_g)_{\mathrm{ab}} = \mathbf{Z}^{2g}$, and the compact nonorientable surfaces N_h, $h \geq 1$, are distinct, $\pi_1(N_g)_{\mathrm{ab}} = \mathbf{Z}^g \times \mathbf{Z}/2$.

Corollary 4.11. *Let M be a connected manifold of dimension ≥ 3. Then $\pi_1(M - \{x\}) = \pi_1(M)$ for any point $x \in M$.*

Proof. Apply van Kampen to $M = M - \{x\} \cup D^n$, $M - \{x\} \cap D^n \simeq S^{n-1}$ and remember that S^{n-1} is simply connected when $n \geq 3$. □

Which groups can be realized as fundamental groups of spaces? For instance, $C_\infty = S^1$ and $C_m = S^1 \cup_m D^2$ so that any finitely generated abelian group can be realized as the fundamental group of a product of these spaces.

Corollary 4.12. *For any group G there is a 2-dimensional CW-complex X_G such that $\pi_1(X_G) \cong G$.*

Proof. Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$ and let

$$X_G = D^0 \cup \coprod_{\{g_\alpha\}} D^1 \cup \coprod_{\{r_\beta\}} D^2$$

be the 2-dimensional CW-complex whose 1-skeleton is a wedge of circles, one for each generator, with 2-discs attached along the relations. □

Observe that $X_{H \amalg G} = X_H \vee X_G$. Also, $X_{\pi_1(M_g)} = M_g$, $X_{\pi_1(N_g)} = N_g$, $g \geq 1$.

4.1. **Fundamental groups of knot and link complements.** The complement of a pair of unlinked circles in $\mathbf{R}^3$ deformation retracts to $S^1 \vee S^1 \vee S^2 \vee S^2$ and a pair of linked circles to $(S^1 \times S^1) \vee S^2$. The fundamental groups are $\mathbf{Z} * \mathbf{Z}$ and $\mathbf{Z} \times \mathbf{Z}$, respectively. Thus the two complements are not homeomorphic.

Let m and n be relatively prime natural numbers and $K = K_{mn}$ the (m, n)-torus knot. We want to compute the knot group $\pi_1(\mathbf{R}^3 - K)$.

According to (4.11), $\pi_1(\mathbf{R}^3 - K) = \pi_1(S^3 - K)$. Now

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2$$

is the union of two solid tori intersecting in a torus $S^1 \times S^1$. Let K be embedded in this middle torus. Then

$$S^3 - K = (\partial D^2 \times D^2 - K) \cup (D^2 \times \partial D^2 - K), \qquad (\partial D^2 \times D^2 - K) \cap (D^2 \times \partial D^2 - K) = S^1 \times S^1 - K$$

and van Kampen says (if we ignore[1] the condition that the subsets should be open)

$$\pi_1(S^3 - K) = \frac{\pi_1(\partial D^2 \times D^2 - K) \amalg \pi_1(D^2 \times \partial D^2 - K)}{\pi_1(S^1 \times S^1 - K)}$$

Here, $\partial D^2 \times D^2 - K$ deformation retracts onto the core circle $\partial D^2 \times \{0\}$, and $S^1 \times S^1 - K$ (the torus minus the knot) is an annulus $S^1 \times (0, 1)$. (Take an open strip $[0, 1] \times (O, 1)$ and wrap it around the torus so that the end $0 \times (0, 1)$ meets the end $1 \times (0, 1)$). The image of the generator of this infinite cyclic group is the m power of a generator, respectively the nth power. Hence

$$\pi_1(S^3 - K) = \langle a, b \mid a^m = b^n \rangle = G_{mn}$$

It is now a matter of group theory to tell us that if $G_{m_1 n_1}$ and $G_{m_2 n_2}$ are isomorphic then $\{m_1, n_1\} = \{m_2, n_2\}$. In order to analyze this group, note that $a^m = b^n$ is in the center. Let C be the central group generated by this element. The quotient group

$$G_{mn}/C = \langle a, b \mid a^m, b^n \rangle = \mathbf{Z}/m \amalg \mathbf{Z}/n$$

has no center. (In general the free product $G \amalg H$ of two nontrivial groups has no center because the elements are words in elements from G alternating with elements from H.) Therefore C is precisely the center of G_{mn}. Thus we can recover mn as the order of the abelianization of $G/Z(G)$. Also, any element of finite order in $\mathbf{Z}/m \amalg \mathbf{Z}/n$ is conjugate to an element of $\mathbf{Z}/m$ or $\mathbf{Z}/n$. Thus we can recover the largest of m, n as the maximal order of a torsion element in $G/Z(G)$. Thus we can recover the set $\{m, n\}$.

Corollary 4.13. *There are infinitely many knots. (Here are some of them.)*

Another way of saying this is that $\partial D^2 \times D^2 - K$ deformation retracts onto the mapping cylinder of the degree m, respectively n, map $S^1 \to S^1$. Thus the union of these two spaces, $S^3 - K$, deformation retracts onto the union of the two mapping cylinders, which is the double mapping cylinder X_{mn} for the two maps.

Thus X_{mn} embeds in S^3 and $\mathbf{R}^3$ when $(m, n) = 1$. On the other hand X_{22} is the union of two Möbius bands. A Möbius band is $\mathbf{R}P^2$ minus an open 2-disc, so $X_{22} = \mathbf{R}P^2 \# \mathbf{R}P^2$, the Klein bottle, which does not embed in $\mathbf{R}^3$.

[1] To fix this, thicken the knot and enlarge the two solid tori a little so that they overlap.

5. Categories

A *category* $\mathcal{C}$ consists of [7]

- *Objects* $a, b, \dots$
- For each pair of objects a and b a set of *morphisms* $\mathcal{C}(a,b)$ with domain a and codomain b
- A composition function $\mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)$ that to each pair of morphisms g and f with $\mathrm{dom}(g) = \mathrm{cod}(f)$ associates a morphism $g \circ f$ with $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$

We require

Identity: For each object a the morphism set $\mathcal{C}(a,a)$ contains a morphism id_a such that $g \circ \mathrm{id}_a = g$ and $\mathrm{id}_a \circ f = f$ whenever these compositions are defined

Associativity : $h \circ (g \circ f) = (h \circ g) \circ f$ whenever these compositions are defined

A morphism $f \in \mathcal{C}(a,b)$ with domain a and codomain b is sometimes written $f\colon a \to b$. A morphism $f\colon a \to b$ is an *isomorphism* if there exists a morphism $g\colon b \to a$ such that the two possible compositions are the respective identities.

Definition 5.1. A group is a category with one object where all morphisms are isomorphisms. A groupoid is a category where all morphisms are isomorphisms.

Example 5.2. In the category **Top** of topological spaces, the objects are topological spaces, the morphisms are continuous maps, and composition is the usual composition of maps. In the category **hoTop**, the objects are topological spaces, the morphisms are homotopy classes of continuous maps, and composition is induced by the usual composition of maps. In the category **Grp** of groups, the objects are groups, the morphisms are groups homomorphisms, and composition is the usual composition of group homomorphisms. In the category $\mathbf{Mat}_R$ the objects are the natural numbers $\mathbf{Z}_+$, the set of morphisms $m \to n$ consists of all n by m matrices with entries in the commutative ring R, and composition is matrix multiplication. The fundamental groupoid $\pi(X)$ of a topological space X is a groupoid where the objects are the points of X and the morphisms $x \to y$ are the homotopy classes $\pi(X)(x,y)$ of paths from x to y, and composition is composition of path homotopy classes.

A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ associates to each object a of $\mathcal{C}$ an object $F(a)$ of $\mathcal{D}$ and to each morphism $f\colon a \to b$ in $\mathcal{C}$ a morphism $F(f)\colon F(a) \to F(b)$ in $\mathcal{D}$ such that $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A *natural transformation* $\tau\colon F \implies G\colon \mathcal{C} \to \mathcal{D}$ between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{D}$-morphism $\tau(a) \in \mathcal{D}(Fa, Ga)$ for each object a of $\mathcal{C}$ such that the diagrams

$$\begin{array}{c} a \\ {\scriptstyle f}\downarrow \\ b \end{array} \qquad \begin{array}{ccc} Fa & \xrightarrow{\tau(a)} & Ga \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fb & \xrightarrow{\tau(b)} & Gb \end{array}$$

commute for all morphisms $f \in \mathcal{C}(a,b)$ in $\mathcal{C}$. A natural transformation τ is a *natural isomorphism* if all the components $\tau(a)$, $a \in \mathrm{Ob}(\mathcal{C})$, are $\mathcal{D}$-isomorphisms.

Example 5.3. The fundamental group is a functor from the category of based topological spaces and based homotopy classes of maps to the category of groups.

The fundamental groupoid is a functor from the category of topological spaces to the category of groupoids. Any homotopy $h\colon f_0 \simeq f_1$ induces a natural isomorphism $h\colon \pi(f_0) \implies \pi(f_1)\colon \pi(X) \to \pi(Y)$ between functors between fundamental groupoids (Lemma 1.12).

Definition 5.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The functor category $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ is the category whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural transformations.

Definition 5.5. Two categories, $\mathcal{C}$ and $\mathcal{D}$, are *isomorphic* (*equivalent*) when there are functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ such that the composite functors are (naturally isomorphic to) the respective identity functors.

Lemma 5.6. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an* equivalence *of categories if and only if*

- *any object of $\mathcal{D}$ is isomorphic to an object of the form $F(a)$ for some object a of $\mathcal{C}$*

- *F is bijective on morphism sets: The maps $\mathcal{C}(a,b) \xrightarrow{f\to F(f)} \mathcal{D}(F(a),F(b))$ are bijections for all objects a and b of $\mathcal{C}$*

Proof. Suppose that $F\colon \mathcal{C}\to\mathcal{D}$ is an equivalence of categories. Then there is a functor G in the other direction and natural isomorphisms $\sigma\colon GF \Longrightarrow 1_{\mathcal{C}}$ and $\tau\colon FG \Longrightarrow 1_{\mathcal{D}}$. Let d be any object of $\mathcal{D}$. The isomorphism $\tau_d\colon FG(d) \xrightarrow{\cong} d$ shows that d is isomorphic to Fa for $a = Gd$. Let a,b be objects of $\mathcal{C}$. We note first that $\mathcal{C}(a,b)\to\mathcal{D}(Fa,Fb)\to\mathcal{C}(GFa,GFb)$ is injective for the commutative diagram

$$\begin{array}{ccccc} a & \quad & GFa & \xrightarrow[\cong]{\sigma_a} & a \\ {\scriptstyle f}\downarrow & & {\scriptstyle GFf}\downarrow & & \downarrow{\scriptstyle f} \\ b & & GFb & \xrightarrow[\cong]{\sigma_b} & b \end{array}$$

shows that $f = \sigma_b\circ GFf\circ\sigma_a^{-1}$ can be recovered from GFf. Thus $\mathcal{C}(a,b)\to\mathcal{D}(Fa,Fb)$ is injective. Symmetrically, also the functor G is injective on morphism sets. To show that F is surjective on morphism sets let g be any $\mathcal{D}$-morphism $Fa\to Fb$. Put $f = \sigma_b\circ Gg\circ\sigma_b^{-1}$. The commutative diagram

$$\begin{array}{ccccccccc} Fa & \quad & GFa & \xrightarrow[\cong]{\sigma_a} & a & \xleftarrow[\cong]{\sigma_a} & GFa & \quad & a \\ {\scriptstyle g}\downarrow & & {\scriptstyle Gg}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle GFf} & & \downarrow{\scriptstyle f} \\ Fb & & GFb & \xrightarrow[\cong]{\sigma_b} & b & \xleftarrow[\cong]{\sigma_b} & GFb & & b \end{array}$$

shows that $GFf = Gg$ and so $Ff = g$ since G is injective on morphism sets.

Conversely, suppose that $F\colon\mathcal{C}\to\mathcal{D}$ is a functor satisfying the two conditions. We must construct a functor G in the other direction and natural isomorphisms $\tau\colon FG\Longrightarrow 1_{\mathcal{D}}$ and $\sigma\colon GF\Longrightarrow 1_{\mathcal{C}}$. By the first condition, for every object $d\in\mathcal{D}$, we can find an object $Gd\in\mathcal{C}$ and an isomorphism $\tau_d\colon FGd\to d$. By the second condition, $\mathcal{C}(Gc,Gd)\cong\mathcal{D}(FGc,FGd)$ for any two objects c and d of $\mathcal{D}$. Here, $\mathcal{D}(c,d)\cong\mathcal{D}(FGc,FGd)$ because $FGc\cong c$ and $FGd\cong d$. Thus we have $\mathcal{D}(c,d)\cong\mathcal{D}(FGc,FGd)\cong\mathcal{C}(Gc,Gd)$. This means that for every $\mathcal{D}$-morphism $g\colon c\to d$ there is exactly one $\mathcal{C}$-morphism $Gg\colon Gc\to Gd$ such that

$$\begin{array}{ccc} FGc & \xrightarrow[\cong]{\tau_c} & c \\ {\scriptstyle FGg}\downarrow & & \downarrow{\scriptstyle g} \\ FGd & \xrightarrow[\cong]{\tau_d} & d \end{array}$$

commutes. Now G is a functor and τ a natural isomorphism $FG\Longrightarrow 1_{\mathcal{D}}$. What about GF? Well, for any object a of $\mathcal{C}$, $\mathcal{C}(GFa,a)\cong\mathcal{D}(FGFa,Fa)\ni\tau_{Fa}$ so there is a unique isomorphism $\sigma_a\colon GFa\to a$ such that $F\sigma_a = \tau_{Fa}$. This gives the natural isomorphism $\sigma\colon GF\Longrightarrow 1_{\mathcal{C}}$. □

It follows that when $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is an equivalence of categories then there are bijections

$$\mathcal{C}(c,Gd) = \mathcal{D}(Fc,d) \qquad\qquad \mathcal{C}(Gd,c) = \mathcal{C}(d,Fc)$$

of morphism sets.

Lemma 5.7. *If $\mathcal{C},\mathcal{C}'$ and $\mathcal{D},\mathcal{D}'$ are equivalent, then the functor categories* $\mathrm{Func}(\mathcal{C},\mathcal{D})$ *and* $\mathrm{Func}(\mathcal{C}',\mathcal{D}')$ *are equivalent.*

The *full subcategory* generated by some of the objects of $\mathcal{C}$ is the category whose objects are these objects and whose morphisms are all morphisms in $\mathcal{C}$.

Example 5.8. The category of finite sets is equivalent to the full subcategory generated by all sections $S_{<n} = \{x\in\mathbf{Z}_+ \mid x<n\}$, $n\in\mathbf{Z}_+$, of $\mathbf{Z}_+$. The category of finite dimensional real vector spaces is equivalent to the category $\mathbf{Mat}_{\mathbf{R}}$. If $f\colon X\to Y$ is a homeomorphism (homotopy equivalence) then the induced morphism $\pi(f)\colon\pi(X)\to\pi(Y)$ is an isomorphism (equivalence) of categories. The fundamental groupoid of a space is equivalent to the full subcategory generated by a point in each path component.

6. Categories of right G-sets

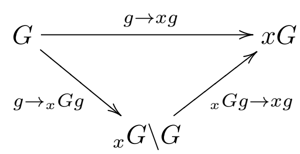
Let G be a topological group and F and Y topological spaces.

Definition 6.1. A *right* action of G on F is a continuous map $F \times G \to F\colon (x,g) \mapsto x \cdot g$, such that $x \cdot e = x$ and $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and all $x \in F$. A topological space equipped with a right G-action is called a right G-space. A continuous map $f\colon F_1 \to F_2$ between two right G-spaces is a G-map if $f(xg) = f(x)g$ for all $g \in G$ and $x \in F_1$.

Definition 6.2. A *left* action of G on Y is a continuous map $G \times Y \to Y\colon (g,y) \mapsto g \cdot y$, such that $e \cdot y = y$ and $(gh) \cdot y = g \cdot (h \cdot y)$ for all $g, h \in G$ and all $y \in Y$. A topological space equipped with a left G-action is called a left G-space. A continuous map $f\colon Y_1 \to Y_2$ between two left G-spaces is a G-map if $f(gy) = gf(y)$ for all $g \in G$ and $x \in Y_1$.

The orbit spaces (with the quotient topologies) are denoted $F/G = \{xG \mid x \in F\}$ for a right action $F \times G \to F$ and $G\backslash Y = \{Gy \mid y \in Y\}$ for a left action $G \times Y \to Y$.

The *orbit* through the point $x \in F$ for the right action $F \times G \to F$ is the sub-right G-space $xG = \{xg \mid g \in G\}$ obtained by hitting x with all elements of G; the *stabilizer* at x is the subgroup ${}_xG = \{g \in G \mid xg = x\}$ of G. The universal property of quotient spaces gives a commutative diagram



of right G-spaces and G-maps (General Topology, 2.81). Note that G-map ${}_xG\backslash G \to xG\colon {}_xGg \to xg$ is bijective. (In particular, the index of the stabilizer subgroup at x equals the cardinality of the orbit through x.) In many cases it is even a homeomorphism so that the orbit xG through x and the coset space ${}_xG\backslash G$ of the isotropy subgroup at x are homeomorphic.

Proposition 6.3 (G-orbits as coset spaces)**.** *Suppose that F is a right G-space and x a point of F. Then*

$$ {}_xG\backslash G \to xG \text{ is a homeomorphism} \iff G \xrightarrow{g \to xg} xG \text{ is a quotient map} $$

Proof. Use that the a bijective quotient map is a homeomorphism, the composition of two quotient maps is quotient, and if the composition of two maps is quotient than the last map is quotient (General Topology, 2.77). By definition, $G \to {}_xG\backslash G$ is quotient. □

By a right (or left) G-*set* we just mean a right (or left) G-space with the discrete topology. In the following we deal with G-sets rather than G-spaces.

Definition 6.4. G**Set** is the category of right G-sets and G-maps. The objects are right G-sets F and the morphisms $\varphi\colon F_1 \to F_2$ are G-maps (meaning that $\varphi(xg) = \varphi(x)g$ for all $x \in F_1$ and $g \in G$).

6.5. **Transitive right actions.** The right G-set F is *transitive* if F consists of a single orbit. If F is transitive then $F = xG$ for some (hence any) point $x \in F$ so that F and $H\backslash G$ are isomorphic G-sets where H is the stabilizer subgroup at the point x (Proposition 6.3). Thus any transitive right G-set is isomorphic to the G-set $H\backslash G$ of right H-cosets for some subgroup H of G.

Definition 6.6. The orbit category of G is the full subcategory $\mathcal{O}_G$ of G**Set** generated by all transitive right G-sets.

The orbit category $\mathcal{O}_G$ of G is equivalent to the full subcategory of G**Set** generated by all G-sets of the form $H\backslash G$ for subgroups H of G. What are the morphisms in the orbit category $\mathcal{O}_G$?

Definition 6.7. Let H_1 and H_2 be subgroups of G. The *transporter* is the set

$$ N_G(H_1, H_2) = \{n \in G \mid nH_1n^{-1} \subset H_2\} $$

of group elements conjugating H_1 into H_2.

The transporter set $N_G(H_1, H_2)$ is a left H_2-set. Let Let $H_2\backslash N_G(H_1, H_2)$ be the set of H_2-orbits.

Proposition 6.8. *There is a bijection*

$$\tau\colon H_2\backslash N_G(H_1,H_2) \to \mathcal{O}_G(H_1\backslash G, H_2\backslash G), \quad \tau(H_2n)(H_1g) = H_2ng$$

This map takes H_2n to left multiplication $H_1\backslash G \xrightarrow{H_1g\to H_2ng} H_2\backslash G$ by H_2n. In case $H_1 = H = H_2$, the map

$$\tau\colon H\backslash N_G(H) \to \mathcal{O}_G(H\backslash G, H\backslash G), \quad \tau(Hn)(Hg) = Hng$$

is a group isomorphism.

Proof. The inverse to τ is the map that takes a G-map $H_1\backslash G \xrightarrow{\varphi} H_2\backslash G$ to its value $\varphi(H_1) = H_2n$ at $H_1 \in H_1\backslash G$. Since $H_2n = \varphi(H_1) = \varphi(H_1H_1) = H_2nH_1$, the group element n conjugates H_1 into H_2. In case $H_1 = H = H_2$ and $n_1, n_2 \in N_G(H)$, we have

$$\tau(Hn_1)\tau(Hn_2)(H) = \tau(Hn_1)(Hn_2) = Hn_1n_2 = \tau(Hn_1n_2)(H)$$

so τ is group homomorphism in this case. □

In particular we see that

- all morphisms in $\mathcal{O}_G$ are epimorphisms
- all endomorphisms in $\mathcal{O}_G$ are automorphisms
- every object $H\backslash G$ of $\mathcal{O}_G$ is equipped with left and right actions

(6.9) $$H\backslash N_G(H) \times H\backslash G \times G = \mathcal{O}_G(H\backslash G, H\backslash G) \times H\backslash G \times G \to H\backslash G\colon Hn \cdot Hg \cdot m = Hngm$$

  where the left action are the G-automorphisms of $H\backslash G$ in $\mathcal{O}_G$.
- the maximal G-orbit is $G = \{e\}\backslash G$ and $\mathcal{O}_G(\{e\}\backslash G, H\backslash G) = H\backslash G$, the minimal G-orbit is $* = G\backslash G$ and $\mathcal{O}_G(H\backslash G, G\backslash G) = *$ ($G\backslash G = *$ is the final object of $\mathcal{O}_G$)

Remark 6.10 (Isomorphism classes of objects of $\mathcal{O}_G$)**.** The set of objects of $\mathcal{O}_G$ corresponds to the set of subgroups of G. The set of isomorphism classes of objects of $\mathcal{O}_G$ corresponds to the set of conjugacy classes of subgroups of G: Two objects $H_1\backslash G$ and $H_2\backslash G$ of the orbit category $\mathcal{O}_G$ are isomorphic if and only if H_1 and H_2 are conjugate: If there exist an inner automorphism that takes H_1 into H_2 and an inner automorphism that takes H_2 into H_1 such that the composite maps are the respective identity maps of $H_1\backslash G$ and $H_2\backslash G$, then these inner automorphisms must in fact give bijections between H_1 and H_2 as the factorizations $H_1 \xrightarrow{\mathrm{Inn}(n_1)} H_2 \xrightarrow{\mathrm{Inn}(n_2)} H_1 \xrightarrow{\mathrm{Inn}(n_1)} H_2$ of the respective identity maps imply that the inner automorphism $\mathrm{Inn}(n_1)$ is a bijection.

7. The classification theorem

In this section we shall see that covering maps are determined by their monodromy functor.

Definition 7.1. $\mathrm{Cov}(X)$ is the category of covering spaces over the space X. The objects are covering maps $Y \to X$ and the morphisms $\mathrm{Cov}(X)(p_1\colon Y_1 \to X, p_2\colon Y_2 \to X)$ are continuous maps $f\colon Y_1 \to Y_2$ over X (meaning that f preserves fibres or $p_1 = p_2f$).

How can we describe the category $\mathrm{Cov}(X)$? We are going to assume from now on that X is *path connected and locally path connected*.

Let $\mathrm{Func}(\pi(X), \mathbf{Set})$ be the category of functors from the fundamental groupoid $\pi(X)$ to the category **Set** of sets. There is a functor

$$\mathrm{Cov}(X) \to \mathrm{Func}(\pi(X), \mathbf{Set})$$

which takes a covering map $p\colon Y \to X$ to its monodromy functor $F(p)\colon \pi(X) \to \mathbf{Set}$ (2.8) and a covering map morphism to the induced natural transformation of functors. Conversely, does any such functor come from a covering space of X?

Suppose that $F\colon \pi(X) \to \mathbf{Set}$ is any functor. Let $Y(F) = \bigcup_{x\in X} F(x)$ be the union of the fibres and let $p(F)\colon Y(F) \to X$ be the obvious map taking $F(x)$ to x for any point $x \in X$.

Definition 7.2. A space X is semi-locally simply connected at the point $x \in X$ if any neighborhood of x contains a neighborhood U of x such that any loop at x in U is contractible in X. The space X is semi-locally simply connected if it is semi-locally simply connected at all its points.

All locally simply connected spaces are semi-locally simply connected.

Lemma 7.3. *Suppose that X is locally path connected and semi-locally simply connected. Then there is a topology on $Y(F)$ such that $p(F)\colon Y(F) \to X$ is a covering map. The monodromy functor of $p(F)\colon Y(F) \to X$ is F.*

Proof. Suppose that x is a point in X and $U \subset X$ an open path connected neighborhood of x such that any loop in U based at x is nullhomotopic in X. Observe that this implies that there is a unique path homotopy class u_z from x to any other point z in U so that

$$U \times F(x) \to p^{-1}(U)\colon (y, z) \to F(u_z)(y)$$

is a bijection.

For each $y \in F(x)$, let $(U, y) \subset Y$ be the image of $U \times \{y\}$ under the above bijection. By assumption, the topological space X has a basis of sets U as above. The sets (U, y) then form a basis for a topology on Y.

The covering map $Y(F) \to X$ determines a fibre functor (2.8) from the fundamental groupoid of X to the category of sets. By construction, this fibre functor is F. □

Definition 7.4. A covering map $p\colon Y \to X$ is universal if Y is simply connected.

According to the Lifting Theorem 2.12, any two universal covering spaces over X are isomorphic in the category $\mathrm{Cov}(X)$ of covering spaces over X. We may therefore speak about *the* universal covering space of X. Is there always a universal covering space of X?

By Corollary 2.9 the fundamental groupoid of $Y(F)$ has the set $Y(F)$ as object set and the morphisms are

$$\pi(Y(F))(y_1, y_2) = \{u \in \pi(X)(x_1, x_2) \mid F(u)y_1 = y_2\} \tag{7.5}$$

for all points $x_1, x_2 \in X$ and $y_1 \in F(x_1)$, $y_2 \in F(x_2)$. In particular, let x_0 be a base point in X. There is a right action $F(x_0) \times \pi_1(X, x_0) \to F(x_0)$ and

$$Y(F) \text{ is path connected} \iff \text{The right action of } \pi_1(X, x_0) \text{ on } F(x_0) \text{ is transitive}$$
$$Y(F) \text{ is simply connected} \iff \text{The right action of } \pi_1(X, x_0) \text{ on } F(x_0) \text{ is simply transitive}$$

We can always find a functor that satisfies the last condition in that

$$F = \pi(X)(x_0, -)\colon \pi(X) \to \mathbf{Set}$$

is a functor and the action of $\pi_1(X, x_0)$ on $F(x_0) = \pi_1(X, x_0)$ is simply transitive.

Corollary 7.6. *X admits a simply connected covering space if and only if X is semi-locally simply connected.*

Proof. The covering space $Y(F)$ of the functor $F = \pi(X)(x_0, -)$ is simply connected.

Conversely, suppose that $p\colon Y \to X$ is a covering map and $U \subset X$ and evenly covered open subspace then $U \to X$ factors through $Y \to X$. If $\pi_1(Y)$ is trivial then $\pi_1(U) \to \pi_1(X)$ is the trivial homomorphism. □

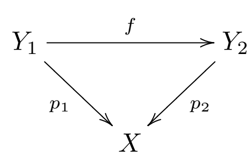
Example 7.7. The Hawaiian Earring $\bigcup_{n \in \mathbf{Z}_+} C_{1/n}$ and the infinite product $\prod S^1$ of circles are connected and locally path connected but not semi-locally simply connected. Thus they have no simply connected covering spaces. The infinite join $\bigvee S^1$ does have a simply connected covering space since it is a CW-complex. Indeed any CW-complex or manifold is locally contractible [5, Appendix], in particular locally simply connected.

Theorem 7.8 (Classification of Covering Maps)**.** *Suppose that X is semi-locally simply connected. The monodoromy functor and the functor $F \to Y(F)$*

$$\mathrm{Cov}(X) \rightleftarrows \mathrm{Func}(\pi(X), \mathbf{Set})$$

are category isomorphisms.

Proof. Let $p_1\colon Y_1 \to X$ and $p_2\colon E_Y \to X$ be covering maps over X with associated functors F_1 and F_2. A covering map



induces a natural transformation $\tau_f\colon F_1 \Longrightarrow F_2$ of functors given by $\tau_f(x) = f|p_1^{-1}x\colon p^{-1}x_1 \to p^{-1}x_2$. Conversely, any natural transformation $\tau\colon F_1 \Longrightarrow F_2$ induces a covering map $Y(f)\colon Y(F_1) \to Y(F_2)$ of the associated covering spaces. □

For example, let $F\colon \pi(X) \to \mathbf{Set}$ be any functor, let $x_0 \in X$ and $y_0 \in F(x_0)$. There is a natural transformation $\pi(X)(x_0, -) \Longrightarrow F$ whose x-component is $\pi(X)(x_0, x) \to F(x)\colon u \to F(u)y_y$ for any point x of X. This confirms that the universal covering space lies above them all.

Corollary 7.9. *The functor*

$$\mathrm{Cov}(X) \to \pi_1(X, x_0)\mathbf{Set}\colon (p\colon Y \to X) \to p^{-1}(x_0)$$

is an equivalence of categories.

Proof. The inclusion $\pi_1(X, x_0) \to \pi(X)$ of the the full subcategory of $\pi(X)$ generated by x_0 into $\pi(X)$ is an equivalence of categories. The induced functor $\mathrm{Func}(\pi(X), \mathbf{Set}) \to \mathrm{Func}(\pi_1(X, x_0), \mathbf{Set})$ is then also an equivalence. But $\mathrm{Func}(\pi_1(X, x_0), \mathbf{Set})$ is simply the category of right $\pi_1(X, x_0)$-sets. □

In particular, the full subcategory $\mathrm{Cov}_0(X)$ of connected covering spaces over X is equivalent to the category of transitive right $\pi_1(X, x_0)$-sets which again is equivalent to the orbit category $\mathcal{O}_{\pi_1(X,x_0)}$ (6.6). The set of covering space morphisms from the connected covering space $p_1\colon Y_1 \to X$ to the connected covering space $p_2\colon Y_2 \to X$ is

$$\begin{aligned}
\mathrm{Cov}(X)(p_1\colon Y_1 \to X, p_2\colon Y_2 \to X) &= \mathrm{Func}(\pi(X), \mathbf{Set})(F(p_1), F(p_2)) \\
&= \pi_1(X)\mathbf{Set}(p_1^{-1}(x_0), p_2^{-1}(x_0)) \\
&= \mathcal{O}_{\pi_1(X)}(\pi_1(Y_1)\backslash\pi_1(X), \pi_1(Y_2)\backslash\pi_1(X)) \\
&= \pi_1(Y_2)\backslash N_{\pi_1(X)}(\pi_1(Y_1), \pi_2(Y_2))
\end{aligned}$$

and, in particular,

$$\mathrm{Cov}(X)(p\colon Y \to X, p\colon Y \to X) = \pi_1(Y)\backslash N_{\pi_1(X)}(\pi_1(Y))$$

for any connected covering space $p\colon Y \to X$ over X. If we map out of the universal covering space $X\langle 1\rangle \to X$ this gives

$$\mathrm{Cov}(X)(X\langle 1\rangle \to X, Y \to X) = \pi_1(Y)\backslash\pi_1(X) \qquad \mathrm{Cov}(X)(X\langle 1\rangle \to X, X\langle 1\rangle \to X) = \pi_1(X)$$

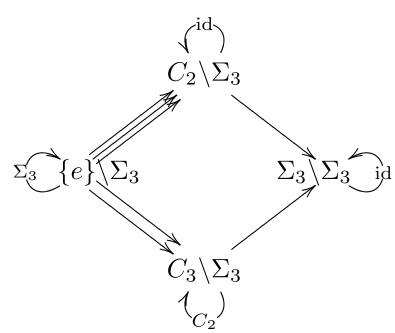
which means that the universal covering space admits a left covering space $\pi_1(X)$-action with orbit space $\pi_1(X)\backslash X\langle 1\rangle \to X = X$.

Corollary 7.10. *Let $G = \pi_1(X)$ for short. The functor*

$$\mathcal{O}_G \to \mathrm{Cov}_0(X)\colon H \to (H\backslash X\langle 1\rangle \to G\backslash X\langle 1\rangle)$$

is an equivalence of categories.

Is this



a picture of the orbit category of symmetric group Σ_3 or is it a picture of the path connected covering spaces over a path connected, locally path connected, and semi-locally simply connected space with fundamental group Σ_3? Both! The space could be X_{Σ_3} from Corollary 4.12; see Example 7.20 for more information.

Here are some examples to illustrate the Classification of Covering Spaces.

Covering spaces of the circle: The category $\mathrm{Cov}_0(S^1) = \mathcal{O}_{C_\infty}$ of path connected covering spaces of the circle $S^1 = \mathbf{Z}\backslash\mathbf{R}$ consists of the covering spaces $n\mathbf{Z}\backslash\mathbf{R} \to \mathbf{Z}\backslash\mathbf{R}$ where $n = 0, 1, 2, \ldots$. There is a covering map $n\mathbf{Z}\backslash\mathbf{R} \to m\mathbf{Z}\backslash\mathbf{R}$ if and only if $m|n$ and in that case there are m such covering maps, namely the maps

$$\begin{array}{ccc} S^1 & \xrightarrow{\zeta z^{m/n}} & S^1 \\ & {\scriptstyle z^n}\searrow \quad \swarrow{\scriptstyle z^m} & \\ & S^1 & \end{array}$$

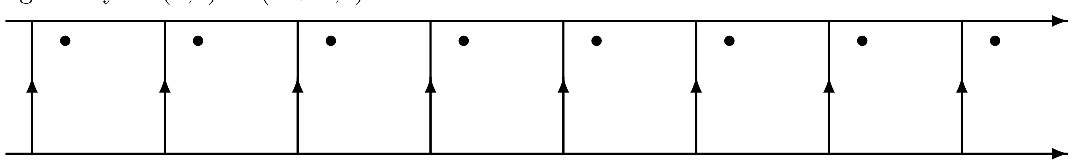
where ζ is any mth root of unity.

Covering spaces of projective spaces: The category $\mathrm{Cov}_0(\mathbf{R}P^n) = \mathcal{O}_{C_2}$ of connected covering spaces of real projective n-space $\mathbf{R}P^n$, $n \geq 2$, has 2 objects, namely the trivial covering map $\mathbf{R}P^n \to \mathbf{R}P^n$ and the universal covering map $S^n \to \mathbf{R}P^n$.

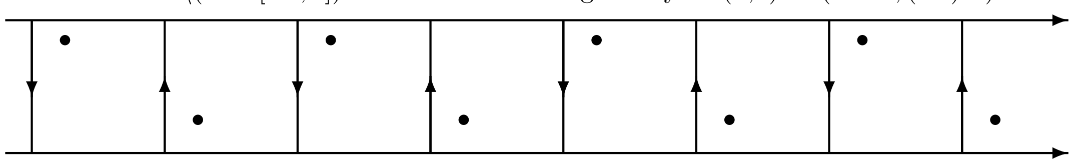
Covering spaces of lense spaces: The universal covering space of the lense space $L^{2n+1}(m) = C_m\backslash S^{2n+1}$, $n \geq 1$, is S^{2n+1}. The other covering spaces are the lense spaces $L^{2n+1}(r) = C_r\backslash S^{2n+1}$ for eah divisor r of m. The category of connected covering spaces of $L^{2n+1}(m)$ is equivalent to the orbit category $\mathcal{O}_{C_m}$.

Covering spaces of surfaces: The category $\mathrm{Cov}_0(M_g) = \mathcal{O}_{\pi_1(M_g)}$ is harder to describe explicitly. Any finite sheeted covering space of a compact surface is again a compact surface. The paper [8] contains information about covering spaces of closed surfaces.

Example 7.11 (Covering spaces of the Möbius band)**.** The cylinder $S^1 \times [-1, 1] = \mathbf{Z}\backslash(\mathbf{R} \times [-1, 1])$ where the action is given by $n \cdot (x, t) \to (x + n, t)$



The Möbius band $\mathrm{MB} = \mathbf{Z}\backslash(\mathbf{R} \times [-1, 1])$ where the action is given by $n \cdot (x, t) \to (x + n, (-1)^n t)$



Every even-sheeted covering space of the Möbius band is a cylinder, every odd-sheeted covering space is a Möbuis band.

Example 7.12 (Covering spaces of $S^1 \cup_m (S^1 \times I)$)**.** Let $X_m = S^1 \cup_m (S^1 \times I)$ be the mapping cylinder of the degree m map of the circle. We can construct X_m in the following way: Take a (codomain) circle of circumference $1/m$ and a square $[0, 1] \times [0, 1]$. Wrap the bottom edge $[0, 1] \times \{0\}$ of the square m times around the circle in a screw motion so that each time the square travels once around the circle it is also being rotated an angle of $2\pi/m$. Finally, glue the two ends, $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$, of the square together. There is a picture of X_m in [5, Example 1.29]. The codomain circle is the core circle and the domain circle is the boundary circle. The fundamental group $\pi_1(X_m)$ is $\mathbf{Z}$ since X_m deformation retracts onto the codomain (core) circle so that the inclusion $S^1 \xrightarrow{i_1} X_m \supset S^1$ is a homotopy equivalence. The inclusion $S^1 \xrightarrow{i_0} X_m \supset S^1$ of the domain (boundary) circle induces multiplication by m on the fundamental groups; this is simply because of the general mapping cylinder diagram which becomes

$$\begin{array}{ccc} S^1 & \xrightarrow{i_0} & (S^1 \times I) \cup_m S^1 = X_m \\ & {\scriptstyle m}\searrow & {\scriptstyle i_1}\uparrow \simeq \downarrow{\scriptstyle \mathrm{DR}} \\ & & S^1 \end{array}$$

in this special case. It may help to envision the boundary circle in X_m sliding towards the core circle.

The universal covering space of X_m is $X_m\langle 1\rangle = C\mathbf{Z}/m \times \mathbf{R}$ where $C\mathbf{Z}/m = (\mathbf{Z}/m \times I)/(\mathbf{Z}/m \times \{1\})$ is the cone on the set $\mathbf{Z}/m$ with m points. ($C\mathbf{Z}/m$ is a starfish with m arms). We may realize $C\mathbf{Z}/m \times \mathbf{R}$ in

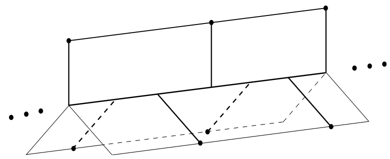


FIGURE 2. The universal covering space of X_m

$\mathbf{R}^3$ with $C\mathbf{Z}/m$ placed horizontally in the XY-plane and $\mathbf{R}$ as the vertical Z-axis. The covering space action of the unit $1 \in \mathbf{Z}$ on $C\mathbf{Z}/m \times \mathbf{R}$ is then the screw motion $([[a]_m, t], x) \to ([[a+1]_m, t], x + 1/m)$ with matrix

$$\begin{pmatrix} \cos(2\pi/m) & -\sin(2\pi/m) & 0 \\ \sin(2\pi/m) & \cos(2\pi/m) & 0 \\ 0 & 0 & 1/m \end{pmatrix}$$

that rotates $C\mathbf{Z}/m$ counterclockwise $1/m$th of a full rotation and moves up along the Z-axis $1/m$th of a unit. (In Figure 7.12 the $\mathbf{R}$-axis isn't exactly vertical since that would take up too much space. The covering space action takes the indicated lines, situated at distance $1/m$, to each other.) What is the lift of the domain and the codomain circles of X_m to the universal covering space $X_m\langle 1\rangle$? (One of them will lift to a loop.)

Since $m \in \mathbf{Z}$ acts trivially on $C\mathbf{Z}/m$ there is an m-sheeted covering map

$$C\mathbf{Z}/m \times S^1 = C\mathbf{Z}/m \times m\mathbf{Z}\backslash\mathbf{R} = m\mathbf{Z}\backslash(C\mathbf{Z}/m \times \mathbf{R}) \to \mathbf{Z}\backslash(C\mathbf{Z}/m \times \mathbf{R}) = X_m$$

with $m\mathbf{Z}\backslash\mathbf{Z}$ as deck transformation group. What is the lift of the domain and the codomain circles to this m-fold covering space?

Let $X = X_1 \cup X_2$ be a CW-complex that is the union of two connected subcomplexes X_1 and X_2 with connected intersection $X_1 \cap X_2$. According to van Kampen, the fundamental group $G = \pi_1(X) = G_1 \amalg_A G_2$ is the free product of $G_1 = \pi_1(X_1)$ and $G_2 = \pi_1(X_2)$ with $A = \pi_1(X_1 \cap X_2)$ amalgamated. We will assume that the homomorphisms $G_1 \leftarrow A \to G_2$ are injective. Then also the homomorphisms $G_1 \to G \leftarrow G_2$ of the push-out diagram

$$\begin{array}{ccc} A & \longrightarrow & G_2 \\ \downarrow & & \downarrow \\ G_1 & \longrightarrow & G \end{array}$$

are injective according to the Normal Form Theorem for Free Products with Amalgamation [6, Thm 2.6].

Let $X\langle 1\rangle$ be the universal covering space of $X = G\backslash X\langle 1\rangle$ and let $p\colon X\langle 1\rangle \to X$ be the covering projection map. The spaces $p^{-1}(X_1)$ and $p^{-1}(X_2)$ are left G-spaces with intersection $p^{-1}(X_1) \cap p^{-1}(X_1) = p^{-1}(X_1 \cap X_2)$. Let $y_0 \in p^{-1}(X_1 \cap X_2)$ be a base point. The commutative diagram [3, II.7.5]

$$\begin{array}{ccc} \pi_1(p^{-1}X_1, y_0) & \longrightarrow & \pi_1(p^{-1}X, y_0) = \{1\} \\ \downarrow & & \downarrow \\ \pi_1(X_1, p(y_0)) & \hookrightarrow & \pi_1(X, p(y_0)) \end{array}$$

tells us that the component of $p^{-1}(X_1)$ containing y_0 is simply connected so it is the universal covering space $X_1\langle 1\rangle$ of $X_1 = G_1\backslash X_1\langle 1\rangle$. We see from this that there is a homeomorphism of left G-spaces

$$G \times_{G_1} X_1\langle 1\rangle \xrightarrow{\cong} p^{-1}(X_1)$$

induced by the map $G \times X_1\langle 1\rangle \to p^{-1}(X_1)$ sending (g, y) to gy. Similar arguments apply to $p^{-1}(X_2)$ and $p^{-1}(X_1 \cap X_2)$, of course, and hence

$$X\langle 1\rangle = G \times_{G_1} X_1\langle 1\rangle \cup_{G \times_A (X_1 \cap X_2)\langle 1\rangle} G \times_{G_2} X_2\langle 1\rangle$$

is the union of the two G-spaces $G \times_{G_i} X_i\langle 1\rangle$, $i = 1, 2$. This means that the universal covering space of X is the union of the G-translates of the universal covering spaces of X_1 and X_2 joined along G-translates of the universal covering space of $X_1 \cap X_2$. The next example demonstrates this principle.

Example 7.13. [5, 1.24, 1.29, 1.35, 1.44, 3.45] Let $X_{mn} = X_m \cup_{S^1} X_n$ be the double mapping cylinder for the degree m map and the degree n map on the circle. X_{mn} is the union of the two mapping cylinders with their domain (boundary) circles identified, $X_m \cap X_n = S^1$. By van Kampen, the fundamental group has a presentation

$$\pi_1(X_{mn}) = \pi_1(X_m) \amalg_{\pi_1(S^1)} \pi_1(X_n) = \langle a, b \mid a^m = b^n \rangle = G_{mn}$$

with two generators and one relation. We shall now try to build its universal covering space.

We may equip X_{mn} with the structure of a 2-dimensional CW-complex. The 1-skeleton of X_{mn} consists of two circles, a and b, joined by an interval, c, and $X_{mn} = X^1_{mn} \cup_{a^m c \bar{b}^n \bar{c}} D^2$ is obtained by attaching a 2-cell

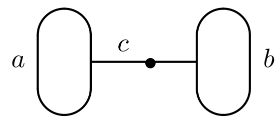


FIGURE 3. 1-skeleton X^1_{mn} of X_{mn}

along the loop $a^m c \bar{b}^n \bar{c}$. (If we use the corollary to van Kampen [5, 1.26] instead of the van Kampen theorem itself we get that $\pi_1(X_{mn}) = \langle a, cb\bar{c} \mid a^m (cb\bar{c})^{-n} \rangle$.)

The universal covering space $X_{mn}\langle 1 \rangle$ is also a 2-dimensional CW-complex. The inverse image in $X_m\langle 1 \rangle$ of the left half of the 1-skeleton is the vertical line $\mathbf{R}$ with spiraling 'rungs' attached $1/m$th of a unit apart. Rungs with vertical distance 1 point in the same direction so they can be joined up with the inverse image in $X_n\langle 1 \rangle$ of the right half of the 1-skeleton. Now fill in 2-cells in each of the rectangles with sides a^m, c,

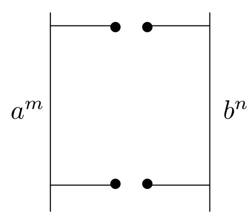


FIGURE 4. Part of 1-skeleton of $X_{mn}\langle 1 \rangle$

b^n and c. Continue this process. There will be similar rectangles shifted up $1/m$th unit along the left axis and rotated $2\pi/m$ or up $1/n$th unit along the right axis and rotated $2\pi/n$. The 2-dimensional CW-complex $X_{mn}\langle 1 \rangle$ built in this way is the universal covering space; it is the product $T_{mn} \times \mathbf{R}$ of a tree T_{mn} and the real line, hence contractible [10, Chp 3, Sec 7, Lemma 1]. The element $a \in G_{mn}$ acts by skew motion around one of the vertical lines in $X_m\langle 1 \rangle$ and $b \in G_{mn}$ acts by skew motion around one of the vertical lines in $X_n\langle 1 \rangle$. Note that $a^m = b^n$ acts by translating one unit up. What is the lift of $X_m \cap X_n$ (the circle parallel to circle a but passing through the point $\bullet$ of the 1-skeleton) to the universal covering space?

What is the universal abelian covering space $G'_{mn} \backslash X_{mn}\langle 1 \rangle$ of X_{mn}? Its deck transformation group is

$$G'_{mn} \backslash G_{mn} = (G_{mn})_{\mathrm{ab}} = \langle a, b | a^m = b^n, ab = ba \rangle = \mathbf{Z} \times \mathbf{Z}/d$$

where $d = (m, n)$ is the greatest common divisor. What is the mn fold covering space with fundamental group equal to the normal closure N of $\langle a^m, aba^{-1}b^{-1} \rangle$ and deck transformation group $N \backslash G = \langle a, b \mid a^m = b^n, a^m, ab = ba \rangle = \langle a, b \mid a^m, b^n, ab = ba \rangle = \mathbf{Z}/m \times \mathbf{Z}/n$? What is the lift of $X_m \cap X_n$ to this covering space?

7.14. **Cayley tables, Cayley graphs, and Cayley complexes.** [6, III.4] [3] For any group presentation $G = \langle g_\alpha \mid r_\beta \rangle$ there exists (Corollary 4.12) a 2-dimensional CW-complex

$$X_{G\backslash G} = D^0 \cup \coprod_{\{g_\alpha\}} D^1 \cup \coprod_{\{r_\beta\}} D^2 = (G\backslash G \times D^0) \cup \coprod_{\{g_\alpha\}} (G\backslash G \times D^1) \cup \coprod_{\{r_\beta\}} (G\backslash G \times D^2)$$

with fundamental group $\pi_1(X_{G\backslash G}) = \langle g_\alpha \mid r_\beta \rangle = G$. This is the most simple space with fundamental group G so it is natural to apply Theorem 7.8 to $X_{G\backslash G}$. So what are the connected covering spaces of X_G? There is an equivalence of categories

$$X_? \colon \mathcal{O}_G \to \mathrm{Cov}_0(X_{G\backslash G}) \colon H\backslash G \to (X_{H\backslash G} \to X_{G\backslash G})$$

and the *Cayley complex* of $H\backslash G$ is the 2-dimensional CW-complex $X_{H\backslash G}$ while the *Cayley graph* is its 1-skeleton. We now define these CW-complexes more explicitly for any object of $\mathcal{O}_G$ (or for any right G-space for that matter) relative to the given presentation of G.

The 0-skeleton of $X_{H\backslash G}$ is the right G-set $X^0_{H\backslash G} = H\backslash G$; this is the fibre of the covering map $X_{H\backslash G} \to X_{H\backslash G}$ as a right G-space. The 1-skeleton of $X_{H\backslash G}$ is the *Cayley graph* for $H\backslash G$, the 1-dimensional $H\backslash N_G(H)$-CW-complex

$$X^1_{H\backslash G} = (H\backslash G \times D^0) \cup \coprod_{\{Hg \to Hgg_\alpha\}} (H\backslash G \times D^1)$$

obtained from the 0-skeleton $H\backslash G$ by attaching to each right coset $Hg \in H\backslash G$ an arrow from Hg to Hgg_α for each generator g_α; note that we have no other choice since the loop g_α in the base space lifts to a path in the total space that goes from Hg in the fibre $H\backslash G$ to Hgg_α in the fibre. (The Cayley graph is simply a graphical presentation of the Cayles table for Cayley table for group multiplication $H\backslash G \times G \to H\backslash G$.) In this way, the Cayley table for $H\backslash G$ is a $|G\colon H|$-fold covering space of the 1-skeleton $\bigvee_{\{g_\alpha\}} S^1$ of X_G. The Cayley graph is connected since each group element g is a product of the generators which means that there is a sequence of arrows connecting the 0-cells He and Hg.

Next attach 2-cells at each $Hg \in H\backslash G$ along the loop r_β for each relation r_β. Since the relation r_β is a factorization of the neutral element e in terms of the g_α, it defines loops $Hg \to Hgr_\beta = Hg$ based at each 0-cell Hg in the Cayley graph $X^1_{H\backslash G}$. The resulting left $H\backslash N_G(H)$-CW-complex

$$X_{H\backslash G} = (H\backslash G \times D^0) \cup \coprod_{\{Hg \to Hgg_\alpha\}} (H\backslash G \times D^1) \cup \coprod_{\{Hg \xrightarrow[r_\beta]{} Hg\}} (H\backslash G \times D^2)$$

is the *Cayley complex* of $H\backslash G$. The Cayley complex is still connected for attaching 2-cells does not alter the set of path components (Corollary 4.9). Clearly, every G-map $H_1\backslash G \to H_2\backslash G$ extends to a covering map $X_{H_1\backslash G} to X_{H_2\backslash G}$.

In particular, taking $H = \{e\}$ to be the trivial group, the Cayley complex for the right G-set $\{e\}\backslash G = G$,

$$X_{\{e\}\backslash G} = (G \times D^0) \cup \coprod_{\{g_\alpha\}} (G \times D^1) \cup \coprod_{\{r_\beta\}} (G \times D^2)$$

is a 2-dimensional left G-CW-complex, the universal covering space of $X_{G\backslash G}$. The 0-skeleton is G, at each $g \in G$ there is an arrow from g to gg_α for each generator g_α and a 2-cell attached by the loop $g \xrightarrow{r_\beta} gr_\beta = g$. In other words, there is one 0-G-cell $G \times D^0$, one G-1-cell $G \times D^1$ for each generator g_α, attached by the left G-map that takes $\{e\} \times \partial D^1 = \{e\} \times \{0,1\}$ to e and g_α, and one G-2-cell $G \times D^2$ for each relation r_β attached by the left G-map that on $\{e\} \times \partial D^2$ is the loop r_β at e. The orbit space under the left action of $H < G$ on $X_G(\{e\}\backslash G)$ is the Cayley complex for the orbit space $H\backslash G$: $H\backslash X_{\{e\}\backslash G} = X_{H\backslash G}$. In particular,

$$G\backslash X_{\{e\}\backslash G} = X_{G\backslash G} = D^0 \cup \coprod_{\{g_\alpha\}} D^1 \cup \coprod_{\{r_\beta\}} D^2 = X_G$$

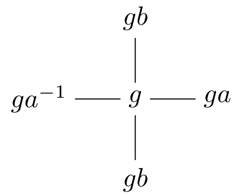
is a point $\{Ge\}$ with an arrow $Ge \xrightarrow{g_\alpha} Ge$ for each generator g_α and with one 2-cell attached along the loop $Ge \xrightarrow{r_\beta} Ge$ for each relation r_β.

It is very instructive to do a few examples. See [4] for information about graph theory.

Example 7.15 (Cayley complexes for cyclic groups)**.** For the infinite cyclic group $G = C_\infty = \langle a \rangle$, $X_{\{e\}\backslash G} = \mathbf{Z} \cup (\mathbf{Z} \times D^1) = \mathbf{R}$ and $X_{G\backslash G} = G\backslash \mathbf{R} = S^1$. For the cyclic group $G = C_2 = \langle a \mid a^2 \rangle$ of order 2, $X_{\{e\}\backslash G} = (C_2 \times D^0) \cup (C_2 \times D^1) \cup (C_2 \times D^2) = S^2$ and $X_{G\backslash G} = G\backslash S^2 = \mathbf{R}P^2$. For the cyclic group $G = C_m = \langle a \mid a^m \rangle$ of order m, $X_{\{e\}\backslash G}$ is a circle with $C_m \times D^2$ attached and and $X_{G\backslash G} = G\backslash X_{\{e\}\backslash G}$ is the mapping cone for $S^1 \xrightarrow{m} S^1$.

Example 7.16 (Cayley graphs for F_2-sets)**.** Let $G = \langle a, b \rangle = \mathbf{Z} \amalg \mathbf{Z}$ be a free group F_2 on two generators. Then $X_{G\backslash G} = S^1 \vee S^1$ and $\mathcal{O}_G = \mathrm{Cov}_0(S^1 \vee S^1)$. Since there are no relations, Cayley complexes for right

G-sets are Cayley graphs. In particular, $X_{\{e\}\backslash G} = (G \times D^0) \cup (G \times D^1 \coprod G \times D^1)$ is the G-graph



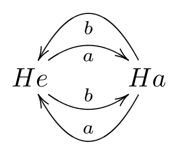
with vertex set G and two edges from g to ga and gb for every vertex $g \in G$, and $X_{G\backslash G} = G\backslash X_{\{e\}\backslash G}$ is the graph, $S^1 \vee S^1$, with one vertex $G\backslash G$ and two edges. In general, for any subgroup H of G, the Cayley graph, $X_{H\backslash G}$, for $H\backslash G$ is the covering space of $X_{G\backslash G} = S^1 \vee S^1$ characterized by any of these three properties:

- $X_{H\backslash G}$ is the Cayley table for $H\backslash G \times G \to H\backslash G$ relative to the generators a and b
- the fibre of $X_{H\backslash G} \to X_{G\backslash G}$ is the right G-set $H\backslash G$
- the image of the monomorphism $\pi_1(X_{H\backslash G}) \to \pi_1(X_{G\backslash G}) = G$ is conjugate to H

Here are some examples:

- If $H = \langle a^2, ab, b^2 \rangle$ then the Cayley table and the Cayley graph of the G-set $H\backslash G = \{He, Ha\}$ are

	a	b
He	Ha	Ha
Ha	He	He



  because $He \xrightarrow{a} Ha$, $He \xrightarrow{b} Hb = Habb^{-2}b = Ha$, $Ha \xrightarrow{a} Haa = He$, and $Ha \xrightarrow{b} Hab = He$. The subgroup H is normal since it has index two. Note that H is free of rank 3.
- If $H = [G, G]$ is the commutator subgroup of G then the Cayley graph gives a tiling of the the plane by squares with edges labeled $aba^{-1}b^{-1}$.
- If $H = G^2$ is the smallest subgroup containing all squares in G, the right cosets are $H\backslash G = \{He, Ha, Hb, Hab\}$ and the Cayley graph is the graph of the Cayley table.
- If H is the smallest normal subgroup containing a^3, b^3, and $(ab)^3$, then the Cayley graph gives a tiling of the plane by hexagons, with edges $ababab$, and triangles with edges aaa or bbb. Observe that $Hxa^3y = Hxa^{-3}x^{-1}xa^3y = Hxy$.

It is, in general, a difficult problem to enumerate the cosets of H in G.

Exercise 7.17. [5, (3) p 58] Let $G =< a, b >$ be the free group on two generators and $H = \langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$. Draw the Cayley graph for $H\backslash G$ with the help of the information provided by this magma session:

```
> G<a,b>:=FreeGroup(2);
>  H:=sub<G|a^2, b^2, a*b*a^-1, b*a*b^-1>;
> Index(G,H);
3
> T,f:=RightTransversal(G,H);
> T;
{@ Id(G), a, b @} //The vertices of the Cayley graph
> E:={@ <v,(v*a)@f,(v*b)@f> : v in T @};
> E;
{@ <Id(G), a, b>, <a, Id(G), a>, <b, b, Id(G)> @} //The edges
>
```

Exercise 7.18. Let G be a free group of finite rank and H a subgroup of G. Show that H is free and that $|G\colon H|(\mathrm{rk}(G) - 1) = \mathrm{rk}(H) - 1$. (This exercise is most easily solved by using the Euler characteristic.)

Example 7.19. When $G = C_m = \langle g \mid g^m \rangle$ is the cyclic group of order $m > 0$, the Cayley complex

$$X_G(\{e\}\backslash G) = (G \times D^0) \cup (G \times D^1) \cup (G \times D^2)$$

is the universal covering space of the mapping cone for the degree m map on the circle. It is the left G-CW-complex consisting of a circle with m 2-discs attached. (When $m = 2$, this is the 2-sphere which is the

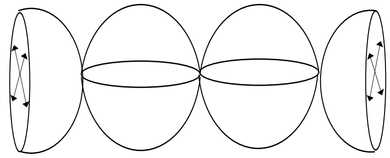
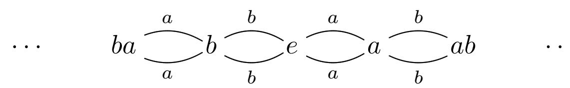


FIGURE 5. $H\backslash X_{\{e\}\backslash G}$ for $H = \langle (ab)^3, a\rangle \leq \langle a, b \mid a^2, b^2\rangle = G$

universal covering space of the mapping cone $\mathbf{R}P^2$ for the degree 2-map of the circle.) What is the covering space action of G on $X_G(\{e\}\backslash G)$?

Example 7.20. [5, Example 1.48, Exercise 1.3.14] Let $G = \langle a, b \mid a^2, b^2\rangle = \mathbf{Z}/2 \amalg \mathbf{Z}/2 \overset{4.2}{=} \mathbf{Z} \rtimes \mathbf{Z}/2$ be the free product of $\mathbf{Z}/2$ with itself. Then $X_{G\backslash G} = \mathbf{R}P^2 \vee \mathbf{R}P^2$ and $\mathrm{Cov}_0(\mathbf{R}P^2 \vee \mathbf{R}P^2) = \mathcal{O}_{C_2 \amalg C_2}$. The total space $X_{\{e\}\backslash G}$ of its universal covering space $X_{\{e\}\backslash G} \to X_{G\backslash G}$ is an infinite string of S^2s. Indeed, the 0-skeleton is G, the 1-skeleton obtained by attaching two 1-discs to each 0-cell, is

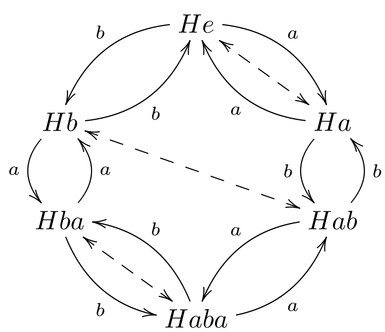


and the 2-skeleton is obtained by attaching two 2-discs at each 0-cell along the maps a^2 and b^2. The left action of $a \in G$ which swaps $e \leftrightarrow a$, $b \leftrightarrow ab$, etc is the antipodal map on the sphere containing e and a.

The subgroup $H = \langle (ab)^3\rangle = 3\mathbf{Z} \subset \mathbf{Z}$ is normal in G so that the orbit set

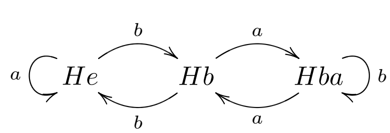
$$H\backslash G = \{He, Ha, Hb, Hab, Hba, Haba\} = 3\mathbf{Z}\backslash\mathbf{Z} \rtimes 2\mathbf{Z}\backslash\mathbf{Z}$$

is actually a group; it is the dihedral group of order 6, isomorphic to Σ_3. The quotient space $H\backslash X_{\{e\}\backslash G} = X_{H\backslash G}$ is a necklace of six S^2s formed from the 1-skeleton



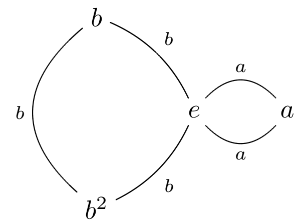
by attaching 2-discs at each vertex along the loops a^2 and b^2. The fundamental group of $H\backslash\widetilde{X}_G$ is H and the deck transformation group is $H\backslash N_G(H) = H\backslash G$ since H is normal. The dashed arrows show the covering space left action by $Ha \in H\backslash G$; the orbit space for this action is the Cayley complex of the next example. The element $Hab \in H\backslash G$ acts by rotating the graph two places in clockwise direction.

For another example, take $H = \langle (ab)^3, a\rangle = 3\mathbf{Z} \rtimes \mathbf{Z}/2$; H is not normal for $N_G(H) = H$, $H\backslash G = \{He, Hb, Hba\}$ has 3 elements, and



is the Cayley graph for $H\backslash G$. The Cayley complex, obtained by attaching six 2-discs along the maps a^2 and b^2 at each vertex, is $\mathbf{R}P^2$, S^2, S^2, $\mathbf{R}P^2$ on a string as shown above. The deck transformation group $H\backslash N_G(H) = H\backslash H$ is trivial.

Example 7.21. Let $G = \mathbf{Z}/2 \amalg \mathbf{Z}/3 \overset{4.2}{=} \mathrm{PSL}(2, \mathbf{Z})$ be the free product of a cyclic group of order two and a cyclic group of order three. This graph



is the beginning of the Cayley complex for G. Describe the left G-CW-complex $X_G(G)$!

7.22. **Normal covering maps.** Let $p\colon Y \to X$ be a covering map between path connected spaces.

Definition 7.23. The covering map $p\colon Y \to X$ is normal if the group $\mathrm{Cov}(X)(Y, Y)$ of deck transformations acts transitively on the fibre $p^{-1}(x)$ over some point of X.

If the action is transitive at some point, then it is transitive at all points. Why are these covering maps called normal covering maps?

Corollary 7.24. *Let X be a path connected, locally path connected and semi-locally simply connected space and $p\colon Y \to X$ a covering map with Y path connected. Then*

$$\text{The covering map } p\colon Y \to X \text{ is normal} \iff \text{The subgroup } \pi_1(Y) \text{ is normal in } \pi_1(X)$$

Proof. The action $H\backslash N_G(H) \times H\backslash G \to H\backslash G$ of the group of covering maps on the fibre is transitive iff and only if H is normal in G. □

All double covering maps are normal since all index two subgroups are normal.

7.25. **Sections in covering maps.** A *section* of a covering $p\colon E \to X$ is a (continuous) map $s\colon X \to E$ such that $s(x)$ lies above x, $ps(x) = x$, for all $x \in X$. In other words, a section is a lift of the identity map of the base space. Each section traces out a copy of the base space in the total space (and that is why it is called a section).

Lemma 7.26. *Let $p\colon E \to X$ be a covering space over a connected, locally path connected and semi-locally simply connected base space X. Then the evaluation map $s \to s(x)$*

$$\{\text{sections of } p\colon E \to X\} \longrightarrow p^{-1}(x)^{\pi_1(X,x)}$$

is a bijection.

Proof. Since X is connected, sections are determined by their value at a single point (2.12), so the map is injective. It is also surjective because any $\pi_1(X, x)$-invariant point corresponds (under the classification of covering spaces over X) to the trivial covering map $X \to X$ which obviously has a section. □

In fact, E contains the trivial covering $p^{-1}(x)^{\pi_1(X,x)} \times X$ as a subcovering.
If either of $Y_1 \to X$ or $Y_2 \to X$ is normal, then

$$\mathrm{Cov}(X)(Y_1, Y_2) = \begin{cases} \pi_1(Y_2)\backslash\pi_1(X) & \pi_1(Y_2) \subset \pi_1(Y_1) \\ \emptyset & \text{otherwise} \end{cases}$$

for the transporter $N_{\pi_1(X)}(\pi_1(Y_1), \pi_1(Y_2))$ equals $\pi_1(X)$ if $\pi_1(Y_1) \subset \pi_1(Y_2)$ and $\emptyset$ otherwise.

8. Universal covering spaces of topological groups

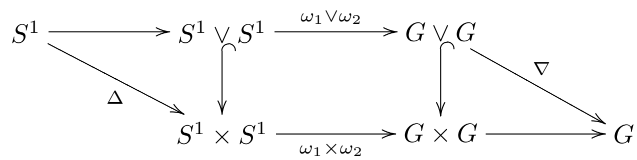
Suppose that G is a connected, locally path connected, and semi-locally simply connected topological group (for instance, a connected Lie group) and let $G\langle 1\rangle$ be the universal covering space (7.4) of G. We can use the group multiplication in G to define a multiplication in $G\langle 1\rangle$ simply by letting the product $[\gamma]\cdot[\eta]$ of two homotopy classes of paths $[\gamma], [\eta] \in G\langle 1\rangle$ equal the homotopy class $[\gamma \cdot \eta] \in G\langle 1\rangle$ of the product path $(\gamma \cdot \eta)(t) = \gamma(t) \cdot \eta(t)$ whose value at any time t is the product of the values $\gamma(t) \in G$ and $\eta(t) \in G$.

Lemma 8.1. *$G\langle 1\rangle$ is a topological group and $G\langle 1\rangle \to G$ is a morphism of topological groups whose kernel is the subgroup $\{[\omega] \mid \omega(0) = \omega(1)\} = \pi_1(G, e)$ of homotopy classes of loops based at the unit $e \in G$.*

The set $\pi_1(G, e)$ is here equipped with the group structure it inherits from $G\langle 1\rangle$ where multiplication of paths is induced from group multiplication in G. However, we have also defined a group structure on $\pi_1(G, e)$ using composition of loops. It turns out that these two structures are identical.

Lemma 8.2. *Let ω_1 and ω_2 be two loops in G based at the unit element e. Then the loops $\omega_1 \cdot \omega_2$ (group multiplication) and $\omega_1\omega_2$ (loop composition) are homotopic loops.*

Proof. There is a homotopy commutative diagram



where Δ is the diagonal and ∇ the folding map. The loop defined by the top edge from S^1 to G is the composite loop $\omega_1\omega_2$ and the loop defined by the bottom edge is the product loop $\omega_1 \cdot \omega_2$. □

One can also show that in this situation $\pi_1(G, e)$ must be abelian.

Let $\mathbf{H} = \mathbf{R}1 \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$ be the quaternion algebra where the rules $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ki$ define the multiplication. Let $\mathrm{Sp}(1)$ denote the topological group of quaternions of norm 1.

$\mathrm{Sp}(1)$ acts in a norm preserving way on the real vector space $\mathbf{H} = \mathbf{R}^4$ by the rule $\alpha \cdot v = \alpha v \alpha^{-1}$ for all $\alpha \in \mathrm{Sp}(1)$ and $v \in \mathbf{R}^4 = \mathbf{H}$. This give a homomorphism $\pi\colon \mathrm{Sp}(1) \to \mathrm{SO}(4)$. Since $\mathbf{R}1$ is invariant under this action, it takes $\mathbf{R}^\perp = \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k = \mathbf{R}^3$ to itself, so there is also a group homomorphism $\pi\colon \mathrm{Sp}(1) \to \mathrm{SO}(3)$ [2, I.6.18, p 88]. The kernel is $\mathbf{R} \cap \mathrm{Sp}(1) = \{\pm 1\}$. Convince yourself that π is surjective (see the computation below and recall that an element of $\mathrm{SO}(3)$ is a rotation around a fixed line), so that $\pi\colon \mathrm{Sp}(1) \to \mathrm{SO}(3) = \{\pm 1\}\backslash \mathrm{Sp}(1)$ is a double covering space.

Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

be the matrix for rotation through angle θ.

Lemma 8.3. *The map $\pi\colon \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is the universal covering map of $\mathrm{SO}(3)$. The fundamental group $\pi_1(\mathrm{SO}(3), E) = \{\pm 1\}$ is generated by the loop*

$$\omega(t) = \pi\alpha(t) = \begin{pmatrix} R(2\pi t) & 0 \\ 0 & 1 \end{pmatrix}, \qquad 0 \le t \le 1,$$

Proof. The topological space $\mathrm{Sp}(1) = S^3$ is simply connected, so $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ is the universal covering space of $\mathrm{SO}(3)$. (We have seen this double covering before: It is the double covering $S^3 \to \mathbf{R}P^3$.)

The fundamental group $\pi_1(\mathrm{SO}(3), E) = C_2$ is generated by the image loop $\omega(t) = \pi\alpha(t)$ of a path $\alpha(t)$ in $\mathrm{Sp}(1)$ from $+1$ to -1. If we take

$$\alpha(t) = \cos(\pi t) + \sin(\pi t)k, \qquad 0 \le t \le 1,$$

then the image in $\mathrm{SO}(3)$ is the loop

$$\omega(t) = \pi\alpha(t) = \begin{pmatrix} R(2\pi t) & 0 \\ 0 & 1 \end{pmatrix}, \qquad 0 \le t \le 1,$$

This follows from the computation

$$\begin{aligned} \alpha(t)i\alpha(t)^{-1} &= (\cos(\pi t) + k\sin(\pi t))i((\cos(\pi t) - k\sin(\pi t)) \\ &= \cos^2(\pi t)i + \cos(\pi t)\sin(\pi t)j + \cos(\pi t)\sin(\pi t)j - \sin^2(\pi t)i = \cos(2\pi t)i + \sin(2\pi t)j \end{aligned}$$

and similarly for $\alpha(t)j\alpha(t)^{-1} = -\sin(2\pi t)i + \cos(2\pi t)j$ and $\alpha(t)k\alpha(t)^{-1} = k$. □

It is also known that the inclusion $\mathrm{SO}(3) \to \mathrm{SO}(n)$ induces an isomorphism on π_1 for $n \geq 3$. We conclude that the fundamental group $\pi_1(\mathrm{SO}(n), E)$ has order two for all $n \geq 3$ and that it is generated by the loop $\omega(t)$ in $\mathrm{SO}(n)$. Thus the topological groups $\mathrm{SO}(n)$, $n \geq 3$, have universal *double* covering spaces that are topological groups.

Definition 8.4. For $n \geq 3$, $\mathrm{Spin}(n) = \mathrm{SO}(n)\langle 1 \rangle$ is the universal covering space of $\mathrm{SO}(n)$ and $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal covering map.

The elements of $\mathrm{Spin}(n)$ are homotopy classes of paths in $\mathrm{SO}(n)$ starting at E and, in particular, $\mathrm{Spin}(3) = \mathrm{Sp}(1)$. The kernel of the homomorphism $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is $\{e, z\}$ where e is the unit element and $z = [\omega]$ is the homotopy class of the loop ω.

Proposition 8.5. *The center of* $\mathrm{Spin}(n)$ *is*

$$Z(\mathrm{Spin}(n)) = \begin{cases} C_2 = \{e, z\} & n \text{ odd} \\ C_2 \times C_2 = \{e, x\} \times \{e, z\} & n \equiv 0 \bmod 4 \\ C_4 = \{e, x, x^2, x^3\} & n \equiv 2 \bmod 4 \end{cases}$$

for $n \geq 3$.

Proof. From Lie group theory we know that the center of $\mathrm{Spin}(n)$ is the inverse image of the center of $\mathrm{SO}(n)$. Thus the center of $\mathrm{Spin}(n)$ has order 2 when n is odd and order 4 when n is even.

Suppose that $n = 2m$ is even. Then $Z(\mathrm{Spin}(2m)) = \{e, z, x, zx\}$ where $x = [\eta]$ is the homotopy class of the path

$$\eta(t) = \mathrm{diag}(R(\pi t), \dots, R(\pi t))$$

from E to $-E$. Note that x^2 is (8.2) represented by the loop

$$\eta(t)^2 = \mathrm{diag}(R(2\pi t), \dots, R(2\pi t))$$

Conjugation with a permutation matrix from $\mathrm{SO}(2n)$ takes

$$\omega(t) = \mathrm{diag}(R(2\pi t), E, E, \dots, E) \text{ to } \mathrm{diag}(E, R(2\pi t), E, \dots, E)$$

and since inner automorphisms are based homotopic to identity maps, both the above loops represent the generating loop ω. It follows that

$$x^2 = [\eta(t)^2] = [\omega(t)^m] = z^m = \begin{cases} e & m \text{ even} \\ z & m \text{ odd} \end{cases}$$

Thus $Z(\mathrm{Spin}(2m)) = \{z\} \times \{x\} = C_2 \times C_2$ if m is even and $Z(\mathrm{Spin}(2m)) = \{x\} = C_4$ if m is odd. □

What is the fundamental group $\pi_1(\mathrm{PSO}(2n))$ of the topological group $\mathrm{PSO}(2n) = \mathrm{SO}(2n)/\langle -E \rangle$?

When will two diagonal matrices in $\mathrm{SO}(n)$ commute in $\mathrm{Spin}(n)$? Let $D = \{\mathrm{diag}(\pm 1, \dots, \pm 1\}$ be the abelian subgroup of diagonal matrices in $\mathrm{SO}(n)$. By computing commutators and squares in $\mathrm{Spin}(n)$ we obtain functions

$$[\ ,\ ]\colon D \times D \to \{e, z\}, \qquad q\colon D \to \{e, z\}$$

given by $q(d) = (\bar{d})^2$ and $[d_1, d_2] = [\bar{d}_1, \bar{d}_2]$ where $\pi(\bar{d}) = d$, $\pi(\bar{d}_1) = d_1$, $\pi(\bar{d}_2) = d_2$. They are related by formula

$$q(d_1 + d_2) = q(d_1) + q(d_2) + [d_1, d_2]$$

which says that $[\ ,\ ]$ records the deviation from q being a group homomorphism (using additive notation here). It suffices to compute q in order to answer the question about commutativity relations.

Proposition 8.6. $q(d) = e$ *iff the number of negative entries in the diagonal matrix* $d \in D$ *is divisible by* 4. $[d_1, d_2] = e$ *iff the number of entries that are negative in both* d_1 *and* d_2 *is even.*

Proof. Note that two elements of D are conjugate iff they have the same number of negative entries. Use permutation matrices and, if necessary, the matrix $\mathrm{diag}(-1, 1, \dots, 1)$. Consider for instance

$$d_1 = \mathrm{diag}(-1, -1, 1, \dots, 1), \qquad d_2 = \mathrm{diag}(-1, -1, -1, -1, 1, \dots, 1)$$

with two, respectively four, negative entries. The paths

$$\bar{d}_1(t) = \mathrm{diag}(R(\pi t), 1, \dots, 1), \qquad \bar{d}_2(t) = \mathrm{diag}(R(\pi t), R(\pi t), 1, \dots, 1)$$

represent lifts of d_1 and d_2 to $\mathrm{Spin}(n)$. Then $(\bar{d}_1)^2 = z = q(d_1)$ and $(\bar{d}_2)^2 = e = q(d_2)$. Computations like these prove the formula for q and the formula for $[\ ,\ ]$ follows. The number of negative entries in $d_1 + d_2$ is the number of negative entries in d_1 plus the number of negative entries in d_2 minus twice the number of entries that are negative in both d_1 and d_2. $\square$

Exercise 8.7. Let $\overline{D}_n \subset \mathrm{Spin}(n)$ be the inverse image of $D \subset \mathrm{SO}(n)$. How many elements of order 4 are there in $\overline{D}_n$? Can you identify the group $\overline{D}_n$? Show that there is a homomorphism $\mathrm{SU}(m) \to \mathrm{Spin}(2m)$. When m is even, what is the image of $-E \in \mathrm{SU}(m)$? What is the image of the center of $\mathrm{SU}(m)$? Describe the covering spaces of $\mathrm{U}(n)$.

The inclusions $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$, $n > 2$, and $\mathrm{SO}(m) \times \mathrm{SO}(n) \subset \mathrm{SO}(m+n)$, $m, n > 2$, of special orthogonal groups lift to inclusions

$$\begin{array}{ccc} \mathrm{Spin}(n) & \hookrightarrow & \mathrm{Spin}(n+1) \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \hookrightarrow & \mathrm{SO}(n+1) \end{array} \qquad \begin{array}{ccc} \mathrm{Spin}(m) \times_{\langle (z_1,z_2)\rangle} \mathrm{Spin}(n) & \hookrightarrow & \mathrm{Spin}(m+n) \\ \downarrow & & \downarrow \\ \mathrm{SO}(m) \times \mathrm{SO}(n) & \hookrightarrow & \mathrm{SO}(m+n) \end{array}$$

of double coverings. (Here, $\mathrm{Spin}(m) \times_{\langle (z_1,z_2)\rangle} \mathrm{Spin}(n)$ stands for $\langle (z_1, z_2)\rangle \backslash (\mathrm{Spin}(m) \times \mathrm{Spin}(n))$).

The inclusion $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, that comes from the identification $\mathbf{C}^n = \mathbf{R}^{2n}$, lifts to an inclusion of double covering spaces as shown in the following diagrams.

$$\begin{array}{ccc} \mathrm{SU}(n) \times_{C_k} \mathrm{U}(1) & \longrightarrow & \mathrm{Spin}(2n) \\ {\scriptstyle (A,z)\to(A,z)} \downarrow & & \downarrow \\ \mathrm{SU}(n) \times_{C_n} \mathrm{U}(1) = \mathrm{U}(n) & \hookrightarrow & \mathrm{SO}(2n) \end{array} \qquad \begin{array}{ccc} \mathrm{SU}(n) \times_{C'_n} \mathrm{U}(1) & \longrightarrow & \mathrm{Spin}(2n) \\ {\scriptstyle (A,z)\to(A,z^2)} \downarrow & & \downarrow \\ \mathrm{SU}(n) \times_{C_n} \mathrm{U}(1) = \mathrm{U}(n) & \hookrightarrow & \mathrm{SO}(2n) \end{array}$$

To the left, $n = 2k$ is even, and to the right, $n = 2k + 1$ is odd; $C_n = \{(\zeta E, \zeta^{-1}) \mid \zeta^n = 1\}$ and $C'_n = \{(\zeta E, \zeta^k) \mid \zeta^n = 1\}$ are cyclic groups of order n and $C_k = \{(\zeta E, \zeta^{-1}) \mid \zeta^k = 1\} \subset C_{2k} = C_n$ is cyclic of order k. The isomorphism $\mathrm{SU}(n) \times_{C_n} \mathrm{U}(1) \to \mathrm{U}(n)$ takes (A, z) to zA. When n is divisible by 4, $z = (-E, -1)$ and $x = (E, -1)$ have order two; when n is even and not divisible by 4, $x = (E, i)$ has order four and $x^2 = (E, -1) = z$. This explains the computation of the center of $\mathrm{Spin}(2n)$. (Is the group in the upper left corner of the right diagram isomorphic to $\mathrm{U}(n)$? See [1] for more information.)

There is a double covering map $\mathrm{pin}(n) \to \mathrm{O}(n)$ obtained as the pullback of $\mathrm{Spin}(2n) \to \mathrm{SO}(2n)$ along the inclusion homomorphism $\mathrm{O}(n) \subset \mathrm{SO}(2n)$.

Example 8.8. The inclusion $\mathrm{U}(2) \subset \mathrm{SO}(4)$ lifts to an inclusion $\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{Spin}(4)$. Let $G_{16} \subset \mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{Spin}(4)$ be the group

$$G_{16} = \left\langle \left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, i \right), \left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, -i \right) \right\rangle$$

G_{16} has order 16, center $Z(G_{16}) = \{x, z\} = C_2 \times C_2 = Z(\mathrm{Spin}(4))$, and derived group $[G_{16}, G_{16}] = \{xz\} = C_2$. Its image under the covering maps

$$\begin{array}{ccccc} & \nearrow & \mathrm{Spin}(4)/\langle z\rangle = \mathrm{SO}(4) & \searrow & \\ \mathrm{Spin}(4) & \longrightarrow & \mathrm{Spin}(4)/\langle xz\rangle & \longrightarrow & \mathrm{Spin}(4)/\langle x, z\rangle = \mathrm{PSO}(4) \\ & \searrow & \mathrm{Spin}(4)/\langle x\rangle = \mathrm{SSpin}(4) & \nearrow & \end{array}$$

is dihedral D_8 in $\mathrm{SO}(4)$, abelian $C_4 \times C_2$ in $\mathrm{Spin}(4)/\langle xz\rangle$, quaternion Q_8 in the semi-spin group $\mathrm{SSpin}(4) = \mathrm{Spin}(4)/\langle x\rangle$, and elementary abelian $C_2 \times C_2$ in $\mathrm{PSO}(4)$. (All proper subgroups of G_{16} are abelian but itself and some of its quotient groups are nonabelian.)

Example 8.9. There exists a covering space homomorphisms of topological groups

$$\mathrm{U}(1)\times \mathrm{SU}(n)\to \mathrm{U}(n)\colon (z,A)\to zA$$

with kernel $C_n=\{(z,z^{-1}E)\mid z^n=1\}=\big\langle(\zeta,\zeta^{-1}E)\big\rangle$ where $\zeta_n=e^{2\pi i/n}$. The universal covering space homomorphism is $\mathbf{R}\times\mathrm{SU}(n)\to\mathrm{U}(n)\colon (t,A)\to\zeta_n^tA$ with kernel $C_\infty=\big\langle(1,\zeta_n^{-1}E)\big\rangle$. Any covering space of $\mathrm{U}(n)$ is of the form $\big\langle(k,\zeta_n^{-k})\big\rangle\backslash(\mathbf{R}\times\mathrm{SU}(n))$ for some integer $k\geq 0$.

Similarly, let $S(\mathrm{U}(m)\times\mathrm{U}(n))$ denote the closed topological subgroup $(\mathrm{U}(m)\times\mathrm{U}(n))\cap\mathrm{SU}(m+n)$ of $\mathrm{U}(m+n)$. There exists a covering space homomorphisms of topological groups

$$\mathrm{U}(1)\times\mathrm{U}(m)\times\mathrm{U}(n)\to S(\mathrm{U}(m)\times\mathrm{U}(n))\colon (z,A,B)\to \mathrm{diag}(z^{n_1}A,z^{-m_1}B)$$

with kernel $C_{\mathrm{lcm}(m,n)}=\{(z,z^{-n_1}E,z^{m_1}E)\mid z^{\mathrm{lcm}(m,n)}=1\}=\big\langle(\zeta_{\mathrm{lcm}(m,n)},\zeta_m^{-1},\zeta_n)\big\rangle$ where $m_1=m/\gcd(m,n)=\mathrm{lcm}(m,n)/n$ and $n_1=n/\gcd(m,n)=\mathrm{lcm}(m,n)/m$. The universal covering space homomorphism of $S(\mathrm{U}(m)\times\mathrm{U}(n))$ is $\mathbf{R}\times\mathrm{SU}(m)\times\mathrm{SU}(n)\to S(\mathrm{U}(m)\times\mathrm{U}(n))\colon (t,A,B)\to(\zeta_m^tA,\zeta_n^tB)$ with kernel $C_\infty=\big\langle(1,\zeta_m^{-1},\zeta_n)\big\rangle$. Any covering space of $S(\mathrm{U}(m)\times\mathrm{U}(n)$ is of the form $\big\langle(k,\zeta_m^{-k},\zeta_n^k)\big\rangle\backslash(\mathbf{R}\times\mathrm{SU}(m)\times\mathrm{SU}(n))$ for some integer $k\geq 0$.

All finite covering spaces of $\mathrm{U}(n)$ are covered by $\mathrm{U}(1)\times\mathrm{SU}(n)$. To see this, let n and k be integers and put $k_1=k/\gcd(n,k)$. Then there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{U}(1)\times\mathrm{SU}(n) & \dashrightarrow & \mathbf{R}\times_{\langle(k,\zeta^{-k})\rangle}\mathrm{SU}(n) \\
{\scriptstyle (z,A)\to(z^{k_1},A)}\downarrow & & \downarrow \\
\mathrm{U}(1)\times\mathrm{SU}(n) & \xrightarrow[(\zeta_n^t,A)\to(t,A)]{} & \mathbf{R}\times_{\langle(1,\zeta^{-1})\rangle}\mathrm{SU}(n)=\mathrm{U}(n)
\end{array}$$

of covering space homomorphisms.

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