A PARAMETERIZED INDEX THEOREM FOR THE ALGEBRAIC K-THEORY EULER CLASS

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Abstract. A Riemann-Roch theorem asserts that some algebraically defined wrong–way map in $K$-theory agrees with a topologically defined one [BFM]. Bismut and Lott [BiLo] proved a Riemann–Roch theorem for smooth fiber bundles in which the topologically defined wrong–way map is the homotopy transfer of Becker–Gottlieb and Dold. We generalize their theorem, refine it, and prove a converse stating that an appropriate Riemann–Roch equation holds for a compact topological manifold bundle if and only if up to fiber homotopy the bundle has a fiberwise smooth structure. We obtain a similar characterization of fibrations which are fibre homotopy equivalent to compact smooth manifold bundles. In the process, we prove a family index theorem for fiber bundles with compact topological manifold fibers, a theorem in which the relevant index equation involves algebraic $K$-theory.

0. Introduction

The classical setting. Suppose that $p : E \to B$ is a smooth fiber bundle with compact fiber $F$; in other words $F$ is a smooth manifold (possibly with boundary) and the structure group of $p$ is the topological group of diffeomorphisms of $F$. Let $V$ be a complex vector bundle on $E$ with discrete structure group (flat complex vector bundle for short), and $V_i$ the complex vector bundle on $B$ whose fiber over $b \in B$ is the local–coefficient homology group $H_i(p^{-1}(b);V)$. Denote by $K(\mathbb{C}_t)$ the $K$-theory space of the topological ring $\mathbb{C}$, so that $K(\mathbb{C}_t) \simeq \mathbb{Z} \times BU$ and for a space $X$ the group of homotopy classes $[X,K(\mathbb{C}_t)]$ is the topological complex $K$-theory of $X$. The vector bundles $V, V_i$ give classes $[V] \in [E,K(\mathbb{C}_t)], [V_i] \in [B,K(\mathbb{C}_t)]$, and the Atiyah–Singer index theorem for families leads to the equation [BecSch]

$$(0\cdots 1) \quad \mathrm{tr}^*[V] = \sum (-1)^i[V_i] \in [B,K(\mathbb{C}_t)].$$

Here $\mathrm{tr}$ is the homotopy transfer of Becker–Gottlieb and Dold, which is a stable map from $B$ to $E$ determined by $p$, and $\mathrm{tr}^*[V]$ is the image of $[V]$ under the map on topological $K$-theory induced by $\mathrm{tr}$.
The Bismut–Lott theorem. Let $K(C)$ be the $K$-theory space of the discrete ring $C$, in other words, the space whose homotopy groups are the algebraic $K$-groups of $C$. Like $K(C_t)$, $K(C)$ represents a cohomology theory, and the natural ring homomorphism from $C$ with the discrete topology to $C$ with the ordinary topology induces a map $K(C) \rightarrow K(C_t)$. Since the vector bundles $V$ and $V_i$ above are flat, the elements $[V]$ and $[V_i]$ lift back to similarly denoted elements in $[E, K(C)]$ and $[B, K(C)]$ respectively. It is natural to ask whether equation (0–1) holds in $[B, K(C)]$.

Bismut and Lott have given strong evidence that this is the case. There are certain characteristic classes for flat bundles, constructed by Kamber and Tondeur, which for odd $k$ yield homomorphisms $c_k : [B, K(C)] \rightarrow H^k(B; \mathbb{R})$. The Riemann–Roch theorem of Bismut and Lott [BiLo] then states that equation (0–1) holds in $[B, K(C)]$ after the homomorphisms $c_k$ are applied:

\[ (0–2) \quad c_k (\text{tr}^*[V]) = \sum (-1)^i c_k[V_i] \in H^k(B; \mathbb{R}). \]

To prove this Bismut and Lott use Bismut’s local version of the Atiyah–Singer theorem for families. (They also assume that the fibers of $p$ are closed manifolds, but we have been told that this was mostly to simplify the presentation.) Lott issued the challenge to topologists to find a more topological proof.

An improvement. We do find a topological proof, of what is in fact a more general result. Let $R$ be an arbitrary ring with associated $K$-theory space $K(R)$. Suppose that $V$ is a bundle of finitely generated projective left $R$–modules on $E$. As above, $V$ determines an element $[V] \in [E, K(R)]$. Each fiber $F_b = p^{-1}(b)$ of $p$ has local–coefficient homology groups $H_*(F_b; V)$, and for fixed $i$ these homology groups form a bundle $V_i$ of left $R$–modules on $B$. We assume that the fibers of $V_i$ are again projective, in which case they are also finitely generated. Each $V_i$ determines an element $[V_i] \in [B, K(R)]$. In §9 we prove

\[ (0–3) \quad \text{tr}^*[V] = \sum (-1)^i [V_i] \in [B, K(R)]. \]

Smoothness is an essential hypothesis; in another paper we shall show that (0–2) and hence (0–3) can fail for a fiber bundle whose fibers are closed topological manifolds.

Further improvement, and a converse. There is something striking about (0–3), namely, both sides of the equation are homotopy theoretic in nature. In other words, the equation (true or false) makes sense if we merely suppose that $p : E \rightarrow B$ is a fibration in which the fibers have appropriate finiteness properties. It is tempting to look for a geometric characterization of those fibrations $p : E \rightarrow B$ for which (0–3) is valid, with the expectation that a simple characterization will exist only if parameters such as $R$ and $V$ are chosen in some universal way. It turns out that there is such a characterization, and that it amounts to a converse for the result above: equation (0–3) holds in the universal case if and only if the fibration $p$ is fiber homotopy equivalent to a smooth bundle with compact smooth manifold fibers. Formulating this more precisely requires identifying the appropriate universal case.
Suppose to begin with that \( F \) is a finite complex and that \( B \) is a point, so that \( F = E \). Temporarily ignore questions of path connectivity and basepoint. Let \( R \) and \( V \) be as above (so that \( V \) is a module over \( \mathbb{Z}[\pi_1X] \)) and consider the maps

\[
K(S^0[\Omega F]) \to K(\mathbb{Z}[\pi_1F]) \to K(R).
\]

Here \( \Omega F \) is the loop space of \( F \), considered as a topological or simplicial group, and \( S^0[\Omega F] \) (introduced by Waldhausen) is the group ring of the loop space over the sphere spectrum \( S^0 \). The first map is induced by the homomorphism \( S^0[\Omega F] \to \pi_0S^0[\Omega F] = \mathbb{Z}[\pi_1F] \), and the second by tensoring over \( \mathbb{Z}[\pi_1F] \) with \( V \). The group \( \Omega F \) acts on the path space \( PF \) in such a way that the unreduced suspension spectrum \( S^0 \land PF_+ \) becomes the appropriate analogue of a finite projective chain complex over \( S^0[\Omega F] \); one way to see this is to filter \( PF \) by inverse images of the skeleta of \( F \) and observe that the filtration quotients suspend to finitely generated free modules over \( S^0[\Omega F] \). Let \( [S^0 \land PF_+] \in \pi_0K(S^0[\Omega F]) \) denote the K-theory class that corresponds to this “chain complex”; this is our candidate for the universal right-hand side of (0–3). It can actually be defined as long as \( F \) is finitely dominated. Under the above K-theory maps, \( [S^0 \land PF_+] \) is carried first to the class \( [C_*(\tilde{F};\mathbb{Z})] \in \pi_0K(\mathbb{Z}[\pi_1X]) \) which projects in \( K_0(\mathbb{Z}[\pi_1X])/K_0\mathbb{Z} \) to the Wall finiteness obstruction of \( F \). In fact this identifies \( [S^0 \land PF_+] \), since the map \( \pi_0K(S^0[\Omega F]) \to \pi_0K(\mathbb{Z}[\pi_1F]) \) is an isomorphism. The further image in \( \pi_0K(R) \) is the class represented by the finite projective chain complex \( C_*(F;V) \), or equivalently, by the sum \( \sum (-1)^i[V_i] \).

The universal left-hand side of (0–3) for \( B = * \) is the homotopy class

\[
\text{tr} : * \to Q(F_+) = Q(B\Omega F_+)
\]

provided by the transfer. The target of this map is the zero space in the spectrum \( S^0 \land F_+ \), that is, the space whose homotopy groups are the unreduced stable homotopy of \( F \). In particular \( \pi_0Q(F_+) \cong \mathbb{Z} \), and it is a classical result that the image of the map \( \text{tr} \) lies in the component given by the Euler characteristic of \( F \).

Waldhausen describes a map \( \iota : Q(B\Omega F)_+ \to K(S^0[\Omega F]) \) which can be interpreted as arising from the inclusion of a certain monomial or permutation matrix subgroup of \( \text{GL}(S^0[\Omega F]) \). On components, \( \iota \) is the map \( \mathbb{Z} = K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\pi_1F]) \), and its image is the subgroup of \( K_0(\mathbb{Z}[\pi_1F]) \) generated by free modules. The universal case of (0–3) in the case \( B = * \) is the equation \( \iota \cdot (\text{tr}) = [S^0 \land PF_+] \), which the reader may by now recognize as expressing the fact that the Wall finiteness obstruction of \( F \) vanishes. The theorem, due to Wall, is that this equality holds if and only if \( F \) is homotopy equivalent to a compact smooth manifold (possibly with boundary).

For a general fibration \( p : E \to B \) with finitely dominated fiber \( F \), we construct fibrations \( \cup_bQ(F_b)_+ \to B \) and \( \cup_bK(S^0[\Omega F_b]) \to B \) by fiberwise application of \( p \) of stable-homotopy and K-theory functors respectively. The transfer gives a section \( \text{tr} \) of the first fibration, and the map \( b \mapsto [S^0 \land (PF_b)_+] \) (suitably rigidified) a section \( \chi \) of the second. There is a map \( \iota \) over \( B \) from one fibration to the other, and the universal case of (0–3) is the equation \( \iota \cdot (\text{tr}) \cong \chi \), where “\( \cong \)” denotes homotopy between sections. The theorem is that this equation holds if and only if \( p \) is fiber homotopy equivalent to a compact smooth manifold bundle.
Of course there are details to handle. For instance $K(S^0[\Omega F_0])$ has to be constructed in a way which doesn’t depend upon choice of a basepoint in $F$; Waldhausen has done this with the functor $A(F)$. The description of $[S^0 \wedge PF_+]$ in this setting is particularly appealing. We also start with the assumption that the fiber $F$ is homotopy equivalent to a finite complex; this simplifies the exposition, although it conceals the significance of the theory if $B$ is a point.

*An interpretation of the results.* We offer the following conceptual interpretation of what the homotopy theoretic results in this paper mean. Let $p : E \to B$ be a fibration with fiber $F$, $F$ a finite complex. We view the section $\chi$ above as capturing the cellular monodromy of the fibration $p$. If $p$ is equivalent to a compact smooth manifold bundle, then $\chi$ lifts back to a section of a fibration whose fiber $Q(B\Omega F)^+$ is built from permutation matrices: diffeomorphisms permute the cells. If $p$ is equivalent to a compact topological manifold bundle, $\chi$ lifts back to a section of a fibration with fiber $\Omega^\infty(F_+ \wedge A(\ast))$ (see below): homeomorphisms shift the cells and lay them on top of one another with coefficients from $A(\ast)$. In general $\chi$ is simply a section of a fibration with fiber $K(S^0[\Omega F]) = A(F)$: self-homotopy equivalences can entangle the cells in an arbitrary, nonlocal way (nonlocal, in that $A(F)$ is not a homological functor of $F$).

**Strategy of the proof: the index theorem.** Recall that Atiyah–Singer [AtS, §4] deduce the Hirzebruch–Riemann–Roch theorem from their index theorem. We adopt a similar strategy. The appropriate index theorem for us is a generalization of the Hopf–Pontryagin (HP) index theorem stating that $e(\tau^M) \cap [M, \partial M] = \chi(M)$. Here $M$ is a compact manifold with fundamental class $[M, \partial M]$, and $e(\tau^M)$ is the (relative) Euler class of the tangent bundle. Roughly speaking, we identify the section “$\chi$” above as an Euler characteristic and the section “$\text{tr}$” as the fiberwise Poincaré dual of an Euler class; the equality between them is the generalization of the HP index theorem. In formulating this generalization, we adopt the language of index theory: symbols, topological index, analytic index, and so on. However, our proof does not use operators, and in fact involves compact topological manifolds instead of smooth ones. We begin by updating the main ingredients in the HP theorem: Euler classes and Euler characteristics.

**Euler classes, classical.** For a space $X$ and an $n$–dimensional real vector bundle $\xi$ on $X$, the Euler class $e(\xi)$ lives in $H^n(X; \mathbb{Z}^l)$, the cohomology of $X$ with possibly twisted integral coefficients. When $X$ is a closed smooth $n$–manifold, and $\xi$ is its tangent bundle, then the Poincaré dual of $e(\xi)$ lives in $H_0(X; \mathbb{Z})$. Notice that we are now in homology with untwisted integer coefficients, even though $X$ might be non–orientable. More generally, $X$ could be a compact smooth $n$–manifold with boundary, and $\xi$ its tangent bundle; then $e(\xi) \in H^n(X, \partial X; \mathbb{Z}^l)$ and the Poincaré dual again lives in $H_0(X, \mathbb{Z})$.

**Euler classes of vector bundles as obstructions.** Suppose that $X$ is a space equipped with a Riemannian vector bundle $\xi$ of fiber dimension $n$. Let $\varepsilon^k$ be the trivial $k$–dimensional vector bundle on $X$, for some $k \gg 0$. Taking automorphisms
fiberwise produces an inclusion of bundles on $X$:

$$
\begin{align*}
\frac{O(\varepsilon^{k+1})}{O(\varepsilon^k)} & \longrightarrow \frac{O(\xi \oplus \varepsilon^{k+1})}{O(\xi \oplus \varepsilon^k)} \\
\downarrow & \downarrow \downarrow \\
X & \to \Omega^k_X (\frac{O(\xi \oplus \varepsilon^{k+1})}{O(\xi \oplus \varepsilon^k)})
\end{align*}
$$

The fibers of the bundle on the left are homeomorphic to $S^k$, and the fibers of the bundle on the right are homeomorphic to $S^{n+k}$. Taking adjoints fiberwise produces another inclusion of bundles on $X$,

$$
\begin{align*}
\mathbb{S}^0 \times X & \longrightarrow \Omega^k_X (\frac{O(\xi \oplus \varepsilon^{k+1})}{O(\xi \oplus \varepsilon^k)}) \\
\downarrow & \downarrow \downarrow \\
X & \to \Omega^k_X (\frac{O(\xi \oplus \varepsilon^{k+1})}{O(\xi \oplus \varepsilon^k)})
\end{align*}
$$

which we interpret as two sections of the new right–hand bundle with fibers homeomorphic to $\Omega^k_S S^{n+k}$. One of these is the zero section, the other is the Euler section $e^{sm}(\xi)$ (and its homotopy class is the Euler class). Letting $k$ tend to infinity, one can interpret the Euler class as an element in the $n$–th cohomology of $X$ with twisted coefficients in the sphere spectrum. This construction is equivalent to one given by Becker [Be]. Becker gives the following application. If $X$ is a CW–space of dimension $\leq 2n - 1$, then the structure group of $\xi$ can be reduced from $O(n)$ to $O(n - 1)$ if and only if the class of $e(\xi)$ is zero.

If $X$ is a compact smooth $n$–manifold, with tangent bundle $\xi$, then the Poincaré dual of the Euler class is in the zero–th homology of $X$ with untwisted coefficients in the sphere spectrum. Actually, there is no compelling reason for passing to homotopy classes. The Poincaré dual of the Euler section of the tangent bundle of $X$ is sufficiently well defined as a point in $\Omega^\infty(X_+ \wedge \mathbb{S}^0)$.

**Euler classes of euclidean bundles as obstructions.** Suppose that $X$ is a space equipped with a fiber bundle $\xi$ whose fibers are homeomorphic $\mathbb{R}^n$ for some $n$ (a euclidean bundle for short). Proceeding as above, we get an inclusion of bundles

$$
\begin{align*}
\frac{O(\varepsilon^{k+1})}{O(\varepsilon^k)} & \longrightarrow \frac{\text{TOP}(\xi \oplus \varepsilon^{k+1})}{\text{TOP}(\xi \oplus \varepsilon^k)} \\
\downarrow & \downarrow \downarrow \\
X & \to \Omega^k_X (\frac{\text{TOP}(\xi \oplus \varepsilon^{k+1})}{\text{TOP}(\xi \oplus \varepsilon^k)})
\end{align*}
$$

and taking adjoints fiberwise we have

$$
\begin{align*}
\mathbb{S}^0 \times X & \longrightarrow \Omega^k_X (\frac{\text{TOP}(\xi \oplus \varepsilon^{k+1})}{\text{TOP}(\xi \oplus \varepsilon^k)}) \\
\downarrow & \downarrow \downarrow \\
X & \to \Omega^k_X (\frac{\text{TOP}(\xi \oplus \varepsilon^{k+1})}{\text{TOP}(\xi \oplus \varepsilon^k)})
\end{align*}
$$

The corresponding sections of the right–hand bundle are zero section and Euler section $e^{bp}(\xi)$. The Euler section defines a class in the $n$–th cohomology of $X$ with twisted coefficients in a spectrum $A(*)$ whose $i$–th term is the space $\text{TOP}(i + 1)/\text{TOP}(i)$. (Note the similarity with the sphere spectrum, whose $i$–th term is $O(i + 1)/O(i)$.) **Warning:** the twist is not easy to identify.

If $X$ is a CW–space of dimension $\leq (4n - 5)/3$, and $n > 5$, then the structure group of $\xi$ can be reduced from $\text{TOP}(n)$ to $\text{TOP}(n - 1)$ if and only if the class of $e(\xi)$ is zero. See lemma 8.1.
If $X$ is a compact topological $n$–manifold, and $\xi$ is its tangent bundle, then the Poincaré dual of the Euler section is a point in $\Omega^\infty(X_+ \wedge A(\ast))$. As the warning just above suggests, this is harder to prove than the corresponding statement for vector bundles. See 7.3.

It is known that the inclusion of spectra $S^0 \to A(\ast)$ splits up to homotopy. Therefore $e^{tp}(\xi)$ determines $e^{sm}(\xi)$ if $\xi$ is a vector bundle.

**Euler characteristics.** We adopt the Waldhausen point of view right from the start. Accordingly, an Euler characteristic can be associated to any object in a category with cofibrations and weak equivalences $\mathcal{C}$. Such a category has a zero object, and in addition there are notions of short exact sequence (unofficial terminology) and weak equivalence. Let $K_0(\mathcal{C})$ be the abelian group with one generator $[C]$ for each object $C$ of $\mathcal{C}$, and with relations

- $[C_2] = [C_1] + [C_3]$ if there exists a short exact sequence $C_1 \to C_2 \to C_3$;
- $[C_1] = [C_2]$ if there exists a weak equivalence from $C_1$ to $C_2$.

For example, $\mathcal{C}$ might be the category of pointed spaces homotopy equivalent to compact CW–spaces. Then $K_0(\mathcal{C}) \cong \mathbb{Z}$. Each space $Y$ homotopy equivalent to a compact CW–space (but without base point) determines an element $[Y_+]$ in $K_0(\mathcal{C})$, where the subscript $+$ denotes an added disjoint base point. This element is the Euler characteristic of $Y$.

Waldhausen [Wald3] noted that the defining relations for $K_0(\mathcal{C})$ could also be used, with more care, as defining relations for a space $K(\mathcal{C})$ with an addition law. The addition law is not strictly commutative, but it does have the properties needed to make $K(\mathcal{C})$ into an infinite loop space. In the case where $\mathcal{C} = \mathcal{C}_*$ is the category of pointed spaces homotopy equivalent to compact CW–spaces, and $Y$ is a space homotopy equivalent to a compact CW–space, we can define the Euler characteristic of $Y$ as $[Y_+] \in K(\mathcal{C}_*)$ (not a component, but a point).

For a further refinement, let $\mathcal{C}_Y$ be the category of retractive spaces over $Y$ (subject to certain finiteness conditions). This is a category with cofibrations and weak equivalences. We (re–)define the Euler characteristic as

$$[Y \amalg Y] \in K(\mathcal{C}_Y).$$

(In the body of the paper, this will be denoted by $\langle Y \rangle$, not $\chi(Y)$, to avoid ambiguity.) Obviously this definition of Euler characteristic lifts the preceding one, in the sense that the map from $Y$ to a point $*$ induces a functor $\mathcal{C}_Y \to \mathcal{C}_*$ and ultimately a map $K(\mathcal{C}_Y) \to K(\mathcal{C}_*)$ which takes the last definition to the preceding definition.

Waldhausen’s notation is $K(\mathcal{C}_Y) = A(Y)$, and we denote the corresponding spectrum by $A(Y)$. This notation is in apparent conflict with our earlier definition of $A(\ast)$ as a spectrum with $i$–th term $\text{TOP}(i + 1)/\text{TOP}(i)$. It is a deep theorem of Waldhausen that the two definitions are homotopy equivalent (by a specific homotopy equivalence).

**Assembly.** Suppose that $F$ is a homotopy invariant functor from compact CW–spaces to spectra. More generally, $F(Y)$ might be defined for any $Y$ homotopy equivalent to a compact CW–space. Then there exists an essentially unique assembly transformation

$$\alpha_Y : Y_+ \wedge F(\ast) \longrightarrow F(Y).$$
which is the obvious isomorphism when $Y$ is a point. (Strictly speaking, the domain of the assembly transformation is not exactly what is written; it is naturally homotopy equivalent to what is written.) See [WWAs] for details. In modern treatments of index theory, assembly often appears as the vehicle by which one passes from the space in which the symbols live to the space in which the analytic indices live.

0.1. Theorem (Index Theorem). Let $M$ be a compact topological manifold with boundary. The assembly

$$\alpha_M : \Omega^\infty (M_+ \wedge A(\ast)) \longrightarrow A(M)$$

carry the Poincaré dual $D(e(\tau^M))$ of $e(\tau^M)$ to the Euler characteristic $\langle M \rangle$ of $M$, in the sense that there is a canonical path in $A(M)$ between $\alpha(D(e(\tau^M)))$ and $\langle M \rangle$. This statement is also true for families (bundles of compact topological manifolds).

In the family version, the index theorem implies the Riemann–Roch formula (0–3), as follows. The assumptions on $p : E \to B$ are as in (0–3). There is a commutative diagram In the lefthand column of the diagram, the unions are taken over all fibers $F$ of $p : E \to B$. There is some topology involved, so that for example $\bigcup A(F)$ is the total space of a fibration on $B$ with fibers of the form $A(F)$. The arrows labeled $\langle p \rangle$ and $D(e(\tau))$ are obtained by taking Euler characteristic and Poincaré dual of Euler class of the tangent bundle fiberwise, respectively. Maps labeled $\iota$ are induced by the unit map from the sphere spectrum to the ring spectrum $A(\ast)$. Maps labeled $\alpha$ are assembly maps or fiberwise assembly maps. The map from $A(E)$ to $K(R)$ is obtained by associating to a retractive space $Y$ over $E$ its relative singular chain complex $C(Y,E;V)$ (coefficients in $V$, pulled back from $E$ to $Y$ by means of the retraction).

The lower chain in the diagram, from $B$ to $K(R)$, represents the righthand side $\sum(-1)^i[V_i]$ of the Riemann–Roch formula (0–3). The upper diagonal arrow labeled $D(e(\tau))$ followed by the top horizontal arrow is the homotopy transfer of Becker–Gottlieb and Dold, from $B$ to $\Omega^\infty \Sigma^\infty (E_+)$. See Appendix A. The arrow from $\Omega^\infty \Sigma^\infty (E_+)$ to $K(R)$ is determined by the other arrows in the diagram; it is a map of infinite loop spaces, so it is enough to know its restriction to $E$. This is easily seen to be the map $[V]$. □

The map $B \to \bigcup \Omega^\infty (F_+ \wedge S^0) := \bigcup Q(F_+)$ is a mild refinement of the Becker–Gottlieb/Dold transfer $B \to Q(E_+)$ which can still be defined in homotopy invariant terms. That is, it can be defined for any fibration $p : E \to B$ where the fibers are homotopy equivalent to compact CW–spaces and the base is a CW–space. As such we still denote it by $\text{tr}$. It should be clear from the sketch proof above that the following is on the one hand a corollary of the index theorem above, and on the other hand, a refinement of the Riemann–Roch formula (0–3).

0.2. Corollary (The A-theory Riemann–Roch formula). Suppose that $p : E \to B$ is a smooth bundle with compact fibers $F$. The fiberwise Euler characteristic $\chi : B \to \bigcup A(F)$ is homotopic (by a canonical homotopy) to the composition

$$B \xrightarrow{\text{tr}} \bigcup Q(F_+) \xrightarrow{\iota} \bigcup A(F).$$
This last version of Riemann–Roch has a converse. Suppose again that \( p : E \to B \) is merely a fibration where the fibers are homotopy equivalent to compact CW–spaces. By a \textit{structure} on \( p \), we mean a fiber homotopy equivalence \( E' \to E \), where \( E' \) is the total space of a bundle on \( B \) with smooth compact fibers. We can make a “moduli space” of such structures, \( S^{\text{sm}}(p) \). We can make another space, \( L^{\text{sm}}(p) \), which is the homotopy pullback of

\[
B \xrightarrow{(p)} \bigcup A(F) \leftarrow \bigcup Q(F_+).
\]

From the \( A \)–theory Riemann–Roch formula, or directly from the index theorem, we get a map \( S^{\text{sm}}(p) \to L^{\text{sm}}(p) \).

\textbf{0.3. Theorem.} The map \( S^{\text{sm}}(p) \to L^{\text{sm}}(p) \) is a homotopy equivalence.

The structure space \( S^{\text{sm}}(p) \) has a topological analog \( S^{\text{tp}}(p) \) (replace all smooth compact manifolds in sight by topological compact manifolds). From the index theorem, we get a map \( S^{\text{tp}}(p) \to L^{\text{tp}}(p) \) where \( L^{\text{tp}}(p) \) is the homotopy pullback of

\[
B \xrightarrow{(p)} \bigcup A(F) \leftarrow \bigcup \Omega^\infty(F_+ \wedge A(\ast)).
\]

\textbf{0.4. Theorem.} The map \( S^{\text{tp}}(p) \to L^{\text{tp}}(p) \) is a homotopy equivalence.

\textbf{0.5. Remark (on Reidemeister torsion).} Suppose that \( p : E \to B \) is a fibration with fibers \( F_b \) homotopy equivalent to compact CW–spaces. Suppose also that for each \( b \in B \) and each \( i \geq 0 \) the homology group \( H_i(F_b; V) \) is zero. Equivalently, the singular chain complexes of the fibers, with coefficients in \( V \), are contractible. Then the image of the Euler characteristic \( \langle F_b \rangle \) under the composition

\[ A(F_b) \xrightarrow{\subseteq} A(E) \xrightarrow{\lambda_V} K(R) \]

is “trivialized”, in other words: we have lifted \( \langle F_b \rangle \) to

\[ (0–5) \text{ hofiber } [A(F_b) \to \cdots \to K(R)] \]

where \textit{hofiber} means homotopy fiber. We call this lift the \textit{homotopy Reidemeister torsion} of \( F_b \). Varying \( b \), we have the parameterized homotopy Reidemeister torsion, a refinement of \( \langle p \rangle \) in (0–4). It is a section of a certain fibration on \( B \) whose fiber over \( b \in B \) is the space \( (0–5) \), for each \( b \). If in addition \( p : E \to B \) is a smooth fiber bundle with compact fibers, then by theorem 0.1 or diagram (0–4) we have a more subtle refinement of \( \langle p \rangle \): a section of a fibration on \( B \) whose fiber over \( b \in B \) is

\[ \text{hofiber } [Q((F_b)_+) \xrightarrow{\iota} \cdots \to K(R)] \]

for each \( b \). This would be called the parameterized \textit{smooth Reidemeister torsion} of the smooth bundle \( p \). Note that it depends on \( V \).

Earlier, Igusa and Klein \cite{IgK} used parameterized generalized Morse functions to define the parameterized Reidemeister torsion of a smooth fiber bundle. Theirs
belongs to exactly the same section space as ours. Our philosophy is much closer to that of [BiLo], in fact identical to it. However, it is not clear to us at this stage whether the parameterized Reidemeister torsions produced by [BiLo] are in agreement with ours. Put differently, we don’t know whether the explicit homotopy they produce to establish (0–2) (given by the differential form $T_*$ in [BiLo, Thm.0.2]) is always in agreement with the one we produce. It certainly is in agreement if the base space of the fiber bundle is a point; this is the theorem of Cheeger and Müller, [Che], [Mü]. Put differently again, our theorem 0.1 and [BiLo] together make it possible to state a family version of the Cheeger–Müller theorem, and we would like to know whether this is in fact true.

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Why is this paper so long? One of the biggest challenges for a topologist who wants to use algebraic $K$-theory is the need to pass from continuously varying geometric information (moduli) to the type of discrete information that the algebraic $K$-theory functor “as is” can process. The question is: Should geometry be made discrete, or should algebraic $K$-theory be made continuous? Our instincts may tell us that the second option is better, but experience and tradition and the sheer nastiness of algebraic $K$-theory have decided in favor of the first option.

The good news is that geometry can sometimes be made discrete. Suppose for example that $M$ is a closed topological manifold. Let $\text{TOP}(M)$ be the topological group of homeomorphisms from $M$ to $M$, and let $\text{TOP}^\delta(M)$ be the underlying discrete group. A theorem due to McDuff–Mather–Thurston–Segal [McD] states that the inclusion of classifying spaces

$$|\text{TOP}^\delta(M)| \longrightarrow |\text{TOP}(M)|$$

is a homology equivalence. This result and some variations of it are central to the present paper, and they explain why topological manifolds appear in the proof of a result about smooth manifolds, such as (0–3).

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Section headings

1. Fibrations and Fiber Bundles
2. Euler characteristics and microcharacteristics
3. Poincaré duality by scanning
4. Discrete models of tangent bundles
5. Microcharacteristics for locally compact spaces
6. The index theorem
7. Euler sections revisited
8. Smoothing theory and Euler sections
Abstract of §1. Let $|S|$ be the classifying space for fibrations $p : E \to B$ where $B$ is a CW–space, the fibers are homotopy equivalent to compact CW–spaces, and $E$ is equipped with a microbundle $\gamma$. Let $|T_n|$ be the classifying space for fiber bundles $p : E \to B$ where the fibers are compact topological $n$–manifolds, and let $|T|$ be the direct limit of the $|T_n|$ under stabilization, i.e. product with $[0, 1]$. A compact manifold $M^n$ determines a fibration $M \to \ast$, which is classified by a map $\ast \to |S|$. Hence $M$ determines a point in $|S|$. What is the homotopy fiber of the inclusion $|T| \hookrightarrow |S|$ over this point? Answer: it is homotopy equivalent to $\Omega \text{Wh}^{\text{TOP}}(M)$, the stabilized $h$–cobordism space of $M$.

Abstract of §2. Part 2.1 explains how Euler characteristics (as defined above) can be extracted from a family of spaces which are homotopy equivalent to finite CW–spaces. Here family refers to the set of fibers of a fibration $p : E \to B$.

Part 2.2 explains why this family–wise Euler characteristic sometimes looks like the righthand side of the Riemann–Roch formula (0–3).

Part 2.3 explains what assembly is about, and introduces a more refined Euler characteristic, called microcharacteristic, for compact euclidean neighborhood retracts (ENR’s). This lives in the domain of an assembly map, and assembly takes it to the ordinary Euler characteristic. The microcharacteristic can be extracted family–wise when the family is the set of fibers of a bundle; this uses the McDuff et al. theorem.

The microcharacteristic concept makes it possible to state an index theorem which is stronger than theorem 0.1. Namely, the Poincaré dual of the Euler section of a compact topological manifold lives in the same space as the microcharacteristic of the manifold, and the two are connected by a canonical path. We refer to this as the micro–index theorem. It is also valid for families. We prove it in §§3–7.

Part 2.4 uses §1 and 2.3 to give a direct construction of the map in theorem 0.4 which does not use the index theorem. The map is then shown to be a homotopy equivalence. In this sense, theorem 0.4 is proved in §2 and should therefore be regarded as much easier than theorem 0.3.

Abstract of §3. This gives a geometric description of Poincaré duality, or rather inverse Poincaré duality, for a compact manifold. The description is geometric in the sense that it uses local properties of the manifold.

Abstract of §4. Let $M$ be an $n$–manifold, let $\mathcal{C}$ be the (discrete) set of embeddings $\mathbb{R}^n \to M$, and let $\mathcal{D}$ be the discrete monoid of embeddings $\mathbb{R}^n \to \mathbb{R}^n$. Then $\mathcal{D}$ acts on $\mathcal{C}$. The map $\mathcal{C} \to \ast$ induces a map of homotopy orbit spaces (Borel constructions) $\mathcal{C}_{h\mathcal{D}} \longrightarrow *_{h\mathcal{D}} = |\mathcal{D}|$. 

\[ \mathcal{C}_{h\mathcal{D}} \longrightarrow *_{h\mathcal{D}} = |\mathcal{D}|. \]
We identify this map, up to homotopy equivalence, with the map \( M \to B\TOP(n) \) which classifies the tangent microbundle. The identification \( |\mathcal{D}| \simeq B\TOP(n) \) is due to McDuff and Segal.

**Abstract of §5.** We show that microcharacteristics can also be defined for non-compact ENR’s. In particular, euclidean space \( \mathbb{R}^n \) has a microcharacteristic. The micro–index theorem (see abstract of §2) can now be stated for non–compact manifolds.

**Abstract of §6.** We reduce the proof of the micro–index theorem for topological manifolds with *trivial ends* to the case of euclidean space. (A manifold has trivial ends if it can be obtained from a compact manifold by deleting some components of the boundary.) We obtain this reduction, not by embedding all manifolds in sight in a euclidean space, but by embedding euclidean spaces in the manifolds.

**Abstract of §7.** Here we prove the micro–index theorem for euclidean space and euclidean upper half space. We have to do this for families of euclidean spaces and euclidean half–spaces, so that the theorem to be proved turns into a statement about equality of certain characteristic classes (better: characteristic sections) for euclidean bundles. Technically, this is the hardest part of the paper.

Using §6, we then have the micro–index theorem for manifolds with trivial ends and therefore the index theorem 0.1 for compact manifolds. Using §1 and §2.4, we also complete the proof of 0.4. The A–theory Riemann–Roch formula (Corollary 0.2) is proved in §7.3.

**Abstract of §8.** This uses smoothing theory to deduce theorem 0.3 from theorem 0.4 and Corollary 0.2.

**Abstract of §9.** Using the results of §2.2, we explain in more detail how the A–theory Riemann–Roch formula (Corollary 0.2) implies the \( K \)-theory Riemann–Roch formula (0–3).

**Abstract of Appendix A–C.** Appendix A is a collection of folklore related to the homotopy transfer of Becker–Gottlieb–Dold. Appendix B and Appendix C are background to §7.

1. **Fibrations and Fiber Bundles**

1.1. **Notation.** We introduce several topological categories. Each of these has a discrete class of objects. The morphism spaces should be regarded as simplicial sets. The categories are quite large, so that their geometric realizations tend to be very large objects to which we nevertheless refer informally as *spaces*.

- The category \( \mathcal{T}_n \) has as objects the compact \( n \)-manifolds \( M \). The space (simplicial set) of morphisms from \( M \) to \( M' \) in \( \mathcal{T}_n \) is the space of homeomorphisms \( M \to M' \). An \( i \)-simplex in \( \text{mor}(M, M') \) is therefore a homeomorphism \( \Delta^i \times M \to \Delta^i \times M' \) respecting the projections to \( \Delta^i \).

- The category \( \mathcal{E}_n \) has the same objects as \( \mathcal{T}_n \), but the space of morphisms from \( M \) to \( M' \) in \( \mathcal{E}_n \) is the space of *invertible* embeddings from \( M \) to \( M' \). An embedding (=injective continuous map) \( e : M \to M' \) is *invertible* if there exists another embedding \( f : M' \to M \) such that \( fe \) and \( ef \) are isotopic to the identity embeddings of \( M \) and \( M' \), respectively.
• The category \( \mathcal{G}_n \) has as objects the compact ENR’s equipped with an \( n \)-microbundle \( \gamma \). The space of morphisms from \( (F, \gamma) \) to \( (F', \gamma') \) is the space of homotopy equivalences \( F \to F' \) covered by microbundle maps \( \gamma \to \gamma' \).

1.2. More notation and some motivation. Note that \( \mathcal{T}_n \subset \mathcal{E}_n \subset \mathcal{G}_n \) where the second “inclusion” is defined by \( M \mapsto (M, \tau^M) \). Product with the identity from \( I \) to \( I \) gives compatible stabilization functors from \( \mathcal{T}_n \) to \( \mathcal{T}_{n+1} \), from \( \mathcal{E}_n \) to \( \mathcal{E}_{n+1} \), and from \( \mathcal{G}_n \) to \( \mathcal{G}_{n+1} \). Let

\[
\mathcal{T}, \mathcal{E}, \mathcal{G}
\]

be the resulting stabilized categories. (For example, an object in \( \mathcal{G} \) is a triple \( (m, F, \gamma) \) where \( m \geq 0 \) and \( (F, \gamma) \) is an object in \( \mathcal{G}_m \). A morphism in \( \mathcal{G} \) from \( (m, F, \gamma) \) to \( (n, F', \gamma') \) is a morphism in \( \mathcal{G}_n \) from \( (F \times I^{n-m}, \gamma \times \varepsilon^{n-m}) \) to \( (F', \gamma') \) provided \( n \geq m \), and if \( n < m \) there is no morphism.) Then

\[
|\mathcal{T}| \simeq \hocolim_n |\mathcal{T}_n|, \\
|\mathcal{E}| \simeq \hocolim_n |\mathcal{E}_n|, \\
|\mathcal{G}| \simeq \hocolim_n |\mathcal{G}_n|
\]

by inspection or by [Tho1]. Here the vertical bars denote geometric realizations alias classifying spaces. (The prefix \( B \) for classifying spaces will not be used unless tradition absolutely requires it.)

Note that \( |\mathcal{T}_n| \) classifies (up to isomorphism) fiber bundles whose fibers are compact \( n \)-manifolds. \( |\mathcal{G}_n| \) classifies (up to fiber homotopy equivalence) fibrations \( p : E \to B \) where the fibers are homotopy equivalent to compact CW–spaces and where \( E \) is equipped with an \( n \)-microbundle \( \gamma \). If \( p : E \to B \) is such a fibration, classified by \( c : B \to |\mathcal{G}_n| \), then the homotopy pullback of

\[
|\mathcal{T}| \leftarrow |\mathcal{G}_n| \leftarrow |\mathcal{T}_n|
\]

can be regarded as a space of *structures* on \( p \) and \( \gamma \). We denote it by \( \mathcal{S}_n^{tp}(p, \gamma) \). When \( \gamma \) is the trivial bundle, we write \( \mathcal{S}_n^{tp}(p) \).

There are similar but more stable interpretations for \( |\mathcal{T}| \) and \( |\mathcal{G}| \). Loosely speaking, \( |\mathcal{T}| \) classifies fiber bundles with compact manifold fibers, and \( |\mathcal{G}| \) classifies fibrations \( p : E \to B \) where the fibers are homotopy equivalent to compact CW–spaces and where \( E \) is equipped with a *stable* microbundle \( \gamma \). If \( p : E \to B \) is such a fibration, classified by \( c : B \to |\mathcal{G}| \), then the homotopy pullback of

\[
|\mathcal{T}| \leftarrow |\mathcal{G}| \leftarrow |\mathcal{T}|
\]

can be regarded as a space of *stabilized structures* on \( p \) and \( \gamma \). We denote it by \( \mathcal{S}_n^{tp}(p, \gamma) \). When \( \gamma \) is the trivial bundle, we write \( \mathcal{S}_n^{tp}(p) \). An easy disk bundle stabilization argument shows that \( \mathcal{S}_n^{tp}(p, \gamma) \simeq \mathcal{S}_n^{tp}(p) \) for any \( \gamma \).

1.3. Proposition. The inclusion \( |\mathcal{E}| \to |\mathcal{G}| \) is a homotopy equivalence.

1.4. Proposition. The homotopy fiber of \( |\mathcal{T}_n| \leftarrow |\mathcal{E}_n| \) over an object \( M^n \) is homotopy equivalent to the \( h \)-cobordism space \( |\mathcal{H}(\partial M)| \).
1.5. Corollary. The homotopy fiber of $|\mathcal{F}| \hookrightarrow |\mathcal{E}|$ over an object $M^n$ is homotopy equivalent to the stabilized $h$–cobordism space

$$\text{hocolim}_k |\mathcal{H}(M \times I^k)| = \Omega \text{Wh}^{\text{TOP}}(M)$$

where $\mathcal{H}(M)$ is the topological category of $h$–cobordisms over $M$.

1.6. Remarks. (i) Strictly speaking we define $\mathcal{H}(\partial M)$ in 1.4 as the topological category of invertible cobordisms on $\partial M$. By definition, a cobordism $(W; \partial M, L)$ is invertible if there exist cobordisms $(V_1; L, \partial M)$ and $(V_2; L, \partial M)$ such that $W \sqcup_L V_1$ and $V_2 \sqcup_{\partial M} W$ are product cobordisms. Invertible cobordisms are $h$–cobordisms [Stal, Thm.2]. We do not know whether the converse is true, but we do know that any counterexample to the three–dimensional Poincaré conjecture gives rise to a non–invertible $h$–cobordism with base $S^2$ (delete two small open disks from the fake 3–sphere).

(ii) Invertible cobordisms arise in the proof of 1.4 as follows: Let $e : M \to M'$ be an invertible embedding, and suppose that $e(M)$ is contained in the interior of $M$ (both $M$ and $M'$ are compact). Then the closure of $M' \setminus e(M)$ is an invertible cobordism from $\partial M$ to $\partial M'$. The proof uses some isotopy extension theory as in [Ce] or [EdKi]. We are indebted to Bill Browder for pointing this out to us.

(iii) Let $(W; K, L)$ be an invertible cobordism $(K$ and $L$ closed). The space of embeddings of $W$ in $K \times [0, 1)$ extending the inclusion of $K \cong K \times \{0\}$ in $K \times [0, 1)$ is contractible. (Use invertibility and the uniqueness of collars, [Bro]).

1.8. Remark. The topological categories $\mathcal{F}_n$ and $\mathcal{F}$ are clearly topological groupoids. Also, $\mathcal{E}_n$, $\mathcal{E}$, $\mathcal{S}_n$ and $\mathcal{S}$ are groupoid–like, that is, they become groupoids when all morphism spaces in sight are replaced by their $\pi_0$.

Proof of 1.3. Let $(F, \gamma)$ be an object of $\mathcal{S}$. Let $M, M'$ be objects of $\mathcal{E}_n$, for some large $n$, such that both $M$ and $M'$ are in the same component of $\mathcal{S}$ as $(F, \gamma)$. Then $M \times I^k$ and $M' \times I^k$ are tangentially homotopy equivalent for large $k$, and it follows easily that there exists an invertible embedding of $M \times I^k$ in $M' \times I^k$. (Compare [Maz] ; use topological immersion theory [Gau] and a general position argument.) Hence the inclusion $|\mathcal{E}| \hookrightarrow |\mathcal{S}|$ is a bijection on components, and we may restrict attention to the component of $|\mathcal{E}|$ containing a particular object $M^n$, say, and to the image component in $|\mathcal{S}|$. Since $\mathcal{E}$ and $\mathcal{S}$ are groupoid–like, these components are homotopy equivalent to the classifying spaces $|\text{end}_\mathcal{E}(M)|$ and $|\text{end}_\mathcal{S}(M)|$. Again, topological immersion theory and a general position argument show that the inclusion $|\text{end}_\mathcal{E}(M)| \hookrightarrow |\text{end}_\mathcal{S}(M)|$ is a homotopy equivalence. Hence the inclusion of classifying spaces is also a homotopy equivalence.

Remark. The classifying space $|\text{end}_\mathcal{E}(M)|$ is homotopy equivalent to the classifying space of the group of stabilized homeomorphisms $\text{int}(M) \to \text{int}(M)$. Taking this into account, we can say that 1.3 is a variant on the open fiber smoothing theorem of Casson and Gottlieb [CaGo].

Proof of 1.4. We shall use Quillen’s Theorem B [Qui], or rather Waldhausen’s simplicial version of it [Wald2, §4], with the “addendum”. We need the following special case: Let $\mathcal{Z} : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between simplicial categories.
Suppose that the simplicial sets \( \text{ob}(\mathcal{C}) \) and \( \text{ob}(\mathcal{D}) \) are discrete. For an object \( D \) in \( \mathcal{D}[0] \), let \( \mathcal{Z}/D \) be the simplicial category whose objects in degree \( n \) are pairs \((C, f)\) where \( C \) is an object of \( \mathcal{C}[0] \) and \( f : \mathcal{Z}(C) \to D \) is a morphism in \( \mathcal{D}[n] \) (that is, a morphism from the \( n \)-fold degeneracy of \( \mathcal{Z}(C) \) to the \( n \)-fold degeneracy of \( D \)). The morphisms in \( \mathcal{Z}/D \) are commutative triangles. Suppose that, for any morphism \( e : D \to D' \) in \( \mathcal{D}[0] \), the transition functor

\[ e_* : \mathcal{Z}/D \to \mathcal{Z}/D' \quad ; \quad (C, f) \mapsto (C, ef) \]

induces a homotopy equivalence of the classifying spaces, \( |(\mathcal{Z}/D)| \simeq |(\mathcal{Z}/D')| \). Then the square

\[
\begin{array}{ccc}
|\mathcal{Z}/D| & \xrightarrow{\text{forget}} & |\mathcal{C}| \\
\downarrow \mathcal{Z} & & \downarrow \mathcal{C} \\
|\text{id}_{\mathcal{D}/D}| & \xrightarrow{\text{forget}} & |\mathcal{D}|
\end{array}
\]

is a homotopy pullback square (and, of course, the lower left-hand term is contractible).

We apply all this with \( \mathcal{C} = \mathcal{F}_n, \mathcal{D} = \mathcal{E}_n \), and \( D = M^n \), and \( \mathcal{Z} \) equal to the inclusion functor. Let \( \mathcal{F}_1 \) be the homotopy functor represented by the space \(|(\mathcal{Z}/M)|\). Then for any CW–space \( X \), an element in \( \mathcal{F}_1(X) \) is represented by a pair \((q : E \to X, \beta)\) where \( q \) is a fiber bundle over \( X \) with compact \( n \)-manifold fibers, and \( \beta \) is a map from \( E \) to \( M \) such that \( \beta \) restricted to any fiber of \( p \) is an invertible embedding. Two such pairs \((q_0, \beta_0)\) and \((q_1, \beta_1)\) are equivalent if there exists a bundle isomorphism \( \iota : E_0 \to E_1 \) such that \( \beta_1 \iota \) is isotopic to \( \beta_0 \). In particular, every element in \( \mathcal{F}_1(X) \) has a representative \((q, \beta)\) such that \( \text{im} (\beta) \cap \partial M = \emptyset \). We call such a representative regular.

Let \( \mathcal{F}_2 \) be the homotopy functor represented by \(|\mathcal{H}(\partial M)|\). For any CW–space \( X \), an element in \( \mathcal{F}_1(X) \) is represented by an \( h \)-cobordism bundle, i.e. a fiber bundle \( \eta : H \to X \) such that \( \eta \) contains as a subbundle the product bundle \( X \times \partial M \to X \) and such that each fiber of \( \eta \) is an (invertible) \( h \)-cobordism with base the copy of \( \partial M \) in that fiber. Another such bundle \( \eta_1 \) represents the same element if there exists a bundle isomorphism \( \eta \to \eta_1 \) extending the identity on the common subbundles \( X \times \partial M \to X \).

Define a natural transformation \( \psi : \mathcal{F}_1 \cong \mathcal{F}_2 \) as follows. Send an element in \( \mathcal{F}_1(X) \) with regular representative \((q : E \to X, \beta)\) to the class of the \( h \)-cobordism bundle on \( X \) whose total space is the closure of the complement of the image of

\[ (q, \beta) : E \to X \times M. \]

The fibers of this so-called \( h \)-cobordism bundle are indeed invertible \( h \)-cobordisms, by remark 1.6, item (ii). It follows from 1.6, item (iii), that \( \psi \) is a natural bijection, so that

\[ |(\mathcal{Z}/M)| \simeq |\mathcal{H}(\partial M)|. \]

Now we can easily verify that the hypothesis of Quillen’s Theorem B is satisfied. Let \( e : M \to M' \) be a morphism in \( \mathcal{E}_n \). The homotopy class of the transition map

\[ e_* : |(\mathcal{Z}/M)| \cong |\mathcal{H}(\partial M)| \quad \to \quad |(\mathcal{Z}/M')| \cong |\mathcal{H}(\partial M')| \]
will only depend on the connected component of \( e \), so we may assume that \( e \) is regular. The closure of the complement of \( \text{im}(e) \) is then an \( h \)-cobordism \( W \) from \( \partial M \) to \( \partial M' \), and \( e_* \) is simply concatenation with this \( h \)-cobordism, as a map from \( |\mathcal{H}(\partial M)| \) to \( |\mathcal{H}(\partial M')| \). Since \( W \) is invertible, \( e_* \) is a homotopy equivalence. Applying Theorem B now, we find that the homotopy fiber in 1.4 is homotopy equivalent to the classifying space of the categorical fiber, which is \(|(\mathbb{Z}/M)| \simeq |\mathcal{H}(\partial M)|\). □

**Remark.** By the remark following the proof of 1.3, the component of \( |E_n| \) containing the object \( M^n \) classifies fiber bundles with fibers homeomorphic to \( \text{int}(M) \). The corresponding component of \( |T_n| \) classifies fiber bundles with fibers homeomorphic to \( M \). This shows that 1.4 becomes the Kuiper–Lashof theorem \([KuiLa1],[KuiLa2]\) when \( M = \mathbb{D}^n \). See also \([Cm]\).

**Proof and explanation of 1.5.** The space \( \Omega\text{Wh}^{\text{TOP}}(M) \) may be defined as the homotopy colimit of the spaces \( |\mathcal{H}(M \times I^k)| \), for \( k \to \infty \), under the upper stabilization maps

\[
|\mathcal{H}(M \times I^k)| \longrightarrow |\mathcal{H}(M \times I^{k+1})|
\]

described in \([Wald2]\). It turns out that \( \Omega\text{Wh}^{\text{TOP}}(M) \) is a homotopy invariant functor in the variable \( M \). For more precision and details, see \([BuLa],[Wald2]\) and \([Wald3]\). From 1.4, we see that the homotopy fiber \( \Phi \) in 1.5 can be described as the homotopy colimit (here telescope) of a diagram

\[
|\mathcal{H}(\partial M)| \to |\mathcal{H}(\partial (M \times I))| \to |\mathcal{H}(\partial (M \times I^2))| \to |\mathcal{H}(\partial (M \times I^3))| \to \cdots
\]

where the maps are given as follows. Go from \( |\mathcal{H}(\partial (M \times I^k))| \) to \( |\mathcal{H}(\partial (M \times I^k) \times I)| \) by upper stabilization; then go from there to \( |\mathcal{H}(\partial (M \times I^{k+1}))| \) using the map induced by the inclusion

\[
\partial (M \times I^k) \times I \subset \partial (M \times I^{k+1})
\]

Since upper stabilization commutes with inclusion–induced maps, we conclude that

\[
\Phi \simeq \text{holim}_k \text{holim}_j |\mathcal{H}(\partial (M \times I^k) \times I^j)| = \text{holim}_k \Omega\text{Wh}^{\text{TOP}}(\partial (M \times I^k)).
\]

Since \( \Omega\text{Wh}^{\text{TOP}} \) is a homotopy functor, and commutes up to homotopy equivalence with the special homotopy colimits that we are using (telescopes), we find

\[
\Phi \simeq \Omega\text{Wh}^{\text{TOP}}(\text{holim}_k (\partial (M \times I^k))) \simeq \Omega\text{Wh}^{\text{TOP}}(M). \quad \square
\]

2. Euler Characteristics and Microcharacteristics

Our goal in this section is to prove Theorem 0.4 in an easier formulation. See the section abstracts. The reader should have an intuitive understanding of homotopy direct and homotopy inverse limits \([BK],[HoVo]\). See also \([WWAs]\). Our notation is: holim for homotopy inverse limits alias homotopy projective limits, and hololim
for homotopy direct limits. Another fundamental concept that we use freely is the $K$-theory of a category with cofibrations and weak equivalences, [Wald3].

Terminology: In this section, *space* means a space homotopy equivalent to a CW–space. A space is *homotopy finite* if it is homotopy equivalent to a compact CW–space. A retractive space over a space $X$ consists of a space $Y$ and maps $s : X \to Y$, $r : Y \to X$ such that $rs = \text{id}_X$ and $s$ is a closed embedding having the homotopy extension property. A map over $X$ and relative to $X$ between retractive spaces over $X$ is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property. It is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence. With these notions of cofibration and weak equivalence, the category of retractive spaces over $X$ is indeed a category with cofibrations and weak equivalences, in the sense of [Wald 3]. This follows from [Str]. A retractive space over $X$ is *homotopy finite* if it is the codomain of a weak equivalence from another retractive space over $X$ which is a CW–space relative to $X$, with finitely many cells. Define $A(X)$ as the $K$-theory of the category of homotopy finite retractive spaces over $X$. This must be regarded as a simplicial *class*, since we have made no effort to downsize the category of homotopy finite retractive spaces over $X$. Even so, a small effort is required to make $A(X)$ into a *functor* of the variable $X$. For example: Fix two large (at least uncountable) disjoint sets $U_1$ and $U_2$. Define $A(X)$ only when the set $X$ is a subset of $U_1$, and in that case insist on retractive spaces $Y \rightrightarrows X$ for which the section $X \to Y$ is an inclusion, and $Y \setminus X$ is a subset of $U_2$. Given a map $f : X \to X'$, where $X, X' \subset U_1$ as sets, and a retractive space $Y \rightrightarrows X$, as above, define a new retractive space $f_*Y \rightrightarrows X'$ where $f_*Y = (Y \setminus X) \cup X'$, with the obvious retraction and section. Then $f_*$ is a functor from retractive spaces over $X$ to retractive spaces over $X'$. Most important: the rule $f \mapsto f_*$ respects composition, so that $g_*f_* = (gf)_*$.

2.1. Euler characteristics

Let $Y$ be a homotopy finite space. Any homotopy finite retractive space over $Y$ determines a point in $A(Y)$; specifically, the retractive space

$$S^0 \times Y \xleftarrow{s} Y \xrightarrow{r} Y$$

where $r$ is the projection and $s$ identifies $Y$ with $\{1\} \times Y$, determines a point

$$\langle Y \rangle \in A(Y).$$

2.1.1. Definition. We call $\langle Y \rangle$ the *Euler characteristic* of $Y$. (Note that we are interested in the point $\langle Y \rangle$, not just in its connected component.)

A homotopy equivalence $f : X \to Y$ between homotopy finite spaces induces another homotopy equivalence $f_* : A(X) \to A(Y)$. It also determines a path $\langle f \rangle$ in $A(Y)$ from $f_* \langle X \rangle$ to $\langle Y \rangle$. Namely, $f_* \langle X \rangle$ is the point in $A(Y)$ corresponding to the retractive space

$$\{-1\} \times X \cup \{1\} \times Y \rightrightarrows Y$$
where the retraction is equal to \( f \) on \( \{-1\} \times X \). Now \( f \) gives a weak equivalence from this retractive space to\[S^0 \times Y \cong Y.\]
The weak equivalence determines a path in \( A(Y) \).

We can continue in this manner, looking e.g. at composable sequences of homotopy equivalences. Perhaps the best way to express the naturality properties of Euler characteristics is to use homotopy inverse limits. Let \( p \) be a covariant functor from a small category \( \mathcal{C} \) to homotopy finite spaces. Suppose \( p \) takes all morphisms in \( \mathcal{C} \) to homotopy equivalences. For each \( c \) in \( \mathcal{C} \), we have a functor from the “over” category \( \mathcal{C} \downarrow c \) to the category of retractive spaces over \( p(c) \) which takes \( b \to c \) to the retractive space \( p(b) \amalg p(c) \). Since this functor takes all morphisms in \( \mathcal{C} \downarrow c \) to weak equivalences, it induces a map
\[
(2–1) \quad |\mathcal{C} \downarrow c| \to A(p(c)).
\]
Note that \( c \mapsto A(p(c)) \) is another functor taking all morphisms in \( \mathcal{C} \) to homotopy equivalences. We can regard \((2–1)\) as a natural transformation between functors in the variable \( c \). By definition, such a natural transformation is a point in the homotopy inverse limit
\[
\text{holim}_{c \in \mathcal{C}} A(p(c)).
\]
By inspection, our point projects to \( (p(c)) \in A(p(c)) \), for each \( c \) in \( \mathcal{C} \). We refer to this type of naturality as \textit{lax naturality}. The Euler characteristic \( \langle Y \rangle \) is lax natural with respect to homotopy equivalences.

We like to think of points in the homotopy inverse limit above as “sections” of a certain “fibration”. This is explained in the next construction.

\textbf{2.1.2. Construction.} Let \( \mathcal{C} \) be a small category. Let \( v(c) = |\mathcal{C} \downarrow c| \) for each \( c \) in \( \mathcal{C} \). Suppose that \( q \) is a covariant functor from \( \mathcal{C} \) to spaces taking all morphisms to homotopy equivalences. Form the commutative square
\[
\begin{array}{ccc}
\text{hocolim} q \times v & \longrightarrow & \text{hocolim} q \\
\downarrow v^* q & & \downarrow q \\
\text{hocolim} v & \longrightarrow & \text{hocolim}^* \\
\end{array}
\]
(all homotopy colimits over \( \mathcal{C} \), and \( v \) and \( q \) are the obvious projections). This is a pullback square where the horizontal arrows are homotopy equivalences and the vertical arrows are \textit{quasifibrations} in the sense of [DoTho]. A point in the homotopy \textit{inverse} limit \( \text{holim} q \) is a natural transformation \( t : v \to q \), giving rise to another natural transformation \( (t,\text{id}) : v \to q \times v \), and this induces a \textit{section} \( t_* \) of the quasifibration on the left. Thus we have maps
\[
\text{holim} q \longrightarrow \Gamma(v^* q) \xleftarrow{\sim} \Gamma(q)
\]
where \( \Gamma \) denotes the section spaces of the associated fibrations. \textit{Exercise:} The left–hand map, from \( \text{holim} q \) to \( \Gamma(v^* q) \), is also a homotopy equivalence (spectral sequence argument ; use the skeleton filtration of \(|\mathcal{C}|\)). In this way, \( \text{holim} q \) is an acceptable model or substitute for \( \Gamma(q) \).
2.1.3. Remark. Again let $p$ from $C$ to spaces be a covariant functor taking all morphisms to homotopy equivalences. Since the projection $p : \text{hocolim} \, p \to |C|$ is a quasifibration in the sense of Dold–Thom, we can convert it into a fibration using the Serre construction, without changing the homotopy types of the fibers. To increase the usefulness of this observation we add another: essentially every fibration arises in this way. Indeed, suppose that $p : E \to B$ is a fibration (where $E$ and $B$ are homotopy equivalent to CW–spaces, according to our conventions). Let $\text{simp}(B)$ be the category of singular simplices of $B$. An object of $\text{simp}(B)$ is a map $f : \Delta^j \to B$ for some $j \geq 0$, and a morphism from $f : \Delta^j \to B$ to $g : \Delta^k \to B$ is a monotone map $v$ from $\{1, \ldots, j\}$ to $\{1, \ldots, k\}$ such that $g \circ v = f$. Define $p$ and $w$ from $\text{simp}(B)$ to spaces by $p(f) = f^*E$ for $f : \Delta^j \to B$, and $w(f) = \Delta^j$. There is a commutative diagram

\[
\begin{array}{ccc}
\text{hocolim} \, p & \leftarrow & \text{hocolim} \, p \\
\downarrow \text{proj.} & & \downarrow p \\
\text{hocolim} \, \ast & \leftarrow & \text{hocolim} \, w \\
\downarrow \text{proj.} & & \downarrow \text{eval.} \\
& & B
\end{array}
\]

where middle arrow is induced by the projections $f^*E \to \Delta^j$ for $f : \Delta^j \to B$, and “eval.” is induced by the evaluation transformation from $w$ to the constant functor with value $B$. (All homotopy colimits are over $\text{simp}(B)$.) Both squares in the diagram are homotopy pullback squares, and the maps in the lower row are homotopy equivalences [Se2].

2.1.4. Example. Let $p$ be a functor from $C$ to spaces taking all morphisms to homotopy equivalences, and such that each $p(c)$ is homotopy finite. Then $c \mapsto A(p(c))$ is another functor taking all morphisms to homotopy equivalences. Now write

\[
B = |C|, \quad E := \text{hocolim} \, p, \quad A_B(E) := \text{hocolim} \, A \cdot p.
\]

Let $p : E \to B$ be the projection and write $\langle p \rangle$ for the distinguished element in $\text{holim} \, A \cdot p$ that we get from (2–1). We call $\langle p \rangle$ the Euler section of $p$. Informally, it is a section $B \to A_B(E)$ of the projection $A_B(E) \to B$. Note that $A_B(E)$ is what we called $\bigcup A(F)$ in the introduction, for example in theorem 0.3.

We can write $E = \text{colim} \, p \times v$ (notation of 2.1.2), which explains the second arrow in

\[\text{hocolim} \, A \cdot p \leftarrow \text{hocolim} \, A \cdot p \to A(E).\]

Informally, (2–2) is a map from $A_B(E)$ to $A(E)$. We mention it here because the composition

\[B \xrightarrow{\langle p \rangle} A_B(E) \to A(E)\]

appears implicitly on the right–hand side of [BiLo, Thm. 0.1]. More details are given below in subsection 2.2 and in section 9.
2.2. Linearized Euler characteristics

Keep the notation of 2.1.4. Suppose in addition that $E$ is equipped with a bundle $V$ of finitely generated projective left $R$–modules, where $R$ is some (discrete) ring. We want to take a closer look at the composition

$$(p)_V : B \xrightarrow{(p)} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda} K(R)$$

where the last map, from $A(E)$ to $K(R)$, is induced by a functor $\lambda$ between categories with cofibrations and weak equivalences. For a homotopy finite retractive space

$$X \xleftarrow{s} E$$

let $\lambda(X \rightrightarrows E)$ be the singular chain complex of the pair $(X, E)$ with (twisted) coefficients in the bundle of modules $r^*(V)$. Note that $\lambda(X \rightrightarrows E)$ is a chain complex of projective $R$–modules, homotopy equivalent to a finitely generated one. Such chain complexes form a category with cofibrations and weak equivalences, and its $K$-theory is $K(R)$. More details can be found below.

**Remark.** John Klein points out that the functor $\lambda$ is not an exact functor. It respects cofibrations and weak equivalences, but it does not respect cobase change. For example, suppose that $X \to Y$ is a cofibration (and also an inclusion) in the domain category of $\lambda$. Then there is a canonical map from $\lambda(X)/\lambda(Y)$ to $\lambda(X/Y)$, but the map is rarely an isomorphism. Fortunately however, it is a chain homotopy equivalence. This suggests the remedy: Instead of using Waldhausen’s $S_\bullet$ construction to define the $K$-theory of a category with cofibrations and weak equivalences, use the variation due to Thomason, given at the end of [Wald3, §1.3]. This is natural with respect to functors which are slightly less than exact. In particular, $\lambda$ qualifies.

Now for the alternative description of the composite map $(p)_V$: For each vertex $c$ in $B$ (=object $c$ in $\mathcal{C}$), we have $p(c) \subset E$ as the fiber over $c$. Let $p_V(c)$ be the singular chain complex of $p(c) \subset E$ with (twisted) coefficients in $V$. Now $p_V$ is a functor from $\mathcal{C}$ to certain chain complexes taking all morphisms to homotopy equivalences. Hence it leads to a map $(p_V)$ from $|\mathcal{C}| = B$ to $K(R)$. The following is true by construction.

**2.2.1. Proposition.** $(p)_V \simeq (p_V)$.

Assuming that each $H_i(p_V(c))$ has a finite length resolution by f.g. projective left $R$–modules, we shall give a homology theoretic description of $(p_V)$. Here is some $K$-theory background.

Recall that an exact category is determined by an embedding of an additive category as a full subcategory of an abelian category $\mathcal{A}$ where $\mathcal{M}$ is closed under extensions in $\mathcal{A}$. See [Qui, p.16] and [Tho2, App.A]. A map $f$ in $\mathcal{M}$ is an admissible monomorphism if it is a monomorphism in $\mathcal{A}$ and the cokernel is isomorphic to an object in $\mathcal{M}$. Dually, a map $f$ in $\mathcal{M}$ is an admissible epimorphism if it is an
epimorphism in $\mathcal{A}$ and the kernel is isomorphic to an object in $\mathcal{M}$. The two main examples for us are

$$\mathcal{M} = \mathcal{PA},$$

the category of projective objects in the abelian category $\mathcal{A}$; and

$$\mathcal{M} = \mathcal{NPA},$$

consisting of those objects in the abelian category $\mathcal{A}$ which have finite length resolutions by objects in $\mathcal{PA}$. The letter $N$ can be read as nearly. If $\mathcal{A}$ is the category of finitely generated left modules over a ring $R$, then we write $\mathcal{P}_R$ and $\mathcal{N}_P_R$ instead of $\mathcal{PA}$ and $\mathcal{NPA}$.

Notice that any morphism in $\mathcal{PA}$ which is epi in $\mathcal{A}$ is admissible. A morphism in $\mathcal{PA}$ which is mono in $\mathcal{A}$ is admissible iff it splits. However, all morphisms in $\mathcal{NPA}$ which are epi/mono in $\mathcal{A}$ are admissible. See [Bass, I.6.2]. Quillen’s $Q$-construction associates to an exact category $\mathcal{M}$ an infinite loop space $K(\mathcal{M})$. See [Qui]. Quillen’s resolution theorem [Qui] implies that the inclusion $K(\mathcal{PA}) \rightarrow K(\mathcal{NPA})$ is a homotopy equivalence.

For any category $\mathcal{D}$ with cofibrations $\text{cof} \mathcal{D}$ and weak equivalences $\text{w} \mathcal{D}$, Waldhausen [Wald3] has constructed an infinite loop space $\Omega|wS_\bullet(\mathcal{D})|$ which we shall denote by $K(\mathcal{D})$. If $\mathcal{M}$ is an exact category, we can make $\mathcal{M}$ into a category with cofibrations and weak equivalences, by letting $\text{cof} \mathcal{M}$ be the admissible monomorphisms in $\mathcal{M}$ and by letting $\text{w} \mathcal{M}$ be the isomorphisms in $\mathcal{M}$. Then there is natural equivalence from $K(\mathcal{M})$ in the sense of Quillen to $K(\mathcal{M})$ in the sense of Waldhausen [Wald3, 1.9], [G, 9.3].

For geometric applications we want to have a chain complex theoretic description of $K(\mathcal{M})$. Let $\text{ch}(\mathcal{M})$ be the category of chain complexes in $\mathcal{M}$ which are graded over $\mathbb{Z}$ and bounded above and below. Homology is defined by first mapping the chain complex into the abelian category $\mathcal{A}$. We make $\text{ch}(\mathcal{M})$ into a category with cofibrations and weak equivalences by letting $\text{cof} \text{ch}(\mathcal{M})$ be the chain maps which are degreewise admissible monomorphisms, and by letting $\text{w} \text{ch}(\mathcal{M})$ be the quasi-isomorphisms, i.e. chain maps which induce isomorphisms in homology. The chain complexes concentrated in degree zero form a full subcategory which we identify with $\mathcal{M}$. In many cases the inclusion of Waldhausen $K$-theories, $K(\mathcal{M}) \rightarrow K(\text{ch}(\mathcal{M}))$, is a homotopy equivalence.

2.2.2. Example. The commutative square of inclusion maps

$$
\begin{array}{c}
K(\mathcal{PA}) \longrightarrow K(\text{ch}(\mathcal{PA})) \\
\downarrow \\
K(\mathcal{NPA}) \longrightarrow K(\text{ch}(\mathcal{NPA}))
\end{array}
$$

consists entirely of homotopy equivalences.

Proof. The upper horizontal arrow is a homotopy equivalence by [Wald3, 1.7.1] and the approximation theorem [Wald3, 1.6.7]. See [Tho2]. The right–hand vertical arrow is a homotopy equivalence by the approximation theorem, and the left–hand
vertical arrow is a homotopy equivalence by Quillen’s resolution theorem, mentioned earlier. □

Remark. Let $\mathcal{A}$ be the category of finitely generated left $R$–modules. Replacing $\text{ch}(\mathcal{P}\mathcal{A})$ by the larger category of chain complexes of left projective $R$–modules which are homotopy equivalent to objects in $\text{ch}(\mathcal{P}\mathcal{A})$ does not change the homotopy type of the $K$-theory space. This follows directly from the approximation theorem.

For any category $\mathcal{D}$ with cofibrations and weak equivalences Waldhausen has constructed an injective map $\psi : |w\mathcal{D}| \to K(\mathcal{D})$, depending functorially on $\mathcal{D}$. Suppose for illustration that $\mathcal{D} = \mathcal{P}_R$ for some ring $R$, and that $V$ is an object in $\mathcal{D}$. Then $K(\mathcal{D}) =: K(R)$, and the composition

$$|\text{aut}(V)| \hookrightarrow |w\mathcal{D}| \xrightarrow{\psi} K(\mathcal{D})$$

induces a homomorphism on $\pi_1$, which is the classical one from $\text{aut}(D)$ to $K_1(R)$.

Returning to an arbitrary exact category $\mathcal{M} \subset \mathcal{A}$, we introduce certain full subcategories of $\text{ch}(\mathcal{M})$. Let $\mathcal{tch}(\mathcal{M})$ consist of the trivial chain complexes (with trivial differential), and let $\mathcal{sch}(\mathcal{M})$ consist of the very special chain complexes $C$ whose homology groups $H_iC$ belong to $\mathcal{M}$ for all $i$. Thus

$$\mathcal{tch}(\mathcal{M}) \subset \mathcal{sch}(\mathcal{M}) \subset \mathcal{ch}(\mathcal{M})$$

and we define $w\mathcal{tch}(\mathcal{M}) := \mathcal{tch}(\mathcal{M}) \cap w\mathcal{M}$, $w\mathcal{sch}(\mathcal{M}) := \mathcal{sch}(\mathcal{M}) \cap w\mathcal{M}$. Note: it is irrelevant to us whether $\mathcal{sch}(\mathcal{M})$ is an exact subcategory of $\mathcal{ch}(\mathcal{M})$ or not.

In the proposition just below, we regard the homology functor $H_*$ as a functor from $\mathcal{sch}(\mathcal{M})$ to $\mathcal{tch}(\mathcal{M})$. We also use a restricted product $\prod'$ of pointed spaces. It consists of those points $(x_i)$ in the honest product for which $x_i \neq *$ for only finitely many $i$.

2.2.3. Proposition. The following diagram commutes up to homotopy:

$$\begin{array}{ccc}
|w\mathcal{sch}(\mathcal{M})| & \xrightarrow{H_*} & |w\mathcal{tch}(\mathcal{M})| \\
\downarrow\psi & & \downarrow\psi \\
K(\mathcal{ch}(\mathcal{M})) & \overset{=}\longrightarrow & K(\mathcal{ch}(\mathcal{M})) \\
\end{array}
\xrightarrow{\text{altern. sum}}
\prod'_{i \in \mathbb{Z}} K(\mathcal{ch}(\mathcal{M})).$$

Proof. Given a chain complex $C$ in $\mathcal{ch}(\mathcal{M})$, let $P_kC$ be the $k$–the Postnikov approximation to $C$. Thus $(P_kC)_i = C_i$ for $i \leq k$, $(P_kC)_{k+1} = \text{im}(\partial : C_{k+1} \to C_k)$, and $(P_kC)_i = 0$ for $i > k + 1$. Let $Q_kC$ be the kernel of the canonical projection from $P_kC$ to $P_{k-1}C$. Then

$$Q_kC \longrightarrow P_kC \longrightarrow P_{k-1}C$$

is a functorial cofibration sequence in $\mathcal{sch}(\mathcal{M})$. The projection

$$Q_kC \longrightarrow H_*(Q_kC)$$
is a weak equivalence in \( \text{sch}(\mathcal{M}) \), and of course \( H_*(Q_k C) \) is concentrated in degree \( k \) and equal to \( H_k C \) there. Using this and an observation [Wald3, 1.3.3] related to the additivity theorem, one finds that the left-hand square in 2.2.3 commutes up to homotopy. Commutativity of the right-hand square is a consequence of [Wald3, 1.6.2]. □

We note that 2.2.3 remains valid in the case \( M = P_R \) if the chain complexes in \( \text{sch}(P_R) \) and \( \text{ch}(R) \) are allowed to be chain complexes of projective left \( R \)-modules which are homotopy equivalent to finitely generated ones.

We return to \( p, p, E, B \) and \( V \) of 2.1.4 and 2.2.1. Let \( H_i(p(c); V) \) be the \( i \)-th homology of \( p(c) \) with twisted coefficients in \( V \). Combining 2.2.3 with 2.2.1, we get the following result.

2.2.4. Proposition. Suppose that each \( H_i(p(c); V) \) has a finite length resolution by \( f.g. \) projective \( R \)-modules. Then the composition

\[
B \xrightarrow{(p)} A_B(E) \rightarrow A(E) \xrightarrow{\lambda} K(R)
\]

is homotopic to the alternating sum \( \Sigma(-1)^i \ldots \) of the maps

\[
B \xrightarrow{k(i)} |\text{iso}(N\mathcal{P}_R)| \xrightarrow{\psi} K(R) \quad (i \geq 0)
\]

where \( N\mathcal{P}_R \) consists of all left \( R \)-modules having a finite resolution by finitely generated projective ones, and \( k(i) \) is induced by \( c \mapsto H_i(p(c); V) \). (Remember \( B = |\mathcal{C}|. \)) □

2.3. Assembly

A functor \( F \) from spaces to CW-spectra is homotopy invariant if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant \( F \) is excisive if \( F(\emptyset) \) is contractible and if \( F \) preserves homotopy pushout squares (alias homotopy cocartesian squares, see [Go1], [Go2]). The excision condition implies that \( F \) preserves finite coproducts, up to homotopy equivalence. Call \( F \) strongly excisive if it preserves arbitrary coproducts, up to homotopy equivalence.

2.3.1. Theorem [WWAs]. For any homotopy invariant \( F \) from spaces to CW-spectra, there exist a strongly excisive (and homotopy invariant) \( F^\% \) from spaces to CW-spectra and a natural transformation

\[
\alpha = \alpha_F : F^\% \rightarrow F,
\]

called “assembly”, such that \( \alpha : F^\%(\ast) \rightarrow F(\ast) \) is a homotopy equivalence. Moreover, \( F^\% \) and \( \alpha_F \) can be made to depend functorially on \( F \), and

\[
F^\%(X) \simeq X_+ \wedge F(\ast)
\]

by a chain of natural homotopy equivalences.
2.3.2. Observation. If $F$ is already excisive, then

$$\alpha : F^\%(Y) \to F(Y)$$

is a homotopy equivalence for all finite $Y$, and if $F$ is strongly excisive, then $\alpha$ is a homotopy equivalence for all $Y$.

Proof. By arguments going back to Eilenberg and Steenrod it is sufficient to verify that $\alpha$ is a homotopy equivalence for $Y = \text{point}$. □

We want to show that $\alpha = \alpha_F$ is the “universal” approximation (from the left) of $F$ by a strongly excisive homotopy invariant functor. Suppose therefore that $\beta : E \to F$ is another natural transformation with strongly excisive and homotopy invariant $E$. The commutative square

$$
\begin{array}{ccc}
E^\% & \xrightarrow{\alpha_E} & E \\
\downarrow{\beta^\%} & & \downarrow{\beta} \\
F^\% & \xrightarrow{\alpha_F} & F
\end{array}
$$

in which the upper horizontal arrow is a homotopy equivalence by 2.7, shows that $\beta$ essentially factors through $\alpha_F$.

Following [AnCoFePe] and [CaPe], [CaPeVo] we introduce control in order to get explicit models for assembly maps in $A$–theory. A control space is a pair consisting of a locally compact Hausdorff space $\bar{E}$ and an open dense subset $E$ of $\bar{E}$. Let $p : X_1 \to \bar{E}$ and $q : X_2 \to E$ be proper spaces over $E$ (which means that $X_1, X_2$ are locally compact, and $p, q$ are proper). A continuous proper map $f : X_1 \to X_2$ is a controlled map if it satisfies the following condition: Given $z \in \bar{E} \setminus E$, and a neighborhood $U$ of $z$ in $\bar{E}$, there exists a smaller neighborhood $U_0$ of $z$ in $\bar{E}$ such that $p(x) \in U_0$ implies $q(f(x)) \in U$, and $q(f(x)) \in U_0$ implies $p(x) \in U$, for all $x \in X_1$.

It is straightforward to define controlled (proper) homotopies between controlled maps, and then controlled (proper) homotopy equivalences between proper spaces over $E$. Denote the homotopy category of proper spaces and controlled maps over $E$ by $\mathcal{H}(E \triangleleft \bar{E})$.

We form the category of proper retractive ENR’s over $E$, where the morphisms are maps over $\bar{E}$ and relative to $E$. Such a morphism is a cofibration if it is injective. Let $A(E \triangleleft \bar{E})$ be the $K$-theory spectrum of this category with cofibrations, allowing as weak equivalences all those morphisms which become invertible in $\mathcal{H}(E \triangleleft \bar{E})$.

We shall also need germs near $\bar{E} \setminus E$ of proper spaces over $E$. Such a germ is represented by a proper space over $W$, where $W \subset E$ is such that $E \setminus W$ is closed in $\bar{E}$. Germs of controlled maps and germs of controlled homotopies are defined in the most obvious way. Hence there is a category $\mathcal{H}((E \triangleleft \bar{E})_\infty$ of germs near $\bar{E} \setminus E$ of proper spaces over $E$ and controlled homotopy classes of controlled map germs.

We form the category of germs near $\bar{E} \setminus E$ of proper retractive ENR’s over $E$, where the morphisms are germs of maps over $E$ and relative to $E$. Such a morphism
is a cofibration if it is the germ of an injection. Let $A(E \triangleleft E)_\infty$ be the $K$-theory spectrum of this category with cofibrations, allowing as weak equivalences all those morphisms which become invertible in $\mathcal{H}(E \triangleleft E)_\infty$. The following is a reformulation of a special case of the main theorem of [Vo2]; see also [PeWei], [AnCoFePe], [Vo3], [Vo4]. Notation: $Y$ is a compact ENR which we sometimes identify with $Y \times \{0\}$.

2.3.3. Theorem [Vo2], [Vo4], [CaPeVo]. The functor $Y \mapsto A(Y \times [0, 1] \triangleleft Y \times [0, 1])_\infty$ is homotopy invariant and excisive.

2.3.4. Theorem [CaPeVo]. The commutative square of inclusion maps

$$
\begin{array}{ccc}
A(Y) & \cong & A(Y \times \{0\}) \\
\downarrow & & \downarrow \\
* & \longrightarrow & A(Y \times [0, 1] \triangleleft Y \times [0, 1])_\infty
\end{array}
$$

is a homotopy pullback square. If $Y = *$, its upper right–hand vertex is contractible.

Comments. Contractibility of $A([0, 1] \triangleleft [0, 1])$ is proved using an Eilenberg swindle. The same Eilenberg swindle shows that $A(Y \times [0, 1] \triangleleft Y \times [0, 1])$ is connected (but it is not always contractible). See the remark following 2.3.5 below. We mention this because Carlsson, Pedersen and Vogell use slightly bigger categories of retractive spaces than we do, with a view to “idempotent completeness”. Consequently they obtain versions of $A$–theory with a potentially bigger $\pi_0$ than we do. However, when $Y = *$ there is no disagreement anywhere. Therefore 2.3.4 is correct when $Y = *$, and we can conclude that $A([0, 1] \triangleleft [0, 1])_\infty$ is homotopy equivalent to $S^1 \wedge A(*)$. (Note that this is the coefficient spectrum in 2.3.3.) Using 2.3.3 now, and assuming without loss of generality that $Y$ is connected, we conclude that the maps

$$
\begin{align*}
A(Y) & \to A(*) \\
A(Y \times [0, 1] \triangleleft Y \times [0, 1]) & \to A([0, 1] \triangleleft [0, 1]) \\
A(Y \times [0, 1] \triangleleft Y \times [0, 1])_\infty & \to A([0, 1] \triangleleft [0, 1])_\infty
\end{align*}
$$

induced by $Y \to *$ are 1–connected, 1–connected and 2–connected, respectively. Hence 2.3.4 is correct for arbitrary $Y$, despite the missing components.

The category of proper retractive ENR’s over $Y \times [0, 1]$ has an endomorphism $t$ induced by the shift map

$$
Y \times [0, 1] \to Y \times [0, 1] \quad ; \quad (y, 1 - u) \mapsto (y, 1 - u/2).
$$

Using $t$ we can manufacture other endomorphisms such as

$$
\sum_{i \geq 0} k_it^i
$$

where $k_0, k_2, k_3, \ldots$ can be any sequence of positive integers and $k_it^i$ is short for a $k_i$–fold coproduct, and the sum sign also denotes a coproduct. A retractive map $f$
between proper retractive ENR’s over \(Y \times [0, 1]\) is a \textit{microequivalence} if \((\sum_i k_i t^i)(f)\) is invertible in \(\mathcal{H}(Y \times [0, 1] \triangleleft Y \times [0, 1])\) for arbitrary positive integers \(k_0, k_1, k_2, \ldots\). For example, any isomorphism between proper retractive ENR’s over \(Y \times [0, 1]\) is a microequivalence. Write \(P(Y)\) for the \(K\)-theory spectrum of the category of proper retractive ENR’s over \(Y \times [0, 1]\), where cofibrations are defined as usual and the weak equivalences are the microequivalences. Then

\[
P(Y) \subset A(Y \times [0, 1] \triangleleft Y \times [0, 1]).
\]

In the following lemma, \(Y\) is still compact.

\textbf{2.3.5. Lemma.} \textit{The spectrum \(P(Y)\) is contractible.}

\textit{Remark.} Notice that \(P(*)\) is all of \(A([0, 1] \triangleleft [0, 1])\), so 2.3.5 implies contractibility of \(A([0, 1] \triangleleft [0, 1])\). For arbitrary \(Y\), the inclusion \(P(Y) \subset A(Y \times [0, 1] \triangleleft Y \times [0, 1])\) induces a surjection on \(\pi_0\) because the underlying inclusion of categories is a bijection on objects. Therefore 2.3.5 implies that \(A(Y \times [0, 1] \triangleleft Y \times [0, 1])\) is connected.

\textit{Proof of 2.3.5.} It is convenient to replace \([0, 1]\) by \([0, \infty]\) via the homeomorphism \(s \mapsto \ln(1 - s)/\ln(1/2)\). Then \(t\) is the endomorphism induced by the shift map \((y, s) \mapsto (y, s + 1)\) from \(Y \times [0, \infty)\) to itself.

Let \(u := \sum t^i\) where the sum is taken over all \(i \geq 0\). The endofunctors \(t\) and \(u\) take microequivalences to microequivalences and respect cofibrations. They therefore induce self–maps of \(P(Y)\). Write \([t]\) and \([u]\) for their homotopy classes. There exists another endofunctor \(t'\) respecting cofibrations and microequivalences, and there exist natural microequivalences

\[
t(X) \hookrightarrow t'(X) \hookrightarrow X
\]

where \(X\) is any retractive ENR over \(Y\). (Details: \(t'(X) = g_*([0, 1] \times X)\) where \(X\) is a retractive space over \(Y \times [0, \infty)\), so that \([0, 1] \times X\) is a retractive space over \([0, 1] \times Y \times [0, \infty]\), and \(g : [0, 1] \times Y \times [0, \infty) \longrightarrow Y \times [0, 1]\) takes \((s, y, t)\) to \((y, s + t)\). Note that \(X \cong g_*([0] \times X)\) and \(t(X) \cong g_*([1] \times X)\).) Therefore

\[
[id] = [t] = [u] - [tu] = [u] - [t][u] = [u] - [u] = [u] .
\]

We are now ready for the microcharacteristic. Our model for \(A^\infty(Y)\) will be the homotopy pullback of the diagram

\[
\begin{array}{ccc}
A(Y) & \xrightarrow{\cong} & P(Y) \\
\downarrow & & \xrightarrow{\cong} \\
\end{array}
\]

By 2.3.3, 2.3.4 and 2.3.5 this is indeed a homotopy invariant and excisive functor of the variable \(Y\), and it comes with a projection to \(A(Y)\) which is a homotopy equivalence when \(Y = \ast\). Hence, by the uniqueness result 2.3.2 and sequel, our
notation $A^\% (Y)$ is fully justified. We write $A^\% (Y)$ for the corresponding infinite loop space; that is, $A^\% (Y)$ is the homotopy pullback of

$$
A(Y) \\
\downarrow \subset \\
P(Y) \xrightarrow{\subset} A(Y \times [0,1) \triangleleft Y \times [0,1]) .
$$

Observe now that the Euler characteristic $\langle Y \rangle \in A(Y)$ actually lives in the subspace $A(Y) \cap P(Y)$ of $A(Y \times [0,1) \triangleleft Y \times [0,1])_\infty$. Therefore it lives in $A^\% (Y)$. As such we denote it by $\langle \langle Y \rangle \rangle$.

2.3.6. Variation. It follows from 2.3.4 that the homotopy fiber of

$$
P(Y) \to A(Y \times [0,1) \triangleleft Y \times [0,1])_\infty
$$

is another possible model for $A^\% (Y)$. The preceding model maps to it (forgetfully) by a homotopy equivalence. With the new model, the microcharacteristic $\langle \langle Y \rangle \rangle$ is particularly easy to describe, as a point in $P(Y)$ which goes to the base point in $A(Y \times [0,1) \triangleleft Y \times [0,1])_\infty$. Drawback of this construction: The relationship with $A(Y)$ and $\langle Y \rangle \in A(Y)$ is less direct.

In the following we write $\langle Y \rangle \in A(Y)$ and $\langle \langle Y \rangle \rangle \in A^\% (Y)$, suppressing any information about the specific models of $A(Y)$ and $A^\% (Y)$ that we had to use to make it all work. — It is easy to verify that $\langle \langle Y \rangle \rangle$ is lax natural for homeomorphisms. Although it is easy, it depends crucially on the fact that isomorphisms between compact retractive ENR’s over $Y \approx Y \times \{0\}$ can be regarded as microequivalences between proper retractive ENR’s over $Y \times [0,1)$. In fact, the same argument shows that $\langle \langle Y \rangle \rangle$ is lax natural for cell–like maps [La1], [La2], [La3] between ENR’s.

2.3.7. Remark. We can now state the micro–index theorem (see section abstracts) in a very informal way. Suppose that $M^n$ is a compact topological manifold. The microcharacteristic $\langle \langle M \rangle \rangle$ lives in $A^\% (Y)$, which we take the liberty to identify with $\Omega^\infty (M_+ \wedge A(\ast))$. The Euler section of $\tau^M$, as defined in the introduction, is trivialized near the boundary of $M$. It represents an element in the $n$–th cohomology of the pair $(M, \partial M)$ with twisted coefficients in $A(\ast)$. Modulo an identification of the twist, which we will do in §7, we can conclude that the Poincaré dual of the Euler section of $\tau^M$ lives in $\Omega^\infty (M_+ \wedge A(\ast))$, just like $\langle \langle M \rangle \rangle$. The micro–index theorem states that Poincaré dual of Euler section and microcharacteristic are connected by a canonical path.

We need this statement for families. In the next subsection, 2.4, we will see among other things how microcharacteristics can be extracted from families.

2.4. Classification of topological bundle structures

As in [WW1] let $\text{TOP}(M)$ be the topological group of homeomorphisms $M \to M$ which agree with the identity near $\partial M$. A difficult theorem due to [McD1] (see also [Se], [Math1], [Math2], [Thu]) asserts that the inclusion of $|\text{TOP}^\delta (M)|$
in $|\text{TOP}(M)|$ is a homology equivalence for $M$, where $\text{TOP}^δ(M)$ is the underlying discrete group and the vertical bars are for classifying spaces. (A map $f : X \to Y$ of connected spaces is a homology equivalence if it induces isomorphisms $f_* : H_*(X; J) \to H_*(Y; J)$ for any $\pi_1(Y)$–module $J$.) Also, the inclusion of $|\text{TOP}^δ(M, \partial M)|$ in $|\text{TOP}(M, \partial M)|$ is a homology equivalence; $\text{TOP}(M, \partial M)$ is the simplicial group of all topological automorphisms of $M$.]

We now give a direct construction of the map $S^\text{tp}(p) \to L^\text{tp}(p)$ in theorem 0.4, without using any index theorem. Actually we will construct a map $S^\text{tp}(p, \gamma) \to L^\text{tp}(p)$, where $\gamma$ can be any microbundle on the total space of $p$. See (1–2).

We can simplify the task and avoid all questions of naturality by concentrating on certain universal examples. In the first example, take $B$ to be $|\mathcal{G}_n|$. See §1. Let $p$ be the tautological quasifibration on $B$ whose fiber over the point corresponding to an object $(F, \gamma)$ in $\mathcal{G}_n$ is $F$ (something canonically homeomorphic to $F$, to be quite honest). In the second example, take $B$ to be $|\mathcal{T}_n|$. See again §1. Let $p$ be the tautological fiber bundle on $B$ with compact $n$–manifold fibers.

**First universal example.** Let $\mathcal{G}_n^δ$ be the discrete category underlying $\mathcal{G}_n$. The inclusion of $|\mathcal{G}_n^δ|$ in $|\mathcal{G}_n|$ is a homotopy equivalence (compare [DwKa], [Fie]). Hence we may take $B = |\mathcal{G}_n^δ|$ instead of $B = |\mathcal{G}_n|$. Define $p, q, q^\%$ from $\mathcal{G}_n^δ$ to spaces by

$$p(F, \gamma) = F, \quad q(F, \gamma) = A(F), \quad q^\%(F, \gamma) = A^\%(F).$$

Let $p, q, q^\%$ be the corresponding quasifibrations. Euler characteristics $F \mapsto \langle F \rangle$ determine an element $\langle p \rangle$ of $\text{holim} \, q \simeq \Gamma(q)$.

We can also try the stable version $|\mathcal{G}_n^δ|$ instead of $|\mathcal{G}_n^δ|$. Then we still have $q$ and $q^\%$ on $\mathcal{G}_n^δ$, given by $(m, F, \gamma) \mapsto A(F)$ and $(m, F, \gamma) \mapsto A^\%(F)$. (These are functors because a morphism $g : (m, F, \gamma) \to (n, F', \gamma')$ in $\mathcal{G}$ induces

$$A(F) \xrightarrow{\times I^{n-m}} A(F \times I^{n-m}) \xrightarrow{g^*} A(F')$$

$$A^\%(F) \xrightarrow{\times I^{n-m}} A^\%(F \times I^{n-m}) \xrightarrow{g^*} A^\%(F'),$$

where $\times I^{n-m}$ is regarded as an exact functor from retractive spaces over $F$ to retractive spaces over $F \times I^{n-m}$. Slight problems are caused by the fact that products in the category of sets are not strictly associative, but given any “category of sets” one can easily construct an equivalent category with an explicit and associative product.) We still have quasifibrations $q, q^\%$ on $|\mathcal{G}_n^δ|$ and a distinguished $\langle p \rangle$ in $\text{holim} \, q \simeq \Gamma(q)$.

**Second universal example.** Let $\mathcal{T}_n^δ$ be the discrete category underlying $\mathcal{T}_n$. The inclusion $|\mathcal{T}_n^δ| \hookrightarrow |\mathcal{T}_n|$ is a homology equivalence, by the McDuff–Mather–Thurston theorem mentioned earlier. It does not follow immediately that we may substitute $|\mathcal{T}_n^δ|$ for $|\mathcal{T}_n|$, but we will do so now and justify later. Define $p, q, q^\%$ from $\mathcal{T}_n^δ$ as before, restricting from $\mathcal{G}_n^δ$. Write $p, q, q^\%$ for the corresponding quasifibrations on $|\mathcal{T}_n^δ|$. Euler characteristics $F \mapsto \langle F \rangle$ determine $\langle p \rangle$ in $\text{holim} \, q \simeq \Gamma(q)$, and microcharacteristics $F \mapsto \langle F \rangle$ determine $\langle p \rangle$ in $\text{holim} \, q^\% \simeq \Gamma(q^\%)$. Since
microcharacteristics lift Euler characteristics, we have \( \alpha \cdot \langle p \rangle = \langle p \rangle \), where \( \alpha \) is the assembly (from \( A\% \)-theory to \( A \)-theory).

Again, we may try the stable version \(|T^\delta|\) instead of \(|T^\delta_n|\). We can then still regard \( q \) and \( q^\% \) as functors on \( T^\delta \), and we get distinguished elements \( \langle p \rangle \in \text{holim} q \) and \( \langle p \rangle \in \text{holim} q^\% \).

**Justification.** Concentrating on the stable case, we use the commutative square
\[
\begin{array}{ccc}
|T^\delta| & \xrightarrow{\subset} & |T^\delta| \\
\subset & \downarrow \cong & \subset \downarrow \cong \\
|T| & \xrightarrow{\subset} & |T| \\
\end{array}
\]

We constructed two quasifibrations \( q \), one on \(|T^\delta|\) and another on \(|T^\delta|\). However, one of these is the restriction of the other, so there is no serious ambiguity. We also constructed two quasifibrations \( q^\% \), one on \(|T^\delta|\) and another on \(|T^\delta|\). But again, one of these is the restriction of the other.

Since the inclusion of simplicial sets \(|T^\delta| \hookrightarrow |T|\) is a homotopy equivalence, the quasifibration \( q \) on \(|T^\delta|\) extends to a quasifibration on \(|T|\) which we still denote by \( q \). (Construct the extension by induction over the relative skeleta.) In the same way, \( q^\% \) extends to a quasifibration on \(|T|\), which we still denote by \( q^\% \). Moreover, if we construct the extension of \( q^\% \) carefully, then the morphism of quasifibrations \( \alpha : q^\% \rightarrow q \) defined over \(|T^\delta|\) extends to the one defined over \(|T|\). These extensions (of \( q, q^\% \) and the morphism from \( q^\% \) to \( q \)) are unique up to contractible choice; details omitted.

Now we have a commutative diagram of spaces of sections
\[
\begin{array}{ccc}
\Gamma(q^\%, |T^\delta|) & \xrightarrow{\text{res}} & \Gamma(q, |T^\delta|) \\
\uparrow_{\text{res}} & & \uparrow_{\text{res}} \\
\Gamma(q^\%, |T|) & \xleftarrow{\text{res}} & \Gamma(q, |T|) \\
\end{array}
\]

(2–3)

where \( \Gamma(q^\%, |T|) \) denotes a space of sections over \(|T|\), for example, and all arrows labeled res are restriction maps. The unlabeled arrows are induced by the morphism \( q^\% \rightarrow q \). We know already that one of the vertical arrows is a homotopy equivalence; we shall see that the other two are also homotopy equivalences. (This will complete the justification, because all we ever wanted was a point in the holim of the lower row of (2–1), and what we have constructed so far is a point in the holim of the upper row of (2–1).)

Suppose therefore that \( f : X \rightarrow Y \) is a homology equivalence between CW–spaces, and that \( r \) is a fibration on \( Y \) with nilpotent fibers [HMR]. Then Postnikov technology implies that the pullback map from the section space \( \Gamma(r) \) to \( \Gamma(f^* r) \) is a homotopy equivalence. Example: \( f \) can be the inclusion of \(|T^\delta|\) in \(|T|\), and \( r \) can be \( q \) or \( q^\% \) restricted to \(|T|\). In this case the fibers are nilpotent because they are infinite loop spaces. \( \square \)

**Remark.** Note that all restriction maps in (2–1) are fibrations. Together with the preceding argument, this shows that the sections \( \langle p \rangle \) and \( \langle p \rangle \) of \( q^\% \) and \( q \), defined
over $|\mathcal{T}\delta|$ and $|\mathcal{G}\delta|$, respectively, can be extended compatibly to $|\mathcal{T}|$ and $|\mathcal{G}|$, respectively. As usual, any choices made are contractible; and we still use the symbols $\langle\langle p\rangle\rangle$ and $\langle p\rangle$ to denote these extended sections.

We have now constructed a map $S^{tp}(p, \gamma) \to \mathcal{L}^{tp}(p)$ by reduction to universal examples (remember the definition of $S^{tp}(p, \gamma)$ given in 1.2, display (1–1)). It remains to show that the map is a homotopy equivalence. Again this can be done at the “universal level”, and the task is then to show that

\[
|\mathcal{T}| \xrightarrow{\langle\langle p\rangle\rangle} E(q^%) \quad \xrightarrow{\subset} \quad |\mathcal{G}| \xrightarrow{(p)} E(q) \quad \xrightarrow{\alpha} \quad E_1(q^%) \xrightarrow{\subset} \quad E(q)_{1}\%
\]

is a homotopy pullback square (where $E(q)$ and $E(q^%)$ are the \textit{total spaces} of the quasifibrations $q^%$ and $q$). Now every component of $|\mathcal{G}|$ contains a point corresponding to an object of the form $(n, M^n, \tau^M)$ where $M$ is a compact $n$–manifold. The map of vertical homotopy fibers in (2–4), over such a point and its image point in $E(q)$, takes the form

\[
\Omega Wh^{\text{TOP}}(M) \to \text{hofiber}_{(M)}[A^ sexes(M) \to A(M)],
\]

by 1.5. The subscript $\langle M\rangle$ means that the homotopy fiber (of the assembly map) must be taken over the point $\langle M\rangle$. (Note, however, that $\alpha$ from $A^%_{\langle M\rangle}(M)$ to $A_{\langle M\rangle}(M)$ is a morphism of infinite loop spaces, so that all its homotopy fibers are canonically homotopy equivalent to that over the zero element.) We see that domain and codomain of (2–5) are “abstractly” homotopy equivalent by Waldhausen’s theorem. But so far we do not know how the homotopy equivalence due to Waldhausen is related to the map (2–5). Moreover, before we can explore the relationship, we need a description of the Waldhausen map.

Fix $M^n$, with boundary $\partial M$. Recall from Corollary 1.5 the topological category of $h$–cobordisms $\mathcal{H}(\partial M)$. Also, let $\mathcal{H}^\ell(\partial M)$ be the topological category whose objects are pairs $(X, \gamma)$ where $X$ is a compact ENR containing $\partial M$, in such a way that the inclusion $\partial M \to X$ is a homotopy equivalence, and where $\gamma$ is an $n$–microbundle on $X$ extending the restriction of $\tau^M$ to $\partial M$. A morphism in $\mathcal{H}^\ell(\partial M)$, say from $(X_1, \gamma_1)$ to $(X_2, \gamma_2)$, is a map $X_1 \to X_2$ relative to $\partial M$, covered by a bundle map $\gamma_1 \to \gamma_2$ relative to $\tau^M$. Then $|\mathcal{H}^\ell(\partial M)|$ is contractible.

To an object $(X, \gamma)$ in $\mathcal{H}^\ell(\partial M)$ we can associate its \textit{relative} Euler characteristic: the point in $A(X)$ corresponding to the retractive space $X \amalg_{\partial M} X$ over $X$. This is lax natural (perhaps not in a continuous way, but we know how to handle such problems). To an $h$–cobordism $(W; \partial M, N)$ we can associate its \textit{relative} microcharacteristic: the point in $A^%(W)$ corresponding to the retractive space $W \amalg_{\partial M} W$. Playing the same game as before, we obtain a diagram similar to (2–4):

\[
|\mathcal{H}(\partial M)| \xrightarrow{} E(q^%) \quad \xrightarrow{\subset} \quad |\mathcal{H}^\ell(\partial M)| \xrightarrow{(p)} E(q)
\]
where $q_1^\%$ and $q_1$ are the (quasi–)fibrations on $|\mathcal{H}^\ell(\partial M)|$ determined by the functors $(X, \gamma) \mapsto A^\%(X)$ and $(X, \gamma) \mapsto A(X)$, respectively. The horizontal maps in (2–6) are essentially the sections determined by relative microcharacteristics and Euler characteristics, respectively, but again we have to use the McDuff–Mather–Thurston theorem to construct these. The map of vertical homotopy fibers in (2–6) takes the form

$$(2–7) \quad |\mathcal{H}(\partial M)| \rightarrow \text{hofiber}[A^\%(\partial M) \rightarrow A(\partial M)]$$

and it is the Waldhausen map, by definition (see Appendix D for more details). As it stands, it is not a homotopy equivalence in general, but it turns into a homotopy equivalence under stabilization.

Now we are in a position to “relate”, and we do it by introducing certain maps, from $|\mathcal{H}^\ell(\partial M)|$ to $|\mathcal{G}|$, and from $|\mathcal{H}(\partial M)|$ to $|\mathcal{J}|$. The first of these, say $\iota$, is induced by the functor

$$(X, \gamma) \mapsto (n, X \amalg_{\partial M} M, \gamma \cup \tau^M).$$

The second is the restriction of the first. Now consider the map from left–hand vertical homotopy fiber to right–hand vertical homotopy fiber in the commutative diagram

$$
\begin{array}{ccc}
|\mathcal{H}(\partial M)| & \longrightarrow & |\mathcal{J}| \\
\downarrow \subset & \downarrow \subset & \downarrow \alpha \\
|\mathcal{H}^\ell(\partial M)| & \longrightarrow & |\mathcal{G}|
\end{array}
\xrightarrow{(\mathbb{p})} 
\begin{array}{c}
E(q^\%) \\
\uparrow_{(2–7)} \\
\text{hofiber}[A^\%(\partial M) \rightarrow A(\partial M)]
\end{array}
\xrightarrow{(\mathbb{p})}
\begin{array}{c}
E(q) \\
\text{hofiber}[A^\%(M) \rightarrow A(M)]
\end{array}
$$

which becomes

$$(2–8) \quad |\mathcal{H}(\partial M)| \longrightarrow \text{hofiber}_{(M)}[A^\%(M) \rightarrow A(M)].$$

Inspection and the additivity theorem in [Wald3] show that

$$|\mathcal{H}(\partial M)| \xrightarrow{(2–8)} \text{hofiber}_{(M)}[A^\%(M) \rightarrow A(M)] \xrightarrow{\subset (2–7)} \text{hofiber}[A^\%(\partial M) \rightarrow A(\partial M)] \xrightarrow{\subset} \text{hofiber}[A^\%(M) \rightarrow A(M)]$$

commutes up to homotopy. Under stabilization (replacing $M$ by $M \times I^k$ for large $k$), the upper horizontal arrow turns into (2–5), and the remaining arrows turn into homotopy equivalences if they are not homotopy equivalences already. It follows that (2–5) is a homotopy equivalence. □

3. Poincaré Duality by Scanning

Let $M^n$ be a closed topological manifold. For $y \in M$ we write $M_y$ to mean the cofiber of the inclusion $M \setminus \{y\} \hookrightarrow M$. Let $X$ be any spectrum, and let $\Gamma(M; X)$ be the section space of the fibration on $M$ whose fiber over $y \in M$ is $\Omega^\infty((M_y) \wedge X)$. For any $y \in M$ we have the inclusion $j_y : \Omega^\infty(M_+ \wedge X) \hookrightarrow \Omega^\infty((M_y) \wedge X)$, which we use to define

$$
\varphi : \Omega^\infty(M_+ \wedge X) \longrightarrow \Gamma(M; X)
\quad z \mapsto (y \mapsto j_y(z)).
$$
3.1. **Proposition.** The map (3–1) is a homotopy equivalence.

**Sketch Proof.** The statement has a generalization to manifolds with countable basis (without boundary, but possibly noncompact): just replace \( \Gamma(M; X) \) by a space \( \Gamma_c(M; X) \) of sections with compact support. The generalized version is easier to prove. First use a direct limit argument to reduce to the case where \( M \) can be covered by finitely many charts. Then use excision properties of \( \Omega^\infty((-)_+ \wedge X) \) and \( \Gamma_c((-); X) \) to reduce to the case where \( M \) is open in \( \mathbb{R}^n \). Triangulate \( M \) and use excision properties once more to reduce to the case where \( M = \mathbb{R}^n \). Then use inspection. □

We would like to call \( \varphi \) a Poincaré duality map (from homology to cohomology). A Poincaré duality map should however admit a description in homotopy theoretic terms, and it is not clear that this one does. (Try to define (3–1) assuming merely that \( M \) is a Poincaré duality space.)

Let \( \nu^k \) be a normal bundle for \( M \), with Thom space \( \text{thom}(\nu) \) and collapse map \( \rho \) from \( S^{n+k} \) to \( \text{thom}(\nu) \). Recall that Milnor–Poincaré duality is the homotopy equivalence

\[
\mu : \text{map}(\text{thom}(\nu), \Sigma^{n+k}X) \longrightarrow \Omega^\infty(M_+ \wedge X)
\]

which to a (pointed) map \( g : \text{thom}(\nu) \to \Sigma^{n+k}X \) associates the composition

\[
S^{n+k} \xrightarrow{\rho} \text{thom}(\nu) \xrightarrow{\text{diag.}} M_+ \wedge \text{thom}(\nu) \xrightarrow{id \wedge g} M_+ \wedge \Sigma^{n+k}X.
\]

(To be quite precise here: \( \Omega^\infty \) of a spectrum is the space of maps from \( S^0 \) to that spectrum, and \( \Sigma^{n+k} \) is the shift operator in the category of spectra.) We shall construct a homotopy from

(3–2) \[
\varphi \mu : \text{map}(\text{thom}(\nu), \Sigma^{n+k}X) \longrightarrow \Gamma(M; X)
\]

to a map which has a description in homotopy theoretic terms. The homotopy is essentially contained in the following lemma.

3.2. **Lemma.** For every \( y \in M \), the composition

\[
S^{n+k} \xrightarrow{\rho} \text{thom}(\nu) \xrightarrow{\text{diag.}} M_+ \wedge \text{thom}(\nu) \xrightarrow{id \wedge g} (M_y) \wedge \text{thom}(\nu)
\]

is canonically homotopic to a map \( f_y \) with image contained in \( (M_y) \wedge \text{thom}(\nu_{|\{y\}}) \). Moreover, \( f_y \) is a homotopy equivalence from \( S^{n+k} \) to \( (M_y) \wedge \text{thom}(\nu_{|\{y\}}) \).

The proof is an exercise. **Digression:** The fiber bundle on \( M \) with fiber \( M_y \) over \( y \in M \) is a spherical fibration with canonical section. It is obviously fiber homotopy equivalent to the fiberwise one–point compactification of “the” tangent bundle of \( M \), and it is therefore not surprising that it should be Whitney inverse to the normal bundle \( \nu \), as a spherical fibration. **End of digression.**

Using the lemma, and the notation of the lemma, we find that \( \varphi \mu \) in (3–2) is homotopic by a canonical homotopy to the map

(3–3) \[
g \mapsto (y \mapsto (id \wedge g)f_y)
\]
where $g$ is any map from $\text{thom}(\nu)$ to $\Sigma^{n+k}X$, and $(\text{id} \land g)f_y$ is the composition

$$\mathbb{S}^{n+k} \xrightarrow{f_y} (M_y) \land \text{thom}(\nu|\{y\}) \xrightarrow{\text{id} \land g} (M_y) \land \Sigma^{n+k}X.$$  

The description (3–3) is in homotopy theoretic terms. Indeed, suppose that $M$ is merely a Poincaré space with Spivak normal fibration $\bar{\nu}$ and with a reduction $\rho : \mathbb{S}^{n+k} \to \text{thom}(\bar{\nu})$. Define the Spivak tangent fibration $\bar{\tau}$ as “the” stable inverse of $\bar{\nu}$. More precisely, choose a spherical fibration $\bar{\tau}$ on $M$ and a trivialization of $\bar{\tau} \oplus \bar{\nu}$. The trivialization will consist of pointed homotopy equivalences

$$f_y : \mathbb{S}^i \to \text{thom}(\bar{\tau}|\{y\}) \land \text{thom}(\bar{\nu}|\{y\})$$

for suitable $i$ and all $y \in M$. Use these to write down (3–3), replacing $M_y$ by $\text{thom}(\bar{\tau}|\{y\})$ throughout.

3.3. Remark. Proposition 3.1 and Lemma 3.2 can be generalized to compact manifolds $M$ with boundary as follows. Define $M_y$ exactly as before, but note that $M_y$ is contractible for $y \in \partial M$ and $M_y \simeq \mathbb{S}^n$ if $y \notin \partial M$. Define $\Gamma(M, \partial M; X)$ as the space of sections of the fibration pair with base pair $(M, \partial M)$ whose fiber over $y \in M$ is $\Omega^{\infty}(M_y \land X)$. Here a fibration pair is a map of pairs

$$(Z_1, Z_2) \longrightarrow (Z_3, Z_4)$$

having the homotopy lifting property with respect to maps of pairs with codomain $(Z_1, Z_2)$. Finally

$$\varphi : \Omega^{\infty}(M_+ \land X) \longrightarrow \Gamma(M, \partial M; X)$$

can be defined as before, by “scanning”. It is a homotopy equivalence, and it can be described in homotopy theoretic terms. We omit the details.

3.4. Remark. There is a further generalization to noncompact manifolds $M$ with boundary. Of course, one such generalization is already implicit in the proof of 3.1, but we mean another which relates cohomology to locally finite homology. For technical reasons we insist on an $\Omega$–spectrum $X$. Define $M_y$ and $\Gamma(M, \partial M; X)$ literally as in 3.3. In particular, there is no “compact support” condition in the definition of $\Gamma(M, \partial M; X)$. However, define $M_+$ as the one–point compactification of $M$. The scanning map then goes from $\Omega^{\infty}(M_+ \land X)$ to $\Gamma(M, \partial M; X)$, and it is a homotopy equivalence. Moreover, it can be reformulated in proper homotopy invariant terms.

We omit the details and refer the reader to [WWPro]. Note in any case that $M_+$ need not be homotopy equivalent to a CW–space; however, $\Omega^{\infty}(M_+ \land X)$ is a CW–space by definition (the geometric realization of the simplicial set of spectrum maps from $S^0$ to $M_+ \land X$). It is explained in [WWPro] why $\Omega^{\infty}(M_+ \land X)$ has the “right” homotopy groups when $X$ is an $\Omega$–spectrum. These groups must be thought of as the locally finite homology groups of $M$ with coefficients in $X$.

Warning: The introduction to [WWPro] gives a wrong definition (line −14) of locally finite homology with integer coefficients. To correct it, replace the inverse limit of homology groups by the homology group of a homotopy inverse limit of singular chain complexes.
4. Discrete models

Fix a topological manifold $M^n$. Let $\mathcal{C}$ be the set of embeddings $f : \mathbb{R}^n \to M$. For $f, g \in \mathcal{C}$, a morphism from $f$ to $g$ is an embedding $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = g\lambda$. Such a morphism from $f$ to $g$ is of course unique if it exists. At any rate, $\mathcal{C}$ is a category, and our goal here is to prove

$$|\mathcal{C}| \simeq M.$$

For a more precise statement, we introduce an open subset $E \subset |\mathcal{C}| \times M$. We say $(x, y) \in E$ if the (open) cell of $|\mathcal{C}|$ containing $x$ corresponds to a nondegenerate simplex (diagram in $\mathcal{C}$)

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

such that the image of $f_k$ contains $y$.

4.1. Proposition. The projections $|\mathcal{C}| \leftarrow E \rightarrow M$ are homotopy equivalences.

The proof is a double application of the following lemma, which is only a mild improvement on [Se, App.].

4.2. Lemma. A microgibki map (explanation follows) with contractible fibers is a Serre fibration.

Explanation. A map $q : X \to Y$ is microgibki if it has the homotopy micro-lifting property of [Gro]): For every space $W$, every map $f : W \to X$ and every homotopy $h : W \times I \to Y$ with $h(w, 0) = qf(w)$ for all $w \in W$, there exists a map $\tilde{h}$ from a neighbourhood of $W \times \{0\}$ in $W \times I$ to $X$ such that $q\tilde{h}$ agrees with $h$ where defined.

Example: The projections $E \to |\mathcal{C}|$ and $E \to M$ in 4.1 are microgibki, since $E$ is open in the product $|\mathcal{C}| \times M$. It is obvious that $E \to |\mathcal{C}|$ has contractible fibers. We shall verify that the fibers of the second projection, $q : E \to M$, are also contractible. Fix $y \in M$, and consider the subspaces

$$|\mathcal{C}_y| \subset E_y$$

of $|\mathcal{C}|$ defined as follows: $|\mathcal{C}_y|$ is the union of all (open) cells corresponding to nondegenerate simplices

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

where $y \in \text{im}(f_0)$, and $E_y$ is the union of all cells corresponding to nondegenerate simplices

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k$$

where $y \in \text{im}(f_k)$. Note the following:

1. $E_y$ is homeomorphic to $q^{-1}(y)$.
2. $|\mathcal{C}_y|$ is a deformation retract of $E_y$ (details follow).
3. $|\mathcal{C}_y|$ is the classifying space of a full subcategory $\mathcal{C}_y \subset \mathcal{C}$. The subcategory consists of all objects $f$ such that $y \in \text{im}(f)$. 
(For the deformation retraction in (2), suppose that \( x \) in \( E_y \) belongs to a cell corresponding to a simplex

\[ f_0 \to f_1 \to \cdots \to f_{k-1} \to f_k \]

with \( y \in \text{im}(f_k) \). Let \( (x_0, x_1, \ldots, x_k) \) be the barycentric coordinates of \( x \), all \( x_i > 0 \), and let \( j \leq k \) be the least integer such that \( y \in \text{im}(f_j) \). Let

\[
h_{1-t}(x) := (tx_{no} + x_{yes})^{-1}(tx_0, tx_1, \ldots, tx_{j-1}, x_j, \ldots, x_k)
\]

\[
x_{no} := \sum_{i<j} x_i \quad x_{yes} := \sum_{i \geq j} x_i
\]

for \( t \in [0, 1] \), using barycentric coordinates in the same simplex.)

It only remains to prove that \(|E_y|\) is contractible: For any finite collection of objects \( f_1, f_2, \ldots, f_k \) in \( E_y \), there exists another object \( f_0 \in E_y \) such that \( f_0 \) is an initial object in the full subcategory generated by \( f_0, f_1, \ldots, f_k \). (Just make sure that \( \text{im}(f_0) \subset \text{im}(f_i) \) for \( 1 \leq i \leq k \).)

We conclude that the two projections in 4.1 are weak homotopy equivalences. But open subspaces of CW–spaces are homotopy equivalent to CW–spaces [Mil2], so that \( E \) is homotopy equivalent to a CW–space. Hence 4.2 implies 4.1. □

Proof of 4.2. Suppose throughout this proof that \( p : X \to Y \) is microgibki. Then \( p^I : X^I \to Y^I \) is also microgibki (mapping spaces with the compact–open topology).

This uses the adjunctions

\[
\text{mor}(W \times I, X) \cong \text{mor}(W, X^I), \quad \text{mor}(W \times I, Y) \cong \text{mor}(W, Y^I)
\]

where \( \text{mor} \) denotes sets of continuous maps. See [MaL, VII.8]. Recall now that a map is a Serre fibration if it has the homotopy lifting property for maps from cubes \( I^k \), for any \( k \geq 0 \). Hence it is sufficient to prove the following.

1. If the fibers of \( p \) are weakly contractible, then \( p \) has the homotopy lifting property for maps from a point. That is, any path \( \omega : I \to Y \) has a lift \( \bar{\omega} : I \to X \), and \( \bar{\omega}(0) \) can be prescribed arbitrarily in the fiber over \( \omega(0) \).
2. If the fibers of \( p \) are weakly contractible (explanation follows), then the fibers of \( p^I \) are weakly contractible also.

A space is weakly contractible if it is nonempty, and any map from a sphere to it is homotopic to a constant map. For the proof of (1) we need the following observation:

3. For continuous \( \mu : I \to X \), any vertical homotopy of \( \mu|\{0\} \) can be extended to a vertical homotopy of \( \mu \).

(A vertical homotopy is a homotopy which turns into a constant homotopy when composed with \( p : X \to Y \).) Note: in (3), we do not assume that the fibers are weakly contractible. The proof of (3) is easy, so we concentrate on (1) and (2).

Assume that the fibers of \( p \) are weakly contractible. Given \( \omega : I \to Y \), we can find a subdivision

\[
0 = t_0 < t_1 \cdots < t_{k-1} < t_k = 1,
\]
of \( I \) such that the restriction of \( \omega \) to \([t_i, t_{i+1}]\) has a lift to \( X \), for \( 0 \leq i < k \). This uses only the microgibki property, and the fact that the fibers of \( p \) are nonempty. Using the weak contractibility of the fibers over the division points \( t_i \), and (3), we can then construct a lift \( \bar{\omega} : I \to X \) of \( \omega \), with any prescribed value at \( t_0 = 0 \). This proves (1).

Let \( \Phi \) be the fiber of \( p_I : X^I \to Y^I \) over some \( \omega : I \to Y \). We know already that \( \Phi \neq \emptyset \). For \( t \in I \), we have the evaluation map

\[
\varepsilon_t : \Phi \to p^{-1}(\omega(t)) ; \quad \bar{\omega} \mapsto \bar{\omega}(t).
\]

Given \( f : S^{n-1} \to \Phi \), we must try to extend \( f \) to a map \( \mathbb{D}^n \to \Phi \). Such extensions may be regarded as sections of another microgibki map \( r \) with target \( I \), whose fiber over \( t \in I \) is the space of extensions to \( \mathbb{D}^n \) of \( \varepsilon_t f \). The fibers of \( r \) are again weakly contractible, and the base is \( I \), so we know from (1) that \( r \) admits a section. \( \square \)

Let \( \mathcal{D} \) be the discrete monoid of embeddings \( \mathbb{R}^n \to \mathbb{R}^n \). A monoid may be regarded as a category with one object, so \( \mathcal{D} \) is a category. Define a functor

\[
\tan : \mathcal{C} \to \mathcal{D}
\]

by sending a morphism \( f \to g \) in \( \mathcal{C} \) to “itself”, that is, to the unique \( \lambda : \mathbb{R}^n \to \mathbb{R}^n \) such that \( f = g\lambda \). The functor induces a map of spaces

\[
(4-1) \quad |\tan| : |\mathcal{C}| \to |\mathcal{D}|
\]

which we may regard as the classifying map for the tangent bundle of \( M \). Indeed, we have verified that \( |\mathcal{C}| \simeq M \), and [McD2, Cor.of Thm.A], [Se] show that

\[
(4-2) \quad |\mathcal{D}| \simeq B\text{TOP}(\mathbb{R}^n).
\]

However, we should also verify that \( \tan \) is the right map. Of course, we cannot do it without explaining to some extent why (4-2) holds. To begin, \( |\mathcal{D}| \) is the codomain of a canonical microgibki map with fibers homeomorphic to \( \mathbb{R}^n \), say \( E_u \to |\mathcal{D}| \). The description of \( E_u \) is similar to that of \( E \) in proposition 4.1. Namely, \( E_u \) is the space

\[
\coprod_s \text{star}(s) \times \mathbb{R}^n / \sim
\]

where \( \text{star}(s) \) is the open star of a nondegenerate simplex \( s \) of \( |\mathcal{D}| \). The equivalence relation identifies \((x, v) \in \text{star}(s) \times \mathbb{R}^n \) with \((x, v) \in \text{star}(\partial_i s) \times \mathbb{R}^n \) for \( i < \text{dim}(s) \); if \( i = \text{dim}(s) \), it identifies \((x, s_i(v)) \in \text{star}(s) \times \mathbb{R}^n \) with \((x, v) \in \text{star}(\partial_i s) \times \mathbb{R}^n \), where \( s_i : \mathbb{R}^n \hookrightarrow \mathbb{R}^n \) is the last component of \( s \). (Think of \( s \) as a string of \( i \) embeddings from \( \mathbb{R}^n \) to itself.) The projection \( E_u \to |\mathcal{D}| \) is obvious.

If \( f : X \to |\mathcal{D}| \) is any map where \( X \) is a CW–space, then \( f^*E_u \to X \) is another microgibki map with fibers homeomorphic to \( \mathbb{R}^n \). The projection \( f^*E_u \to X \) is a homotopy equivalence by 4.2. It has no preferred section, but if we are willing to replace \( X \) by \( f^*E_u \), then we have

\[
(4-3) \quad f^*E_u \times_X f^*E_u \to f^*E_u ; \quad (z_1, z_2) \mapsto z_1
\]
which has a preferred section (the diagonal), and qualifies therefore as a microbundle. This is how maps to \( \mathcal{D} \) give rise to microbundles on the domain, and it is not a very serious objection that we had to modify the domain in order to see a microbundle on it.

Returning to (4–1), suppose that \( f \) is the geometric realization of \( \tan : \mathcal{C} \to \mathcal{D} \). Then \( f^*E_u \) is what we previously called \( E \). We now have to find an isomorphism from the microbundle \((4–3)\) on \( f^*E_u = E \) to the pullback of \( \tau_M \) under the projection \( E \to M \). This is easy. □

4.3. Remark. So far we have not paid much attention to the boundary \( \partial M \). This is however easy to do if we assume that \( M \) is equipped with a collar, by which we understand an embedding \( \partial M \times [-\infty, +\infty) \to M \) extending the identification \( \partial M \times \{ -\infty \} \cong \partial M \). Writing \( \mathcal{C}(\partial M) \) and \( \mathcal{C}(M) \) as well as \( \mathcal{D}(\mathbb{R}^{n-1}) \) and \( \mathcal{D}(\mathbb{R}^n) \) for better distinction, we then have a rather obvious commutative diagram of categories and functors

\[
\begin{array}{ccc}
\mathcal{C}(\partial M) & \xrightarrow{\subset} & \mathcal{C}(M) \\
\downarrow\tan & & \downarrow\tan \\
\mathcal{D}(\mathbb{R}^{n-1}) & \xrightarrow{\subset} & \mathcal{D}(\mathbb{R}^n).
\end{array}
\]

In addition, there are homotopy equivalences of pairs

\[
(M, \partial M) \simeq (|\mathcal{C}(M)|, |\mathcal{C}(\partial M)|)
\]

\[
(|\text{TOP}(\mathbb{R}^n)|, |\text{TOP}(\mathbb{R}^{n-1})|) \simeq (|\mathcal{D}(\mathbb{R}^n)|, |\mathcal{D}(\mathbb{R}^{n-1})|)
\]

and so we may think of the diagram of nerves obtained from (4–4) as the (pairwise) classifying map for the tangent bundle pair \((\tau_M, \tau_{\partial M})\).

5. Microcharacteristics for noncompact spaces

In §2, we defined \( A^\%(Y) \) and \( \langle Y \rangle \in A^\%(Y) \), assuming that \( Y \) is a compact ENR. These definitions, specifically in the formulation 2.3.6, make perfectly good sense for noncompact \( Y \). In the noncompact case, however, we write \( A^\%_{ij}(Y) \) instead of \( A^\%(Y) \) for reasons given in 5.1 and 5.2. In technical terms,

\[
A^\%_{ij}(Y) := \text{hofiber}[P(Y) \to A(Y \times [0, \infty) \ltimes Y \times [0, \infty])_\infty]
\]

and \( P(Y) \) is contractible (by the appropriate generalization of 2.3.4). What follows is the appropriate generalization of 2.3.3, justifying the notation.

5.1. Theorem [CaPe], [CaPeVo]. The functor

\[
Y \mapsto F(Y) := A(Y \times [0, \infty) \ltimes Y \times [0, \infty])_\infty
\]
on the category of ENR’s and proper maps is homotopy invariant and pro-excisive (details follow).

Details. The meaning of *homotopy invariance* in 5.1 is that $F$ takes proper homotopy equivalences to homotopy equivalences. The meaning of *pro-excisive* in 5.1 is as follows. Consider a commutative diagram of ENR’s and proper continuous maps

$$
\begin{array}{ccc}
Y_1 & \longrightarrow & Y_2 \\
\downarrow & & \downarrow \\
Y_3 & \longrightarrow & Y_4
\end{array}
$$

(5–1)

Let $Y_0$ be the homotopy pushout of $Y_3 \leftarrow Y_1 \rightarrow Y_2$. This is an ENR by [Hu]. We say that (5–1) is a *proper* homotopy pushout square if the map from $Y_0$ to $Y_4$ which it determines is a proper homotopy equivalence (=invertible up to proper homotopies). The functor $F$ takes proper homotopy pushout squares of the form (5–1) to homotopy pushout squares of spectra, and it takes $\emptyset$ to a contractible spectrum. Finally, the canonical graded homomorphism

$$\pi_* F(N) \longrightarrow \prod_{i \in \mathbb{N}} \pi_* [F(N \setminus \{i\}) \rightarrow F(N)]$$

is an isomorphism.

For the proof of 5.1, we urge the reader to take a look at the first chapter of [CaPe], and especially Thm. 1.40 of [CaPe] which is the $K$-theory version of 5.1 above. The A–theory result can be found in [CaPeVo], but it must be pieced together from Lemma 3.1, Thm. 2.21 and Prop. 2.18 in [CaPeVo] (and we can only hope that the numbering is final). We suggest choosing $X = \ast$, $F = Y \cup \ast$ (one–point compactification) and $C = \text{point at infinity}$ in Prop. 2.18 of [CaPeVo]. □

So far we have allowed proper maps as morphisms between our ENR’s, but we can be more generous. Namely, any proper map $Y \rightarrow Y'$ gives rise to a continuous pointed map $Y \cup \ast \rightarrow Y' \cup \ast$ of the one–point compactifications, but not every continuous pointed map $Y \cup \ast \rightarrow Y' \cup \ast$ comes from a proper map $Y \rightarrow Y'$. In the sequel we call a pointed continuous map $Y \cup \ast \rightarrow Y' \cup \ast$ a *morphism* from $Y$ to $Y'$. Equivalently, a morphism from $Y$ to $Y'$ is a proper continuous map $f$ from an *open subset* $X$ of $Y$ to $Y'$. In particular, the inclusion of an open subset $X \subset Y$ is a morphism from $Y \rightarrow X$ (take $Y' = X$ and $f = \text{id}_X$).

It is surprising, but easy to verify, that $A^\%_{\bar{q}}(Y)$, with the technical definition given above, is functorial in $Y$ for *morphisms*. Showing this amounts to defining a restriction map

$$A^\%_{\bar{q}}(Y) \longrightarrow A^\%_{\bar{q}}(X)$$

corresponding to any open subset $X \subset Y$. Now a germ near $Y \times \{\infty\}$ of retractive spaces over $Y \times [0, \infty)$ can be restricted to a germ near $X \times \{\infty\}$ of retractive spaces over $X \times (0, \infty)$. Just take the appropriate inverse images, etc. etc.— it all works out. In particular, restriction takes weak equivalences between germs of proper retractive ENR’s to weak equivalences. Similarly, restriction takes microequivalences between proper retractive ENR’s over $Y \times [0, \infty)$ to microequivalences.
5.2. Proposition. Let $F$ be any functor from ENR’s and their morphisms (!) to CW–spectra. Suppose that $F$ is pro–excisive. Suppose or arrange that $F(*)$ is an $\Omega$–spectrum. Then there exists a chain of natural weak homotopy equivalences

$$F(Y) \simeq \cdots \simeq (Y \cup *) \wedge F(*)$$

where $Y \cup *$ is the one–point compactification. Informally, we call $F$ a “locally finite homology theory”.

Proof. See [WWPro].

5.3. Observation. The microcharacteristic is lax natural for open embeddings $X \to Y$ of ENR’s.

(Recall: an open embedding $X \to Y$ is a morphism $Y \to X$, which induces a map $A^\%_{\ell f}(Y) \to A^\%_{\ell f}(X)$.)

We proceed to the construction of characteristic classes etc. for euclidean bundles, by taking the microcharacteristics of their fibers. Fix a space $V$ homeomorphic to $\mathbb{R}^n$ for some $n$. As in §4, let $\mathcal{D} = \mathcal{D}(V)$ be the discrete monoid of embeddings $V$, so that $|\mathcal{D}| \simeq B\text{TOP}(V)$. Note that $\mathcal{D}^{\text{op}}$, not $\mathcal{D}$, acts on $A^\%_{\ell f}(V)$. The action determines a functor $q$ from $\mathcal{D}^{\text{op}}$ to spaces, taking the unique object to $A^\%_{\ell f}(V)$. Lax naturality of the microcharacteristic $\langle\langle V\rangle\rangle$, as in 5.3, determines a point $e$ in $\text{holim} q \simeq \Gamma(q)$ where $q : \text{hocolim} q \to |\mathcal{D}^{\text{op}}|$ is the projection (see 2.1.2 for notation). Informally, $e$ is a section of $q$. In §7 we shall see that $e$ can be identified with the Euler section of the universal euclidean bundle on $B\text{TOP}(V)$. (See the introduction.)

5.4. Remark. For any $V$ as above, product with $\mathbb{R}$ defines a map

$$(5–2) \quad \mathbb{R} \times : A^\%_{\ell f}(V) \to A^\%_{\ell f}(\mathbb{R} \times V)$$

taking $\langle\langle \mathbb{R}^n\rangle\rangle$ to $\langle\langle \mathbb{R} \times V\rangle\rangle$. Do not confuse (5–2) with the map induced by a certain inclusion $V \to \mathbb{R} \times V$. Here are some more details: The product of $\mathbb{R}$ with a proper retractive ENR over $V \times [0, \infty)$ is a proper retractive ENR over $\mathbb{R} \times V \times [0, \infty)$; similarly for germs, weak equivalences, microequivalences, and so on.

The following facts will be quite important in the sequel:

- (5–2) commutes with the right actions of $\mathcal{D}(V)$, the discrete monoid of embeddings $V \to V$;
- it is equivariantly nullhomotopic, loosely speaking.

We prove the second of these—the first is obvious. Let $\mathbb{R}_\lfloor = [-\infty, +\infty)$. There is a factorization of (5–2) of the form

$$A^\%_{\ell f}(V) \xrightarrow{\lambda} A^\%_{\ell f}(\mathbb{R}_\lfloor \times V) \xrightarrow{\omega} A^\%_{\ell f}(\mathbb{R} \times V)$$

where $\lambda$ is product with the space $\mathbb{R}_\lfloor$, and $\omega$ is induced by the “morphism” from $\mathbb{R}_\lfloor \times V$ to $\mathbb{R} \times V \simeq \mathbb{R} \times V$ which corresponds to the inclusion

$$\mathbb{R} \times V \hookrightarrow \mathbb{R}_\lfloor \times V.$$
5.5. Lemma. Let \( r : \mathbb{R} \times V \to \mathbb{R} \times V \) be the reflection at \( V \). The following is a homotopy pullback square:

\[
\begin{array}{ccc}
A^\varnothing_f(V) & \xrightarrow{\lambda} & A^\varnothing_f(\mathbb{R}_1 \times V) \\
\downarrow \lambda & & \downarrow \omega \\
A^\varnothing_f(\mathbb{R}_1 \times V) & \xrightarrow{r \ast \omega} & A^\varnothing_f(\mathbb{R} \times V) \\
\end{array}
\]

Proof. Let \( \mathbb{R}_1 = [-\infty, +\infty] \). We can factor \( \lambda \) as

\[
(5–3) \quad A^\varnothing_f(V) \longrightarrow A^\varnothing_f(\mathbb{R}_1 \times V) \longrightarrow A^\varnothing_f(\mathbb{R}_1 \times V)
\]

where the first map is given by product with the space \( \mathbb{R}_1 \), and the second is induced by the “morphism” from \( \mathbb{R}_1 \times V \) to \( \mathbb{R}_1 \times V \) corresponding to the inclusion of \( \mathbb{R}_1 \times V \) in \( \mathbb{R}_1 \times V \). The first arrow in (5–3) is clearly a homotopy equivalence. Hence it is enough to show that

\[
(5–4) \quad A^\varnothing_f(\mathbb{R}_1 \times V) \longrightarrow A^\varnothing_f(\mathbb{R}_1 \times V)
\]

\[
\downarrow \omega \\
A^\varnothing_f(\mathbb{R}_1 \times V) \xrightarrow{r \ast \omega} A^\varnothing_f(\mathbb{R} \times V)
\]

is a homotopy pullback square (all maps induced by “morphisms” corresponding to inclusions in the opposite direction). Square (5–4) is the target of an inclusion–induced map from another commutative square

\[
(5–5) \quad A^\varnothing_f(V) \longrightarrow A^\varnothing_f([0, \infty) \times V)
\]

\[
\downarrow \omega \\
A^\varnothing_f((-\infty, 0] \times V) \longrightarrow A^\varnothing_f(\mathbb{R} \times V)
\]

(all maps in (5–5) induced by proper inclusions, no reversal of direction). The various maps connecting vertices of (5–4) with vertices of (5–5) are induced by proper homotopy equivalences, so by 5.1 they are themselves homotopy equivalences. It is therefore enough to show that (5–5) is a homotopy pullback square. But this follows from 5.1. \( \square \)

6. The micro–index theorem: Reduction to the euclidean case

The plan is to prove that microcharacteristics of manifolds (see the remark just below) are Poincaré dual to the Euler sections \( e \) of their tangent bundles as defined in §5. Of course, this needs to be proved for families of manifolds. We therefore consider several cases:

(1) one manifold without boundary at a time;

(2) a “flat” family (= fiber bundle with discrete structure group) of such manifolds;
(3) an arbitrary family (= fiber bundle) of manifolds without boundary;
(4) one manifold with boundary;
(5) a “flat” family (= fiber bundle with discrete structure group) of manifolds with boundary.

(6) an arbitrary family (= fiber bundle) of manifolds with boundary.

In cases (1) and (4), the (provisional) index theorem is essentially true by inspection. Cases (2) and (5) follow by dint of naturality, while the McDuff–Segal–Mather–Thurston theory is needed to make the step from (2) to (3) and from (5) to (6).

Remark on terminology. The McDuff et al. theory works for manifold with trivial ends. A manifold has trivial ends if it can be obtained from a compact manifold by deleting some components of the boundary. For this reason, manifold means manifold with trivial ends in this section.

Case (1). Suppose that $M^n$ has empty boundary, and tangent bundle $\tau$. Define $\mathcal{C}$ and $\mathcal{D}$ as in §4. Then $|\mathcal{C}|$ is a model for the homotopy type of $M$, and $|\text{tan}| : |\mathcal{C}| \to |\mathcal{D}|$ is a model for the classifying map of $\tau$. What are the appropriate models for Euler fibration and Euler section? We saw (sequel of 5.3) that $\mathcal{D}^{\text{op}}$ acts on $A_{ij}^{\%}(\mathbb{R}^n)$ by homotopy automorphisms. The action gives a functor $q$ from $\mathcal{D}^{\text{op}}$ to spaces. Our model for the Euler fibration of $\tau$ is the hocolim$(q \cdot \text{tan}) \to |\mathcal{C}^{\text{op}}|$, and our model for the section space of the Euler fibration of $\tau$ is holim$(q \cdot \text{tan})$ (see 2.1.2). Lax naturality of $\langle\langle \mathbb{R}^n \rangle\rangle$ gives rise to a point $e(\tau)$ in holim$(q \cdot \text{tan})$. This is our model for the Euler section of $\tau$. What is the appropriate model for Poincaré duality? Of course, we take the map $\varphi$ from (3–1) and remark 3.4. In our setup, $X = A(*)$ and we may replace $\Omega^\infty(M_+ \land X)$ by $A_{ij}^{\%}(M)$ as defined in §2 and §5. We may also replace $\Gamma(M; X)$ by holim$(q \cdot \text{tan})$. Then $\varphi$ becomes the “scanning” map

\[
A_{ij}^{\%}(M) \to \lim_{f: \mathbb{R}^n \to M} A_{ij}^{\%}(\mathbb{R}^n) \subset \lim_{f: \mathbb{R}^n \to M} A_{ij}^{\%}(\mathbb{R}^n) = \text{holim}(q \cdot \text{tan})
\]

induced by the various $f: \mathbb{R}^n \to M$, which one must remember are “morphisms” from $M$ to $\mathbb{R}^n$ in the category of ENR’s. Lax naturality of the microcharacteristic provides a canonical path from the image of $\langle\langle M \rangle\rangle$ under this map to $e(\tau)$ as defined above. □

Case (2). It is enough to consider the universal situation

\[
p : ET \times_T M \to |T|
\]

where $T = \text{TOP}^\delta(M)$. As before, we must translate everything into categories, nerves, and so on. In particular we need the category $\mathcal{E}T$ with object set $T$, and exactly one morphism between any two objects. $T$ acts on $\mathcal{E}T$ by left translation. We still have $\mathcal{C}$ as in Case (1). Then

\[
\mathcal{E}T \times_T \mathcal{C} = (\mathcal{E}T \times \mathcal{C})/T
\]
is a category whose classifying space is our model for $ET \times_T M$. Note that the object set of $ET \times_T \mathcal{C}$ is canonically identified with that of $\mathcal{C}$, but a morphism in $ET \times_T \mathcal{C}$ from $f : \mathbb{R}^n \to M$ to $g : \mathbb{R}^n \to M$ is a pair $(\lambda, h)$ where $h \in T$, $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is an embedding, and $f = hg\lambda$. Composition of morphisms is defined by $(\lambda_1, h_1)(\lambda_2, h_2) = (\lambda_2\lambda_1, h_1h_2)$. The rule $(\lambda, h) \mapsto \lambda$ is a functor from $ET \times_T \mathcal{C}$ to the monoid $\mathcal{D}$ of embeddings from $\mathbb{R}^n \to \mathbb{R}^n$. We denote it by $\tan_v$ to stress the analogy with Case (1).

The induced map of classifying spaces,

\[(6–3) \quad |ET \times_T \mathcal{C}| \longrightarrow |\mathcal{D}|,
\]

is our model for the map classifying the vertical tangent bundle $\tau_v$ of the manifold bundle (6–2). As before, $\mathcal{D}^{op}$ acts on $A^\%_f(\mathbb{R}^n)$ by homotopy automorphisms, which gives rise to a functor $q$ on $\mathcal{D}^{op}$; our model for the Euler fibration of $\tau_v$ is

\[\text{hocolim}(q \cdot \tan_v) \longrightarrow |ET \times_T \mathcal{C}|\]

and our model for the Euler section is a certain point $e(\tau_v)$ in $\text{holim}(q \cdot \tan_v)$ which we get from the lax naturality of $\langle \langle \mathbb{R}^n \rangle \rangle$. Finally our model for fiberwise Poincaré duality is essentially still the scanning map (6–1). But now we think of it as a $T$–map, or as a map between bundles on $|T|$, and focus on the induced map between section spaces. Using homotopy limits as models for section spaces, this takes the form

\[(A^\%_f(M))^{hT} \longrightarrow \text{holim}(q \cdot \tan_v)\]

or equivalently

\[(A^\%_f(M))^{hT} \longrightarrow (\text{holim}(q \cdot \tan))^{hT}\]

because the homotopy limit of a group action is a space of homotopy fixed points. Lax naturality provides a canonical path from the image of the fiberwise microcharacteristic section, now denoted $\langle \langle p \rangle \rangle$, to the Euler section $e(\tau_v)$. □

**Case (3).** Here the “vertical bars” notation for classifying spaces too confusing, so we make an exception and use the prefix $B$. Keeping the notation from Case (2), we let $T' = \text{TOP}(M)$ and

\[p' : ET' \times_{T'} M \longrightarrow BT',\]

with vertical tangent bundle $\tau'$. We construct $BT'$ by thinking of $T'$ as a simplicial group. Then $BT'$ is the geometric realization of a bisimplicial set, and contains $BT$ as its vertical 0–skeleton. Recall the McDuff–Mather–Thurston theorem to the effect that the inclusion $BT \subset BT'$ is a homology equivalence. Suppose for the moment that the actions of $T$ on $A^\%_f(M)$ and on $\text{holim}(q \cdot \tan)$ can be extended in a canonical way to $T'$. Then an easy obstruction theory argument shows that the forgetful vertical maps in the would–be square

\[\begin{array}{ccc}
(A^\%_f(M))^{hT} & \xrightarrow{\varphi} & (\text{holim}(q \cdot \tan))^{hT} \\
\uparrow{\phi_1} & & \uparrow{\phi_2} \\
(A^\%_f(M))^{hT'} & & (\text{holim}(q \cdot \tan))^{hT'}
\end{array}\]

for fiberwise Poincaré duality is essentially still the scanning map (6–1). But now we think of it as a $T$–map, or as a map between bundles on $|T|$, and focus on the induced map between section spaces. Using homotopy limits as models for section spaces, this takes the form

\[(A^\%_f(M))^{hT} \longrightarrow \text{holim}(q \cdot \tan_v)\]

and our model for the Euler section is a certain point $e(\tau_v)$ in $\text{holim}(q \cdot \tan_v)$ which we get from the lax naturality of $\langle \langle \mathbb{R}^n \rangle \rangle$. Finally our model for fiberwise Poincaré duality is essentially still the scanning map (6–1). But now we think of it as a $T$–map, or as a map between bundles on $|T|$, and focus on the induced map between section spaces. Using homotopy limits as models for section spaces, this takes the form

\[(A^\%_f(M))^{hT} \longrightarrow \text{holim}(q \cdot \tan_v)\]

and our model for the Euler section is a certain point $e(\tau_v)$ in $\text{holim}(q \cdot \tan_v)$ which we get from the lax naturality of $\langle \langle \mathbb{R}^n \rangle \rangle$. Finally our model for fiberwise Poincaré duality is essentially still the scanning map (6–1). But now we think of it as a $T$–map, or as a map between bundles on $|T|$, and focus on the induced map between section spaces. Using homotopy limits as models for section spaces, this takes the form

\[(A^\%_f(M))^{hT} \longrightarrow \text{holim}(q \cdot \tan_v)\]

and our model for the Euler section is a certain point $e(\tau_v)$ in $\text{holim}(q \cdot \tan_v)$ which we get from the lax naturality of $\langle \langle \mathbb{R}^n \rangle \rangle$. Finally our model for fiberwise Poincaré duality is essentially still the scanning map (6–1). But now we think of it as a $T$–map, or as a map between bundles on $|T|$, and focus on the induced map between section spaces. Using homotopy limits as models for section spaces, this takes the form

\[(A^\%_f(M))^{hT} \longrightarrow \text{holim}(q \cdot \tan_v)\]

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\[(A^\%_f(M))^{hT} \longrightarrow \text{holim}(q \cdot \tan_v)\]
are homotopy equivalences. Hence the missing lower horizontal arrow can be filled in, and this is our definition of the fiberwise Poincaré duality map for the manifold bundle $p'$. In the same spirit, we define the microcharacteristic section $\langle \langle p' \rangle \rangle$ as the essentially unique point in the homotopy fiber of $\phi_1$ over $\langle p \rangle$, and we define the Euler section of the vertical tangent bundle of $p'$ as the essentially unique point in the homotopy fiber of $\phi_2$ over the Euler section of the vertical tangent bundle of $p$. Then by definition, fiberwise Poincaré duality takes the microcharacteristic section $\langle \langle p' \rangle \rangle$ to the Euler section of the vertical tangent bundle of $p'$, up to a canonical homotopy of sections. In short, the problem has been defined a way.

It remains to be seen why the actions of $T$ on $A^%_{\ell f}(M)$ and on $\text{holim}(q \cdot \text{tan})$ can be extended to $T'$. As one might expect, this requires a “change of models”. The forgetful map

$$\text{map}_T(T', A^%_{\ell f}(M)) \to \text{map}_T(T, A^%_{\ell f}(M)) \cong A^%(M)$$

is a homotopy equivalence by obstruction theory. (Its homotopy fiber is a space of sections of a fiber bundle with acyclic base $T'/T$ and nilpotent fiber $A^%(M)$; moreover the action of $\pi_1$ of the base on the homotopy groups of the fiber is trivial.) The domain of this forgetful map is our new model for $A^%_{\ell f}(M)$, and clearly it comes with an action of $T'$. We can subject the $T$–space $\text{holim}(q \cdot \text{tan})$ to exactly the same treatment. □

### 6.1. Notation.

Suppose that $(X, Y)$ is a pair of spaces, and that $q, \partial q$ are fibrations or just quasifibrations on $X, Y$, respectively, related by a map $t: \partial q \to q|_Y$ over $Y$. We call $(q, \partial q)$ a pair of quasifibrations on $(X, Y)$, and define the space of sections $\Gamma(q, \partial q)$ as the holim (= homotopy pullback) of

$$\Gamma(\partial q) \xrightarrow{t^*} \Gamma(\imath^* q) \xleftarrow{\imath^* q} \Gamma(q)$$

where $\imath: Y \to X$ is the inclusion. (This is in the notation of 2.1.2.) Note that the homotopy pullback contains the strict pullback.

**Case (4).** Here we assume that $M$ is compact and equipped with a collar (details as in 4.3). Recall the categories $\mathcal{C}(M)$ and $\mathcal{C}(\partial M)$ from 4.3. Our model for the “relative” Euler fibration associated with the tangent bundle pair $\tau = (\tau^M, \tau^{\partial M})$ is a pair of quasifibrations $(q(\tau), \partial q(\tau))$ on the pair $(|\mathcal{C}(M)|, |\mathcal{C}(\partial M)|)$. Here $q(\tau)$ is constructed from the functor

$$q: \mathcal{C}(M) \to \text{spectra}$$

$$(f: \mathbb{R}^n \hookrightarrow M) \mapsto A^%_{\ell f}(\mathbb{R}^n)$$

as in Case (1), and $\partial q(\tau)$ is constructed from the functor

$$\partial q: \mathcal{C}(\partial M) \to \text{spectra}$$

$$(g: \mathbb{R}^{n-1} \hookrightarrow \partial M) \mapsto A^%_{\ell f}(\mathbb{R}^{1} \times \mathbb{R}^{n-1})$$
(notation of 5.4). Note: In general, despite appearances, none of these two functors
is constant—they do not take all morphisms to the identity morphism. We are
dealing with a pair of quasifibrations because there is a map \( t \) from \( \partial q(\tau) \) to \( q(\tau) \)
covering the inclusion of \( |\mathcal{C}(\partial M)| \) in \( |\mathcal{C}(M)| \). It is induced by the inclusion \( \text{int}(\mathbb{R}_1 \times \mathbb{R}^{n-1}) \to \mathbb{R}_1 \times \mathbb{R}^{n-1} \)
which one must remember is a “morphism” from \( \mathbb{R}_1 \times \mathbb{R}^{n-1} \) to \( \text{int}(\mathbb{R}_1 \times \mathbb{R}^{n-1}) \) in the category of ENR’s.— Our model for the section space of the
pair \((q, \partial q)\) is the homotopy pullback of
\[
\text{holim} q \to \text{holim} q \leftarrow \text{holim} \partial q.
\]
In particular, this is where \( e(\tau) \) lives. Our model for Poincaré duality is the scanning
map \( \varphi \) of 3.4, which in our present setup has domain \( A^\mathcal{C}(M) \) and codomain equal
to the homotopy pullback just mentioned. By inspection, this takes \( \langle \langle M \rangle \rangle \) to \( e(\tau) \),
up to a canonical homotopy of sections. □

Case (5). This is of course very much like Case (2), but there is one little point
we must worry about. We used a collar in Case (4). How can we equip \( M \) with a
collar that is invariant under the group of homeomorphisms \( M \to M \)? The answer
is simple: We fix a closed collar
\[
(6–6) \quad \partial M \times [-\infty, +\infty] \hookrightarrow M
\]
and, instead of allowing arbitrary homeomorphisms \( M \to M \), allow only those
which commute with (6–6). The classifying space of the discrete group of these
homeomorphisms still maps to the classifying space of \( \text{TOP}(M) \) by a homology
equivalence; hence it is good enough for our purposes, that is, good enough for
Case (6). □

Case (6). Left to the reader.

7. The micro–index theorem for euclidean space

It has already been explained in the section abstracts how the micro–index the-
orem translates into the statement that our two different ways of associating Eu-
ler sections to euclidean bundles “agree”. One crucial property of Euler sections
of euclidean bundles, say with fibers homeomorphic to \( \mathbb{R}^n \), is the fact that they
are nullhomotopic when the euclidean bundles in question admit a reduction from
structure group \( \text{TOP}(n) \) to structure group \( \text{TOP}(n – 1) \). For the Euler sections
\( e \) defined in the introduction, this is true by inspection. For the competing Euler
sections \( e \), we verified it in §5. The crucial property is needed to show that the
Euler section of the tangent bundle of a manifold \( M \) with boundary is canonically
nullhomotopic over the boundary. It is a direct consequence of two more basic
properties as follows. Let \( \gamma \) be a Euclidean \( n \)–bundle on some space \( X \), and form
\( \gamma’ \) by adding a trivial line bundle. Let \( q(\gamma) \) and \( q(\gamma’) \) be the corresponding Euler
fibrations, defined as in the introduction or as in §5.

(1) There is a map \( q(\gamma) \to q(\gamma’) \) over \( X \) which takes Euler section to Euler
section.

(2) This map is fiberwise nullhomotopic.
Actually the map in (2) is fiberwise nullhomotopic in two preferred ways, giving a
map from \( q(\gamma) \) to \( \Omega_X q(\gamma') \) which happens to be a fiberwise homotopy equivalence.
This map is clearly important to us, in any situation involving stabilization of
euclidean bundles and stabilization of manifolds, such as theorem 0.4. We do not
want to lose it in the abstraction process.

Our point here is that “abstract” Euler sections tend to arise naturally and
intrinsically in connection with spaces having a coordinate free filtration. We are
thinking of spaces such as \( BO \) and \( BTOP \).

### 7.1. Abstract Euler sections

#### 7.1.1. Definitions

Let \( J \) be the category of finite dimensional real vector spaces
\( U, V, W, \ldots \) with inner product (positive definite). The space of morphisms from \( U \)
to \( V \) is the Stiefel manifold of linear maps \( U \rightarrow V \) respecting the inner product ; it
is empty if and only if \( \dim(U) > \dim(V) \). Then \( J \) is a topological category with a
discrete class of objects. We are interested in covariant continuous functors \( Z \)
from \( J \) to spaces. Continuity of \( Z \) means that the evaluation maps \( \text{mor}(U,V) \times Z(U) \rightarrow Z(V) \)
are continuous. Spaces means spaces which are homotopy equivalent to CW–
spaces.

It is a good idea to think of such functors \( Z \) as spaces with a coordinate free
filtration. Example: \( Z(V) = BO(V) \); the (filtered) space that one should think of
is \( BO \). Or take \( Z(V) = BTOP(V) \); the filtered space that one should think of is
\( BTOP \). See [We] for more examples.

#### 7.1.2. Definition

Let \( Z \) be a continuous functor from \( J \) to spaces. An abstract Euler fibration on \( Z \)
consists of another continuous functor \( Y \) from \( J \) to spaces, a
natural transformation \( p : Y \rightarrow Z \), and a natural factorization
\[ Y(V) \xrightarrow{\cong} Z^2(V \times \mathbb{R}) \xrightarrow{\zeta} Y(V \times \mathbb{R}) \]
of the map \( Y(V) \rightarrow Y(V \times \mathbb{R}) \) induced by the inclusion \( V \rightarrow V \times \mathbb{R} \). Condition: the
composition \( p\zeta \) from \( Z^2(V \times \mathbb{R}) \) to \( Z(V \times \mathbb{R}) \) is a homotopy equivalence.

Warning: \( V \mapsto Z^2(V \times \mathbb{R}) \) is intended to be a continuous functor on \( J \) and nothing
else, so that \( Z^2(V) \) is not defined in general.

An abstract Euler section for the abstract Euler fibration above is a natural
transformation \( e : Z \rightarrow Y \) such that \( pe = \text{id}_Z \).

#### 7.1.3. Example

Let \( Z(V) = * \) for all \( V \). Let \( Y(V) = \text{TOP}(\mathbb{R} \oplus V)/\text{TOP}(V) \).
Note that \( Y(V) \) is pointed. The map \( Y(V) \rightarrow Y(V \oplus \mathbb{R}) \) induced by \( V \subset V \oplus \mathbb{R} \)
has a nullhomotopy \( \{h_t\} \), where \( h_t \) for \( 0 \leq t \leq \infty \) is given by
\[ [f] \mapsto [(f \times \text{id}_1) J_t] \]
and \( J_t \) is any linear isometry of \( \mathbb{R} \oplus V \oplus \mathbb{R} \) taking the vector \( (1,0,\ldots,0) \) to a
suitable positive scalar multiple of \( (1,0,\ldots,0,t) \). We may regard the nullhomotopy
as a factorization
\[ Y(V) \rightarrow Z^2(V \oplus \mathbb{R}) \rightarrow Y(V \oplus \mathbb{R}) \]
of the usual map \( Y(V) \rightarrow Y(V \oplus \mathbb{R}) \), where \( Z^2(V \oplus \mathbb{R}) \) is the cone on \( Y(V) \). This
defines an abstract Euler fibration on \( Z \).
An abstract Euler section for this abstract Euler fibration is a natural transformation \( e : Z(V) \to Y(V) \), which is of course fully determined if we specify it for \( V = 0 \). We do so by taking the unique point in \( Z(0) \) to the coset \([-\text{id}]\).

**7.1.4. Example.** This is essentially the same as 7.1.3, but it is technically superior (also less explicit, unfortunately). We let \( Z(V) = * \) and

\[
Y(V) := \text{hofiber}[B\text{TOP}(V) \to B\text{TOP}(\mathbb{R} \oplus V)].
\]

For \( W \) in \( \mathcal{J} \), let \( W^c \) be the one point compactification. It is shown in [We] that there is a binatural and associative transformation \( Y(V) \land W^c \to Y(V \oplus W) \) which is the obvious homeomorphism when \( W = 0 \). (For \( W = \mathbb{R} \), it looks essentially like what we saw in 7.1.3.) In particular, the inclusion–induced map \( Y(V) \to Y(V \oplus \mathbb{R}) \) is canonically nullhomotopic. We obtain an abstract Euler fibration (and an abstract Euler section) much as in 7.1.3. The technical advantages will be seen in 7.1.5.

**7.1.5. Example.** This is the stabilization of 7.1.4. Directly from 7.1.4, we see that \( \{Y(V \oplus \mathbb{R}^i) \mid i \geq 0\} \) is a spectrum \( Y(V) \) for each \( V \), and that we still have a binatural transformation \( Y(V) \land W^c \to Y(V \oplus V) \) which is the obvious identification when \( W = 0 \). In particular, the inclusion–induced map \( Y(V) \to Y(V \oplus \mathbb{R}) \) is naturally nullhomotopic. If we now define \( sY(V) := \Omega^\infty Y(V) \), then \( sY \) with the map \( sY \to * \) and the factorization of \( i : sY(V) \to \sigma(sY)(V) \) through the reduced cone on \( sY(V) \), for all \( V \), constitute an abstract Euler fibration on \( * \). We have a stabilization map from \( Y \) in 7.1.4 to \( sY \), so that \( sY \) inherits the abstract Euler section from \( Y \).

*Notation.* From now on a \( \backslash \) is used to denote homotopy orbit spaces (Borel construction); thus \( G\backslash X \) is the homotopy orbit space of a left action of the group or monoid \( G \) on the space \( X \). Often \( G \) is a topological group or monoid.

**7.1.6. Example.** Keep the notation of 7.1.3, or that of 7.1.4. Note that the group \( \text{TOP}(V) \) acts on the left of each \( Y(V) \) and \( Z^\sharp(V \oplus \mathbb{R}) \), leaving base points fixed. Let

\[
\begin{align*}
\tilde{Z}(V) &:= \text{TOP}(V)\backslash Z(V) = B\text{TOP}(V) \\
\tilde{Y}(V) &:= \text{TOP}(V)\backslash Y(V) \\
\tilde{Z}^\sharp(V \oplus \mathbb{R}) &:= \text{TOP}(V)\backslash CY(V) \amalg_{B\text{TOP}(V)} B\text{TOP}(V \oplus \mathbb{R})
\end{align*}
\]

where \( CY(V) \) is the reduced cone on \( Y(V) \) and the amalgamation identifies the subspace \( \text{TOP}(V)\backslash * \) of \( \text{TOP}(V)\backslash CY(V) \) with \( \tilde{Z}(V) \subset \tilde{Z}(V \oplus \mathbb{R}) \). Then the projection \( Y \to Z \) together with the now obvious factorizations of \( \tilde{Y}(V) \to \tilde{Y}(V \oplus \mathbb{R}) \) through \( \tilde{Z}^\sharp(V \oplus \mathbb{R}) \), for all \( V \), constitute an abstract Euler fibration on \( \tilde{Z} \). Further, \( \tilde{e} := \text{TOP}(V)\backslash e \) from \( \tilde{Z}(V) \) to \( \tilde{Y}(V) \), for all \( V \), defines an abstract Euler section.

**7.1.7. Example.** Replace \( Y \) by \( sY \) in 7.1.6 (and for the meaning of \( sY \), see 7.1.5).

**7.1.8. Example.** For \( V \) in \( \mathcal{J} \) let \( Z(V) = * \) and \( Y(V) = A^\%_\mathcal{J}(V) \). A morphism \( V \to W \) in \( \mathcal{J} \) determines a projection \( W \to V \), and pullback with this is an exact functor between the appropriate categories of retractive spaces which induces \( Y(V) \to Y(W) \). In this way \( Y \) becomes a functor.
Warning 1: The morphism \( V \to W \) also determines a proper inclusion of locally compact spaces \( V \to W \), which in turn leads to a much more obvious map \( A^\%(V) \to A^\%(W) \). But this is not the map we want—explanation in a moment.

Warning 2: We must use a model of the category of sets with a canonical choice of pullbacks such that pullbacks are associative—otherwise \( Y \) will not be a functor. Details are given in the remark just below.

Warning 3: The functor \( Y \) is not continuous. We shall come back to this point. Let \( Z^\%(V \oplus \mathbb{R}) := A^\%(V \times \mathbb{R}) \). The usual map \( Y(V) \to Y(V \oplus \mathbb{R}) \) has a canonical factorization through \( Z^\%(V \oplus \mathbb{R}) \), for all \( V \). (Use the exact functor induced by pullback with the projection \( V \times \mathbb{R} \to V \) to get from \( Y(V) \) to \( Z^\%(V \oplus \mathbb{R}) \), and use the restriction map associated with the open embedding \( V \times \mathbb{R} \to V \times \mathbb{R} \) to get from \( Z^\%(V \oplus \mathbb{R}) \) to \( Y(V \oplus \mathbb{R}) \).) Therefore we have all the ingredients for an abstract Euler fibration on \( Z = * \), except continuity.

To specify an abstract Euler section \( e : Z(V) \to Y(V) \), we specify it for \( V = 0 \), by sending the unique point in \( Z(0) \) to the microcharacteristic \( (\langle 0 \rangle) = (\langle * \rangle) \in A(*) = Y(0) \). For arbitrary \( V \), we may then call the unique point in the image of \( e : Z(V) \to Y(V) \) the ”microcharacteristic” of \( V \), without contradicting earlier definitions too shamelessly. This is the belated justification for making \( Y \) into a functor the way we did.

About the lack of continuity: It is explained in Appendix B how continuity can be enforced. A key observation is that the functor \( Y \) for example can be extended from \( \mathcal{J} \) to a larger category \( \mathcal{J}^{\text{top}} \) which has the same objects as \( \mathcal{J} \). A morphism \( V \to W \) in \( \mathcal{J}^{\text{top}} \) is an equivalence class of homeomorphisms \( h : U \times V \cong W \), where \( U \) is another object in \( \mathcal{J} \); two such, say \( h_1 : U_1 \times V \cong W \) and \( h_2 : U_2 \times V \cong W \), are equivalent if there exists a homeomorphism \( g : U_1 \to U_2 \) such that \( h_2(g \times \text{id}) = h_1 \). Note that a morphism \( V \to W \) in \( \mathcal{J}^{\text{top}} \) still determines a projection \( W \to V \); it does not determine an inclusion \( V \to W \), which makes “Warning 1” above even more appropriate.— In B.7, we replace \( \mathcal{J}^{\text{top}} \) by the equivalent full subcategory with objects \( \mathbb{R}^i \) for \( i \geq 0 \), so that the set of morphisms from \( \mathbb{R}^i \) to \( \mathbb{R}^j \) is \( \text{TOP}(\mathbb{R}^i)/\text{TOP}(\mathbb{R}^{j-i}) \) made discrete.

Remark. Let \( S \) be the full subcategory of the category of sets consisting of all von Neumann ordinals \([V_a] \). A diagram

\[
\begin{array}{ccc}
P & \longrightarrow & S_2 \\
\downarrow & & \downarrow p \\
S_1 & \longrightarrow & S_3 \\
\end{array}
\]

in \( S \) is a special pullback if the two maps issuing from \( P \) induce an order preserving bijection from \( P \) to

\[
\{(a, b) \in S_1 \times S_2 \mid f(a) = p(b)\}
\]

where \( S_1 \times S_2 \) has the lexicographic ordering determined by the orderings of \( S_1 \) and \( S_2 \). In this case, and only in this case, we denote the left–hand vertical arrow in the square by \( f^*p \). One finds that \( f^*p \) is uniquely determined by \( p \) and \( f \) (and it exists when only \( p \) and \( f \) are given). The following properties are easily verified: \( (fg)^*p = g^*(f^*p) \), and \( f^*p = p \) if \( f \) is an identity morphism. This is what
we mean by “associative pullbacks”. Note that our associative pullbacks are not commutative: the domains of \( f^*p \) and \( p^*f \) are not identical, only isomorphic.

### 7.1.9. Example.

For \( V \in J \) let \( \mathcal{D}(V) \) be the discrete monoid of embeddings \( V \to V \). Then \( \mathcal{D}(V) \) acts on the right of the spaces \( Z(V) \), \( Y(V) \) and \( Y^\sharp(V) \) in example 7.1.8. We therefore obtain a new abstract Euler fibration with abstract Euler section (in the sense of 7.1.7) by using 7.1.8 and letting

\[
\bar{Z}(V) := \mathcal{D}(V)^{\text{op}} \downarrow Z(V) \cong \mathcal{D}(V) \\
\bar{Y}(V) := \mathcal{D}(V)^{\text{op}} \downarrow Y(V) \\
\bar{Z}^\sharp(V \oplus \mathbb{R}) := \mathcal{D}(V)^{\text{op}} \downarrow Z^\sharp(V \oplus \mathbb{R}) \mathcal{I}_{\mathcal{D}(V)} | \mathcal{D}(V \oplus \mathbb{R}) \\
\bar{e} := \mathcal{D}(V)^{\text{op}} \downarrow e.
\]

Note that \( \bar{e} \) is essentially \( e \) from §5. Remarks about continuity or absence of continuity made in 7.1.8 apply also here.

There is a notion of morphism between two abstract Euler fibrations. If the abstract Euler fibrations consist of \( p_i : Y_i \to Z_i \) and the natural factorizations

\[
Y_i(V) \to Z^\sharp_i(V \oplus \mathbb{R}) \to Y_i(V \oplus \mathbb{R})
\]

for \( i = 1, 2 \) and all \( V \), then a morphism would consist of natural transformations \( Z_1 \to Z_2 \), \( Y_1 \to Y_2 \), \( Z^\sharp_1(\_ \times \mathbb{R}) \to Z^\sharp_2(\_ \times \mathbb{R}) \) making certain diagrams commute. If domain and codomain of this morphism are equipped with abstract Euler sections, and the morphism respects these, then we would call it a morphism between abstract Euler fibrations with abstract Euler section. Such a morphism is an equivalence if all the maps it consists of are homotopy equivalences. Two abstract Euler fibrations with Euler section are equivalent if they can be related by a chain of equivalences.

### 7.1.10. Theorem.

The abstract Euler fibrations and Euler sections given by 7.1.7 and 7.1.9 are equivalent.

Implicit in 7.1.10 is the claim that \( B\text{TOP}(V) \) can be related to \( |\mathcal{D}(V)| \) by a chain of homotopy equivalences, natural in \( V \); but we know this already from §4.

The proof of 7.1.10 is given in the next subsection, 7.2, and appendix B and C. One property shared by the abstract Euler fibrations 7.1.7 and 7.1.9 that will help us establish their equivalence is stability. To define this notion, we suppose for simplicity that \( Z \) in 7.1.2 is a (continuous) functor from \( J \) to connected pointed spaces. Let \( Y_0(V) \) and \( Z^\sharp_0(V \times \mathbb{R}) \) be the homotopy fibers of

\[
p : Y(V) \to Z(V), \quad p\zeta : Z^\sharp_0(V \times \mathbb{R}) \to Z(V \times \mathbb{R}),
\]

respectively. Then \( Y_0 \) and the factorizations of \( Y_0(V) \to Y_0(V \oplus \mathbb{R}) \) through \( Z^\sharp_0(V \times \mathbb{R}) \) constitute an abstract Euler fibration on \( Z_0 \) where \( Z_0(V) = \ast \).
7.1.11. Definition. The abstract Euler fibration in 7.1.2 is \textit{stable} if

\begin{equation}
\begin{array}{c}
Y_0(V) \xrightarrow{i} Z_0^\sharp(V \times \mathbb{R}) \\
\downarrow \quad \downarrow \zeta
\end{array}
\end{equation}

\textit{is a homotopy pullback square for every } V. \text{ Here } \rho : V \times \mathbb{R} \to V \times \mathbb{R} \text{ is the reflection at the hyperplane } V.

It is clear that 7.1.5 is stable; therefore 7.1.7 is stable. We verified stability of 7.1.8 in 5.5. Therefore 7.1.9 is also stable. Examples 7.1.3, 7.1.4 and 7.1.6 are not stable.

7.1.12. Remarks on 7.1.2. (i) Much of the fun with “concrete” Euler sections (or Euler classes) comes from comparing the Euler sections with suitable zero sections. But where does our abstraction 7.1.2 mention zero sections? The truth is that \( Y(V) \to Z(V) \) does not have a zero section in general. However, \( p : Y(V \oplus \mathbb{R}) \to Z(V \oplus \mathbb{R}) \) has something like a preferred zero section in the shape of \( \zeta : Z^\sharp(V \oplus \mathbb{R}) \to Y(V \oplus \mathbb{R}) \). (Remember that \( p \zeta : Y(V \oplus \mathbb{R}) \to Z(V \oplus \mathbb{R}) \) is a homotopy equivalence.) For us, this is enough.

(ii) Why do we insist on the coordinate free aspect? We would be happy with less, but we do not know how to prove theorem 7.2.1 below with less.

7.2. A characterization

7.2.1. Theorem. Let \( Z(V) = B \text{TOP}(V) \). For an abstract Euler fibration with abstract Euler section on \( Z \), let \( c_n \) be the connectivity of the map between vertical homotopy fibers in the commutative square (notation of 7.1.2)

\begin{equation}
\begin{array}{c}
Z(V) \xrightarrow{\iota \cdot e} Z_0^\sharp(V \times \mathbb{R}) \\
\downarrow \quad \downarrow \zeta
\end{array}
\end{equation}

where \( \dim(V) = n \). Up to equivalence relative to \( Z \), there exists a unique stable abstract Euler fibration with abstract Euler section such that \( \lim_{n \to \infty} (c_n - n) = \infty \).

Proof and explanation. See Appendix C for the proof. Often in an equivalence between abstract Euler fibrations, the base functor \( Z \) is kept fixed. Such an equivalence would be called \textit{relative to } \( Z \). The meaning of \textit{up to equivalence relative to } \( Z \) is: there exists a chain of equivalences \( \ldots \), relative to \( Z \). Rather than just showing existence, we shall specify such a chain.

We must now verify that examples 7.1.7 and 7.1.9 satisfy the connectivity assumption in 7.2.1. In the case of 7.1.7, this is very easy. Modulo straightforward identifications, the map between vertical homotopy fibers in question becomes the stabilization map

\[
\text{TOP}(\mathbb{R} \times V)/\text{TOP}(V) \to \text{hocolim}_{k>0} \Omega^k (\text{TOP}(\mathbb{R} \times V \times \mathbb{R}^k)/\text{TOP}(V \times \mathbb{R}^k)) .
\]

Its connectivity is approximately \((4/3) \cdot \dim(V)\) by [Ig], plus Morlet smoothing theory. See also [Wal2, §2].
7.2.2. Lemma. Example 7.1.9 also satisfies the connectivity assumption in 7.2.1.

Preliminaries and notation. We assume \( V = \mathbb{R}^n \) but continue to write \( V \) because it is more convenient. Let \( Y^\sharp(V \times \mathbb{R}) \) be the homotopy pushout of

\[
\begin{array}{ccc}
Z(V) & \overset{v \cdot e}{\longrightarrow} & Z^\sharp(V \times \mathbb{R}) \\
\downarrow & & \\
Z(V \times \mathbb{R})
\end{array}
\]

so that we have a canonical map \( Y^\sharp_0(V \times \mathbb{R}) \to Y_0(V \times \mathbb{R}) \), from the diagram in 7.2.1. Think of it as a map over \( Z(V \times \mathbb{R}) \), and take homotopy fibers over the base point: this gives a map

\( \beta : Y^\sharp_0(V \times \mathbb{R}) \to Y_0(V \times \mathbb{R}) \).

Note that the domain of \( \beta \) is homotopy equivalent to the join

\[
S_0 \ast \text{TOP}(V \times \mathbb{R}) / \text{TOP}(V),
\]

and the codomain is homotopy equivalent to an \((n + 1)\)-fold delooping of \( A(*) \), where \( n = \dim(V) \). It is enough to show that the connectivity of \( \beta \) exceeds \( n \) by a quantity which goes to infinity with \( n \).

Proof of 7.2.2. Let \( a(*) \) be the suspension spectrum generated in degree \( n + 1 \) by \( Y^\sharp_0(V \times \mathbb{R}) \). By Waldhausen, and by Igusa stability, we know that \( a(*) \) is \( j \)-equivalent to \( A(*) \) where \( j = 4n/3 \) approximately. Let \( F \) be any compact CW–space, equipped with the trivial euclidean bundle \( \varepsilon^{n+1} \) with fiber \( V \times \mathbb{R} \). Recall from §1 the space \( S^\text{tp}_{n+1}(F, \varepsilon^{n+1}) \) of compact manifold structures on \( F \). Up to homotopy equivalence we can recast this as a simplicial set, as follows. A 0–simplex in \( S^\text{tp}_{n+1}(F, \varepsilon^{n+1}) \) is a compact manifold \( M^{n+1} \) with a homotopy equivalence to \( F \), and so on; we construct the (relative) Euler section \( e(\tau) \) as in Case (1) or Case (4) of §6, think of it as a map from \( M / \partial M \) to an \((n + 1)\)-fold delooping of \( A(*) \), and apply Poincaré duality to land in \( \Omega^\infty(M_+ \wedge A(*)) \simeq \Omega^\infty(F_+ \wedge A(*)) \). Map (1) is a much more primitive construction: Starting with the same 0–simplex in \( S^\text{tp}_{n+1}(F, \varepsilon^{n+1}) \) corresponding to \( M^{n+1} \) with a homotopy equivalence to \( F \), and so on, we note that near \( \partial M \) the structure group of \( \tau^M \) has a canonical reduction from \( \text{TOP}(V \times \mathbb{R}) \) to \( \text{TOP}(V) \). This
in conjunction with the trivialization of $\tau^M$ leads to a pointed map from $M/\partial M$ to the join $S^0 \ast (\TOP(V \times \mathbb{R})/\TOP(V))$ taking the complement of a collar about $\partial M$ to the nontrivial point in $S^0$. Apply Poincaré duality to land in $\Omega^\infty(M_+ \wedge a(\ast)) \simeq \Omega^\infty(F_+ \wedge A(\ast))$. Note that Poincaré duality is easy because we are dealing with stably parallelized manifolds. Last not least, map (3) is the microcharacteristic map that we know from §2. By the index theorem, to the extent that we have it in §6, the maps (2) and (3) are homotopic.

Taking a look at the commutative square in 7.2.1, we find that map (2) factors through map (1) up to homotopy. More precisely, (2) is the composition of (1) with

$$\text{id}_F \wedge \beta : F_+ \wedge a(\ast) \longrightarrow F_+ \wedge A(\ast)$$

where $\beta$ is the map $\beta$ from the preliminaries, now regarded as a map of spectra.

We seem to know nothing about $\beta$. But try taking $F = \ast$. Then it is clear that map (3) maps the unique component of $S^\mathrm{tp}_{n+1}(F, \varepsilon^{n+1})$ to the unit component of $\Omega^\infty(F_+ \wedge A(\ast))$. Hence, by the relationship just observed, $\beta$ from $a(\ast)$ to $A(\ast)$ is surjective on $\pi_0$. So we know something about $\beta$ after all.

Next, take $F = S^k$. Choose $n \gg k$. There is then an obvious inclusion of the $h$–cobordism space $|\mathcal{H}(F \times S^{n-k})|$ in $S^\mathrm{tp}_{n+1}(F, \varepsilon^{n+1})$, and the following diagram is homotopy commutative up to addition of a constant map with constant value $\langle\langle F\rangle\rangle$:

$$\begin{array}{ccc}
S^\mathcal{H}(F \times S^{n-k}) & \xrightarrow{\subset} & S^\mathrm{tp}_{n+1}(F, \varepsilon^{n+1}) \\
\downarrow \text{Waldhausen} & & \downarrow (3) \\
A^\mathcal{H}(F \times S^{n-k}) & \longrightarrow & A^\mathcal{H}(F) \simeq \Omega^\infty(F_+ \wedge A(\ast)).
\end{array}$$

(The arrow labeled Waldhausen is Waldhausen’s forgetful map, compare (2–5) and (2–7), and the lower horizontal arrow is induced by the projection.) Furthermore,

$$A^\mathcal{H}(F) = A^\mathcal{H}(S^k) \simeq \Omega^\infty S^0 \times \Omega^\infty S^{n-k} \times \Omega^\infty(A(\ast)/S^0) \times \Omega^\infty(A(\ast)/S^0).$$

It is well known [Wald2], [Ig] that the composite map, from upper left in the diagram to lower right via lower left, and onwards to $\Omega^\infty-k(A(\ast)/S^0)$, is split onto on $\pi_i$ for $i < 2k$ approximately. Combining this with the previous observations, we see that

$$\beta : a(\ast) \longrightarrow A(\ast) \simeq S^0 \vee (A(\ast)/S^0)$$

induces epimorphisms in $\pi_i$ for $i < k$ approximately. Moreover, as we have noted, $\pi_i(a(\ast))$ is abstractly isomorphic to $\pi_i(A(\ast))$ (for $i < 4n/3$ approximately) and finitely generated by [Dw]. Hence $\beta$ is approximately $k$–connected. Since $k$ can be arbitrary provided $n$ is big enough, the proof is complete. \qed

### 7.3. The $A$–theory Riemann–Roch Formula

Combining the identification of $e$ and $e = e^\mathrm{tp}$ of §7.2 with the results of §6, we immediately have the micro–index theorem for families of manifolds with trivial ends. As a corollary, we also obtain the index theorem 0.1.
7.3.1. Theorem. Let \( p : E \to B \) be a fiber bundle where the fibers \( F \) are manifolds with trivial ends, and the base \( B \) is a CW–space. Up to a canonical homotopy, the fiberwise microcharacteristic map

\[
B \xrightarrow{\langle p \rangle} \bigcup A^\%_f(F)
\]

agrees with the map obtained by applying Poincaré duality fiberwise to the Euler section \( e^{\text{BP}}(\tau) \), where \( \tau \) is the vertical tangent bundle of \( E \).

7.3.2. Corollary. Let \( p : E \to B \) be a fiber bundle with compact topological \( n \)-manifolds \( F \) as fibers, where \( B \) is a CW–space. Up to a canonical homotopy, the fiberwise Euler characteristic map \( \langle p \rangle : B \to \bigcup A(F) \) agrees with the composition

\[
B \xrightarrow{D(\tau)} \bigcup A^\%_f(F) \xrightarrow{\alpha} \bigcup A(F)
\]

where \( \tau \) is the vertical tangent bundle (on \( E \)), with Euler section \( e(\tau) = e^{\text{BP}}(\tau) \) whose fiberwise Poincaré dual is \( D(\tau) \).

Note that \( \bigcup A(F) \) is the same as \( A_B(E) \). Note also that the identification of the codomain of the fiberwise Poincaré duality map with \( \bigcup A^\%_f(F) \) requires a good understanding of the (concrete) Euler fibration of \( \tau \). This is a fibration on \( E \) whose fibers are infinite loop spaces which as such are homotopy equivalent to \( \Omega^{\infty-n}A(*) \). For us, the good understanding comes from 7.1.10 and 5.1 and 5.2.

In a little while, we shall need an even better understanding of the concrete Euler fibration on \( E \). We note therefore that it is the pullback of a similar fibration on \( B \text{TOP}(n) \), namely, the (concrete) Euler fibration \( \phi^{\text{BP}}(\xi) \) of the universal euclidean \( n \)-bundle \( \xi \) on \( B \text{TOP}(n) \). We understand this, too, in the sense that we can specify a fiber homotopy trivialization of \( \Omega^\xi \phi^{\text{BP}}(\xi) \) as a fibration with infinite loop space fibers. Here \( \Omega^\xi \phi^{\text{BP}}(\xi) \) is the fibration on \( B \text{TOP}(n) \) whose fiber over a point \( x \) is the space of pointed maps from the one–point compactification of \( \xi_x \) to \( \phi^{\text{BP}}(\xi)_x \).

This raises a question which we have to answer before we can deduce the A–theory Riemann–Roch theorem 0.2 from theorem 0.1. Restrict \( \xi \) to \( B \text{O}(n) \subset B \text{TOP}(n) \). The restriction is a vector bundle, which has a (concrete) Euler fibration \( \phi^{\text{sm}}(\xi) \). Its fibers are infinite loop spaces which as such are homotopy equivalent to \( \Omega^{\infty-n}S^0 \). Here the twist is quite easy to identify. In other words, there is a well–known and easy fiber homotopy trivialization of \( \Omega^\xi \phi^{\text{sm}}(\xi) \) on \( B \text{O}(n) \). We must now ask whether this trivialization is in agreement with that of \( \Omega^\xi \phi^{\text{BP}}(\xi) \). Of course, it is enough to check that the inclusion \( \Omega^\xi \phi^{\text{sm}}(\xi) \to \Omega^\xi \phi^{\text{BP}}(\xi) \), defined over \( B \text{O}(n) \), takes the unit section to the unit section (up to a homotopy of sections).

We now check it. Let \( p : E \to B \text{O}(n) \) be the disk bundle associated with \( \xi \). The Euler section \( e^{\text{sm}}(\tau) \) can be regarded as a section of \( \Omega^\xi \phi^{\text{sm}}(\xi) \). By inspection, it is the unit section, if the above trivialization of \( \Omega^\xi \phi^{\text{sm}}(\xi) \) is used. The topological Euler section \( e^{\text{BP}}(\tau) \) is a section of \( \Omega^\xi \phi^{\text{BP}}(\xi) \) over \( B \text{O}(n) \). Using the micro–index theorem, we see that this is also the unit section if we use the trivialization of \( \Omega^\xi \phi^{\text{BP}}(\xi) \) which we have from 7.1.10. Since the inclusion \( \Omega^\xi \phi^{\text{sm}}(\xi) \to \Omega^\xi \phi^{\text{BP}}(\xi) \) must take \( e^{\text{sm}}(\tau) \) to \( e^{\text{BP}}(\tau) \), the check is complete.
7.3.3. Corollary. Suppose that $p : E \to B$ is a smooth bundle with compact fibers $F$. The fiberwise Euler characteristic $\langle p \rangle : B \to \bigcup A(F)$ is homotopic to the composition

$$B \xrightarrow{\text{tr}} \bigcup Q(F_+) \xrightarrow{\iota} \bigcup A(F).$$

Proof. Let $\tau$ be the vertical tangent bundle. By the index theorem 7.3.2, it is enough to show that $D(e^{tp}(\tau)) : B \to \bigcup A^\#(F)$ agrees up to canonical homotopy with the composition

$$B \xrightarrow{D(e^{sm}(\tau))} \bigcup Q(F_+) \xrightarrow{\iota} \bigcup A^\#(F)$$

where $\iota$ is induced by the unit map $S^0 \to A(\ast)$. But this is exactly what the foregoing discussion does. □

8. Smoothing Theory and Euler Sections

Let $\xi$ denote the universal fiber bundle with fibers $\sim \mathbb{R}^n$ on $B\text{TOP}(n)$. Write $E\text{U}^{\text{tp}}(\xi)$ for the total space of the Euler fibration (introduction) associated with $\xi$, with fibers homotopy equivalent to $\Omega^{\infty-n}A(\ast)$. By 7.2.1 and sequel, the square

$$(8\text{–}1) \quad \begin{array}{ccc} B\text{TOP}(n-1) & \longrightarrow & B\text{TOP}(n) \\ \downarrow & & \downarrow \text{zero section} \\ B\text{TOP}(n) & \longrightarrow & E\text{U}^{\text{tp}}(\xi) \end{array}$$

is commutative up to a preferred homotopy, and as such approximately $(4n/3)$–cartesian (i.e., the resulting map from $B\text{TOP}(n-1)$ to the homotopy pullback of the other three terms is approximately $(4n/3)$–connected).

Notation: Let $\eta$ be a fiber bundle with structure group $G$, say on a CW–space $Z$. Suppose that $H \subset G$ is a subgroup. We denote the space of reductions of the structure group of $\eta$ from $G$ to $H$ by

$$\mathcal{R}^G_H(\eta).$$

It can also be described as the homotopy fiber of $BH^Z \hookrightarrow BG^Z$ over $c(\eta)$, the classifying map for $\eta$.

8.1. Lemma. If $F$ is a CW–space of dimension $k$, with a euclidean $n$–bundle $\eta$, then the map resulting from (8–1),

$$\mathcal{R}^{\text{TOP}(n)}_{\text{TOP}(n-1)}(\xi) \longrightarrow \{\text{nullhomotopies of } e(\xi)\},$$

is approximately $((4n/3) - k)$–connected. □

Next, suppose that $M^n$ is a compact topological manifold with boundary, where $n \neq 4, 5$, and that $\tau = \tau^M$ is equipped with a vector bundle structure $\upsilon$. (The
structure group of $\tau$ has been reduced from $\text{TOP}(n)$ to $O(n)$. We want to understand the space $\text{sm}(M, \nu)$ of smooth structures on $M$ inducing the given vector bundle structure on $\tau$. More precisely, there is a forgetful map from the space of smooth structures on $M$ to the space of vector bundle structures on $\tau$, and we want to analyze its homotopy fiber over the point $\nu$. Morlet’s sliced smoothing theory says that this homotopy fiber is contractible if $\partial M = \emptyset$ [Mor1], [KiSi], [BuLa]. In general, sliced smoothing theory still says that everything can be expressed in terms of bundles and reductions: the homotopy fiber in question maps by a homotopy equivalence to the homotopy fiber of

$$R_{O(n)}(\tau_{|\partial M}) \longrightarrow R_{\text{TOP}(n)}(\tau_{|\partial M})$$

over the base point. Combining this with 8.1 (and its analogue for vector bundles), we have an approximately $(n/3)$–cartesian square

$$\text{sm}(M, \nu) \quad \longrightarrow \quad *$$

$$\downarrow \quad \quad \quad \quad \quad \downarrow$$

$$\{\text{nullhomotopies of } e^{\text{sm}}(\tau_{|\partial M})\} \quad \longrightarrow \quad \{\text{nullhomotopies of } e^{\text{tp}}(\tau_{|\partial M})\}$$

where $e^{\text{sm}}$ and $e^{\text{tp}}$ are the smooth and topological Euler sections, respectively. Finally we note that the spaces in the lower row of (8–2) can be identified with the homotopy fibers of

$$(8-3) \quad \Gamma_{\text{in}}(\phi^{\text{sm}}(\tau)) \longrightarrow \Gamma(\phi^{\text{sm}}(\tau)),$$

$$\Gamma_{\text{in}}(\phi^{\text{tp}}(\tau)) \longrightarrow \Gamma(\phi^{\text{tp}}(\tau))$$

over $e^{\text{sm}}(\tau)$ and $e^{\text{tp}}(\tau)$, respectively. Here $\Gamma$ denotes spaces of sections (with support in the interior of $M$ in the case of $\Gamma_{\text{in}}$), and $\phi$ denotes Euler fibrations. If $M \simeq F$, where $F$ is a CW–space of dimension $< 2n/3$, then the right–hand terms in (8–3) are still $n/3$–connected. Hence we may replace the lower row in (8–2) by

$$\Gamma_{\text{in}}(\phi^{\text{sm}}(\tau)) \longrightarrow \Gamma_{\text{in}}(\phi^{\text{tp}}(\tau))$$

and we have proved the following result.

**8.2. Lemma.** The homotopy commutative square

$$\text{sm}(M, \nu) \quad \longrightarrow \quad *$$

$$\downarrow e^{\text{sm}} \quad \quad \quad \quad \quad \downarrow e^{\text{tp}}(\tau)$$

$$\Gamma_{\text{in}}(\phi^{\text{sm}}(\tau)) \longrightarrow \Gamma_{\text{in}}(\phi^{\text{tp}}(\tau))$$

is approximately $(n/3)$–cartesian if $M \simeq F$ where $\text{dim}(F) < 2n/3$.

Using 7.3 we can further simplify the lower row in (8–4). Let $i$ from $Q(M_+) = \Omega^\infty \Sigma^\infty M_+$ to $A^\%(M)$ be induced by the unit map $S^0 \rightarrow A(*)$.
8.3. Lemma. The homotopy commutative square
\[
\begin{array}{ccc}
\text{sm}(M,υ) & \longrightarrow & * \\
Poincaré dual of e^{\text{sm}} & & \text{Poincaré dual of } e^{\text{tp}}(τ) \\
Q(M+) & \overrightarrow{ι} & A%(M)
\end{array}
\]
is approximately \((n/3)\)–cartesian if \(M \simeq F\) where \(\dim(F) < 2n/3\).

Returning now to the notation of the introduction we assume that \(p : E \to B\) is a fibration where base and fibers are homotopy equivalent to compact CW–spaces. We also fix a vector bundle \(γ^n\) on \(E\). If \(\dim(F) < 2n/3\), then by 8.3 the diagram
\[
\begin{array}{ccc}
\mathcal{S}_n^{\text{sm}}(p,γ) & \subset & \mathcal{S}_n^{\text{tp}}(p,γ) \\
\text{Poincaré dual of } e^{\text{sm}} & & \text{Poincaré dual of } e^{\text{tp}} \\
\Gamma(Q_B(E+) \to B) & \overrightarrow{ι'} & \Gamma(A_B%(E) \to B)
\end{array}
\]
is approximately \(((n/3)− \dim(B))\)–cartesian. Here the vertical maps are defined as follows: Fix \(x \in B\). Each \(σ\) in \(\mathcal{S}_n^{\text{tp}}(p,γ)\) determines a manifold \(M^n\) and a homotopy equivalence \(M \to F_x\) where \(F_x\) is \(p^{-1}(x)\). The Poincaré dual of the Euler section of \(τ^M\) is an element in \(A%(M)\) which we push forward to \(A%(F_x)\) using \(M \to F_x\). This element in \(A%(F_x)\) is the value on \(x\) of the section associated to \(σ\).

We leave it to the reader to deduce from §7 that (8–5) can be stabilized with respect to \(n\) (simultaneously, \(γ\) must be stabilized). This amounts to understanding how the Euler section of the tangent bundle of a smooth or topological manifold \(M\) determines the Euler section of the tangent bundle of \(M \times [0,1]\). Both sections should be regarded as sections which vanish near the boundary.

Since \(((n/3)− \dim(B))\) tends to infinity with \(n\), the stabilized version of (8–5) is cartesian (is a homotopy pullback square). Combining this with theorem 0.4, we have the proof of theorem 0.3. □

9. The Lott Challenge

Let \(p : E \to B\) be a fibration where \(B\) is a connected polyhedron, and the fibers are homotopy equivalent to compact CW spaces. Fix also a bundle \(V\) of finitely generated projective left \(R\)–modules on \(E\), where \(R\) is a ring. Assume that for each \(i \geq 0\), the \(i\)–th homology group of each fiber \(F_x\) of \(p\) with coefficients in \(V\) is nearly f. g. projective over \(R\), which means: has a finite length resolution by f. g. projective \(R\)–modules. This is the situation of 2.2.4, where we obtained a description of the composite map
\[
(9–1) \quad B \xrightarrow{(p)} A_B(E) \to A(E) \xrightarrow{\lambda} K(R).
\]

It was found to belong to the homotopy class
\[
\sum_i (-1)^i[V_i]
\]
where \([V_i]\) classifies the bundle \(V_i\) of nearly finitely generated projective \(R\)-modules \(H_i(F_x; V)\) on \(B\).

Suppose now that \(p\) is a smooth fiber bundle with, with smooth compact manifold fibers. (See the introduction for terminology.) Then we have another way to understand (9–1), namely, by using the \(A\)-theory Riemann–Roch theorem 0.2 in the introduction. This tells us that the composition

\[
B \xrightarrow{(p)} A_B(E) \longrightarrow A(E)
\]

is homotopic to the Becker–Gottlieb transfer \(\text{tr} : B \to Q(E)\) followed by the standard map \(\iota : Q(E) \to A(E)\). We now make the crucial observation that the composite map from \(Q(E)\) to \(K(R)\), via \(A(E)\), is a map of infinite loop spaces. Hence it is determined up to homotopy by its restriction to \(E\). By construction, the restricted map \(E \to K(R)\) is the map \([V]\) classifying the bundle \(V\). Briefly, (9–1) is in the homotopy class \(\text{tr}^*[V]\).

9.1. Theorem.

\[
\text{tr}[V] = \sum_i (-1)^i[V_i].
\]

9.2. Remarks. (i) Assume \(R = \mathbb{C}\). There exist characteristic classes \(c_k\) in \(H^k(K(\mathbb{C}); \mathbb{R})\), for odd \(k\), with the following property. For a flat complex vector bundle \(V\) on \(E\), the class \(c_k[V]\) is the \(k\)-th secondary characteristic class for \(V\) defined by Kamber and Tondeur (see [KamT on] and [Du]). Bismut and Lott [BiLo] have used a local version of the Atiyah-Singer index theorem for families of elliptic operators to prove that the equation in 9.1 holds after \(c_k\) has been applied to both sides. Lott’s challenge was to give a purely topological proof of their result.

To be quite precise, the left–hand side of the Bismut–Lott formula is \(\text{tr}^* (c_k[V])\), not \(c_k(\text{tr}^*[V])\). Hence one needs to know that each \(c_k\) is classified by an infinite loop space map from \(K(\mathbb{C})\) to the appropriate rational Eilenberg–MacLane space. This is indeed the case because the classes \(c_k\) are primitive [BiLo, Prop. 1.13]

(ii) There exist fiber bundles with closed topological manifold fibers for which the equation in 9.1 is violated. This is a consequence of Theorem 0.3. We refer to a (short) forthcoming paper for details.

(iii) We could have been a little more generous in 9.1 at no extra cost, by allowing \(V\) to be a bundle of nearly f.g. projective \(R\)-modules on \(E\).

Appendix A: Becker–Gottlieb transfer for smooth fiber bundles

Here we verify that for a smooth fiber bundle \(p : E \to B\) with compact fibers, the Becker–Gottlieb transfer agrees with the fiberwise Poincaré dual of the Euler class \([e^{sm}]\) of the tangent bundle along the fibers.

First we must recall the Becker–Gottlieb transfer. This is defined for any fibration \(p : E \to B\) where \(B\) has the homotopy type of a CW–space and each fiber \(F_b = p^{-1}(b)\) has the homotopy type of a compact CW–space. There is no smoothness assumption anywhere, but initially we shall make another assumption which
may appear drastic: $B = \ast$. Thus we are dealing with a single fiber, $F$. In this case the Becker–Gottlieb transfer is defined as the composition
\[
S^0 \xrightarrow{\eta} F_+ \wedge D(F_+) \xrightarrow{\Delta \wedge \text{id}} (F_+ \wedge F_+) \wedge D(F_+)
\]
(A–1)
\[
\Rightarrow \approx
\]
\[
F_+ \wedge (F_+ \wedge D(F_+)) \xrightarrow{\text{id} \wedge \eta^*} F_+ \wedge S^0.
\]

Details: $D(F_+)$ is a finite spectrum which is Spanier–Whitehead 0–dual to the pointed space $F_+$. Therefore it comes with a map $\eta: S^0 \to F_+ \wedge D(F_+)$ such that for any finite spectrum $L$, slant product with $[\eta]$ is an isomorphism
\[
[S^0 \wedge F_+, L] \xrightarrow{} [S^0 \wedge S^0, L \wedge D(F_+)]
\]
where $[..., ...]$ denotes homotopy classes. The spectrum $J := F_+ \wedge D(F_+)$ is Spanier–Whitehead self–dual. That is, there exists a map of bispectra
\[
\mu: S^0 \wedge S^0 \to J \wedge J
\]
such that slant product with $\mu$ is an isomorphism
\[
[J, L] \xrightarrow{} [S^0 \wedge S^0, L \wedge J]
\]
for any finite spectrum $L$. Namely, $\mu$ can be defined as
\[
S^0 \wedge S^0 \xrightarrow{\sigma} S^0 \wedge S^0 \xrightarrow{\eta \wedge \eta} J \wedge J \xrightarrow{\sigma'} J \wedge J
\]
where the $\sigma'$ on the right is the anti–map switching the two copies of $D(F_+)$, and the $\sigma$ on the left is the anti–map switching the two copies of $S^0$. (We call these maps anti because they interchange left and right suspension. Both are necessary to ensure that the composition is an honest map of bispectra.)

Since $J$ is self–dual, and $S^0$ is also self–dual, the transpose $\eta^*$ of $\eta$ is defined as a map from $J = F_+ \wedge D(F_+)$ to $S^0$. We emphasize that $\eta^*$ is determined up to contractible choice, like everything else so far. To explain this, suppose that $f: U \to V$ is a map between finite spectra. Let
\[
k: S^0 \wedge S^0 \to U \wedge D(U), \quad \lambda: S^0 \wedge S^0 \to V \wedge D(V)
\]
be Spanier–Whitehead dualities. Choose the transpose $f^*$ together with a homotopy $h$ between the two maps
\[
S^0 \wedge S^0 \xrightarrow{k} U \wedge D(U) \xrightarrow{f \wedge \text{id}} V \wedge D(U),
\]
\[
S^0 \wedge S^0 \xrightarrow{\lambda} V \wedge D(V) \xrightarrow{\text{id} \wedge f^*} V \wedge D(U).
\]
Then $(f, h)$ can be interpreted as an element in the homotopy fiber of a certain map between certain spaces of stable maps, and the homotopy fiber is easily seen to be contractible.
So much for the Becker-Gottlieb transfer when \( B \) is a point; when \( B \) is not a point, these constructions must be done “fiberwise”. Thus we start with the construction of a fibered spectrum \( U \) on \( B \) whose fiber \( U_b \) over \( b \in B \) is Spanier-Whitehead 0-dual to \( F_b \), and equipped with explicit duality maps

\[
\eta : S^0 \to (F_b)_+ \wedge U, \quad \eta^*: (F_b)_+ \wedge U \to S^0.
\]

Then (A–1) gives a section of the fibration on \( B \) with fibers \( Q((F_b)_+) \). This is the Becker–Gottlieb transfer. Existence and essential uniqueness of \( U \) is established in [BeGo] and [Cla]. (End of details.)

Back to the case of a single fiber \( F \): if \( F \) is a compact smooth \( n \)-manifold, then we can make “geometric” choices for \( D(F_+) \), \( \eta \) and \( \eta^* \). Embed the pair \((F, \partial F)\) in some \((\mathbb{R}^k, +, \mathbb{R}^k - 1)\), with normal bundle \( \nu \). Here \( \mathbb{R}^k, + \) is the closed upper half space, and we assume the embedding is transverse to the boundary \( \mathbb{R}^k - 1 \). Let \( D(F_+) \) be the \( k \)-fold desuspension of \( t(\nu)/t(\partial \nu) \) where \( t(\ldots) \) denotes Thom spaces and \( \partial \nu \) is the restriction of \( \nu \) to \( \partial F \). Let \( \eta \) be the \( k \)-fold desuspension of

\[
\mathbb{S}^k \xrightarrow{\text{collapse}} t(\nu)/t(\partial \nu) \xrightarrow{\text{diagonal}} F_+ \wedge (t(\nu)/t(\partial \nu)).
\]

Let \( \eta^* \) be the \( k \)-fold desuspension of the map

\[(A-2) \quad F_+ \wedge (t(\nu)/t(\partial \nu)) \longrightarrow \mathbb{D}^k, +/\partial \mathbb{D}^k, + \quad ; \quad (x, y) \mapsto (y - x)/\delta.\]

To make sense of this formula, assume that the total space (pair) of \((\nu, \partial \nu)\) is embedded in \((\mathbb{R}^k, +, \mathbb{R}^k + 1)\) as a tubular neighborhood of \( F \), in such a way that each \( x \in F \) has distance \( > \delta \) from the complement of the tubular neighborhood. Also, identify the codomain of (A–2) with \( \mathbb{S}^k \). And finally, verify that \( \eta^* \) is indeed the transpose of \( \eta \). (Remember that this amounts to specifying a certain homotopy.)

Let us now take a look at the composition (A–1), with these choices of \( D(F_+) \), \( \eta \) and \( \eta^* \). What we get is the \( k \)-fold desuspension of

\[(A-3) \quad \mathbb{S}^k \xrightarrow{\text{collapse}} t(\nu)/t(\partial \nu) \xrightarrow{\tau} t(\varepsilon^k)/t(\varepsilon^{k, -})\]

where \( \tau \) is the tangent bundle, \( \varepsilon^k \) is a trivial \( k \)-dimensional vector bundle on \( F \), \( \varepsilon^{k, -} \) is the corresponding trivial bundle of closed lower half spaces on \( \partial F \), and \( \iota \) is induced by the inclusion of \( \nu \) in \( \tau \oplus \nu \cong \varepsilon^k \). Finally \( t(\varepsilon^{k, -}) \) is just the one point compactification of the total space of \( \varepsilon^{k, -} \). We have to verify that (A–3) is the Poincaré dual of the Euler class.

**CAP PRODUCTS.** Suppose that we have a pair of parameterized spectra [Be], [ClaPu] over a space \( F \),

\[
U = \{r_i : U_i \to F, s_i : F \to U_i, f_i : \Sigma_B U_i \to U_{i+1}\}
\]

\[
U' = \{r'_i : U'_i \to F, s'_i : F \to U'_i, f'_i : \Sigma_B U'_i \to U'_{i+1}\}.
\]
Suppose $g : F \to U_j'$ represents a cohomology class $[g] \in H^j(F;U)$. Define

$$\tilde{g} : U_\ell/F \to (U_j' \wedge F U_\ell)/F$$

by $u \mapsto (g \cdot r_\ell(u), u)$ for $u \in U_\ell$. Suppose further that $f : S^{t+i} \to U_\ell/F$ represents a homology class $[f] \in H_i(f;U)$. Then $[f] \cap [g]$ is represented by the composition

$$S^{t+i} \overset{f}{\to} U_\ell/F \overset{\tilde{g}}{\to} (U_j' \wedge F U_\ell)/F.$$

**Poincaré duality.** Suppose $\xi$ is a $j$-dimensional vector bundle over $F$. Let $U(\xi)$ be the parameterized spectrum over $F$ where $U(\xi)_{j+i}$ is the fiberwise Thom space $t_F(\xi \oplus \varepsilon^j)$. If $F^n$ is a closed, smooth manifold embedded in $R^k$ with normal bundle $\nu$, then the universal fundamental class of $F$ lives in $H_n(F;U(\nu))$ and is represented by the collapse map $c : S^k \to t_F(\nu)/F = t(\nu)$. The Euler class $[e] = [e^\text{sm}(\xi)]$ in $H^j(F;U(\xi))$ is represented by the "zero section" map $e : F \to t_F(\xi)$. Thus the Poincaré dual of $[e^\text{sm}(\xi)]$ is represented by the composition

$$S^k \overset{c}{\to} t(\nu) \overset{\bar{e}}{\to} t(\nu \oplus \xi)$$

where $\bar{e}$ is obtained from the inclusion map of vector bundles $\nu \to \nu \oplus \xi$. More generally, if $F$ is any compact smooth manifold, then $c$ must be written as a map $S^k \to t(\nu)/t(\partial \nu)$. On the other hand, if $\xi$ has a distinguished nonzero section over $\partial F$, then the Euler class $[e^\text{sm}(\xi)]$ is in $H^j(F,\partial F;U(\xi))$. In this case, by a mild generalization of the arguments above, the Poincaré dual of $[e^\text{sm}(\xi)]$ is represented by a composition of the form

$$S^k \overset{c}{\to} t(\nu)/t(\partial \nu) \overset{\bar{e}}{\to} t(\nu \oplus \xi).$$

For $\xi = \tau$, the tangent bundle of $F$, this agrees with (A–3), showing that (A–3) is indeed the Poincaré dual of the Euler class. \qedhere

Again, for a smooth fiber bundle $p : E \to B$ with compact fibers, the above argument goes through fiberwise, showing that the fiberwise Poincaré dual of the fiberwise Euler class is the Becker–Gottlieb transfer. We omit the details.

**Appendix B: Automatic Continuity**

Let $\gamma : \mathcal{C} \to \mathcal{D}$ be a functor between small categories, and let $F$ be a functor from $\mathcal{C}$ to spaces. We ask whether $F$ has an extension to, or factorization thru, $\mathcal{D}$. To be precise, we are willing to replace $F$ by any other functor from $\mathcal{C}$ to spaces which is related to $F$ by a chain of natural weak homotopy equivalences.

For questions of this type one has the theory of Kan extensions, which we now recall briefly. For $D \in \mathcal{D}$ we form the "comma" category $(D \downarrow \gamma)$ whose objects are pairs $(C,f)$ where $C$ is in $\mathcal{C}$ and $f : D \to F(C)$. A morphism from $(C_1,f_1)$ to $(C_2,f_2)$ is a morphism from $C_1$ to $C_2$ making the appropriate triangle commute. Let $P_D$ be the forgetful functor from $(D \downarrow \gamma)$ to $\mathcal{C}$. Let

$$\text{Kan}^\gamma F(D) = \lim P_D$$

(inverse limit). Then $\text{Kan}^\gamma F$ is a functor on $\mathcal{D}$. There is a natural transformation $(\text{Kan}^\gamma F)\gamma \to F$ which has a suitable universal property. $\text{Kan}^\gamma F$ is the right Kan extension of $F$ along $\gamma$. 
B.1. Variation. Let $h\text{-}\text{Kan}^\gamma F(D) = \text{holim} F \cdot P_D$. (Homotopy inverse limits will be recalled in a moment; see [BK].) This would be the homotopy right Kan extension of $F$ along $\gamma$. Drawback: there is no obvious natural transformation $(h\text{-}\text{Kan}^\gamma F)\gamma \rightarrow F$. But there are canonical natural transformations

$$(h\text{-}\text{Kan}^\gamma F)\gamma \rightarrow h\text{-}\text{Kan}^{\text{id}} F \leftarrow F$$

and the second of these is a natural weak homotopy equivalence.

B.2. Variation. Suppose that $D$ above is a topological category, whereas $\mathcal{C}$ is discrete as before. In more detail, we assume that $D$ has a discrete set of objects, but the morphism sets are equipped with a topology and composition of morphisms is continuous. Then it is appropriate to replace the homotopy limits above by topological homotopy limits. To explain this, we recall homotopy inverse limits. Namely, the homotopy inverse limit of a functor $G$ from $A$ to spaces is the totalization of the cosimplicial space

$$m \mapsto \prod_{Z:[m] \rightarrow \mathcal{A}} GZ(0)$$

where the product is taken over the set of all contravariant functors $Z$ from the poset $[m] := \{0, 1, \ldots, m\}$ to $\mathcal{A}$. (At this point we have to make clear what we mean by spaces, since one clearly needs a convenient category of spaces which is cartesian closed [MaL]. [BK] use simplicial sets, [HoVo] use Kelley spaces, and we use virtual spaces as in [WW1].) When $A = (D \downarrow \gamma)$ and $G = FP_D$, then $Z : [m] \rightarrow \mathcal{A}$ is equivalent to a pair consisting of a contravariant $Y : [m] \rightarrow \mathcal{C}$ and an $f : D \rightarrow \gamma Y(m)$, and our cosimplicial set becomes

$$m \mapsto \prod_{Y : [m] \rightarrow \mathcal{C}} FY(0) \prod_{f \in \text{mor}(D, \gamma Y(m))} FY(0).$$

But here two interpretations of $FY(0)^{\text{mor}(D, \gamma Y(m))}$ are possible: as the space of maps from $\text{mor}(D, \gamma Y(m))$ with the discrete topology to $FY(0)$, or as the space of continuous maps from $\text{mor}(D, \gamma Y(m))$ to $FY(0)$. Here we use the second interpretation, totalize and get something we call $h\text{-}\text{Kan}^\gamma F(D)$. There is a canonical and rather forgetful transformation

$$\text{th}\text{-}\text{Kan}^\gamma F(D) \longrightarrow h\text{-}\text{Kan}^\gamma F(D).$$

B.3. Variation. Note that the set of all functors $Y : [m] \rightarrow \mathcal{C}$ could be regarded as a category in which the morphisms are the natural isomorphisms. The rule

$$Y \mapsto FY(0)^{\text{mor}(D, \gamma Y(m))}$$

could be regarded as a functor on that category. Hence

$$\text{holim}_{Y : [m] \rightarrow \mathcal{C}} FY(0)^{\text{mor}(D, \gamma Y(m))}$$

is defined and maps forgetfully to

$$\prod_{Y : [m] \rightarrow \mathcal{C}} FY(0)^{\text{mor}(D, \gamma Y(m))}.$$
B.4. Proposition. The forgetful map \( \text{bth–Kan}^\gamma F(D) \to \text{th–Kan}^\gamma F(D) \) is a homotopy equivalence.

Proof. From the definition, \( \text{bth–Kan}^\gamma F(D) \) is the totalization of a bi–cosimplicial set, say \((m, n) \mapsto X_{mn}\). The forgetful map to \( \text{th–Kan}^\gamma F(D) \) is the forgetful map from the totalization of \( X_{\bullet \bullet} \) to that of \( X_{\bullet 0} \). Hence it is sufficient to show that the degeneracy \( X_{\bullet i} \to X_{\bullet 0} \) is a homotopy equivalence for every \( i \geq 0 \). However, this is a map between topological homotopy limits induced by a “change of indexing category” which happens to be an equivalence of topological categories. \( \square \)

Specializing a little more now, suppose that \( \gamma : \mathcal{C} \to \mathcal{D} \) is a bijection on objects and on morphism sets, so that \( \mathcal{C} \) is simply the discrete category underlying \( \mathcal{D} \). We are looking for criteria which ensure that the forgetful map

\[
\text{bth–Kan}^\gamma F(D) \to \text{bth–Kan}^\text{id} F(D)
\]

is a homotopy equivalence for every object \( D \) (in \( \mathcal{D} \) or in \( \mathcal{C} \), which amounts to the same thing). Note that \( \text{bth–Kan}^\gamma F \) is a continuous functor on \( \mathcal{D} \), and that there is a canonical homotopy equivalence from \( F(D) \) to \( \text{bth–Kan}^\text{id} F(D) \) for every \( D \). It will be enough to ensure that for each \( D \) and \( m \), the forgetful map

\[
\text{holim}_{Y : [m] \to \mathcal{C}} FY(0)^{\text{mor}(D, \gamma Y(m))} \to \text{holim}_{Y : [m] \to \mathcal{C}} FY(0)^{\delta \text{mor}(D, \gamma Y(m))}
\]

is a homotopy equivalence, where \( \delta \text{mor} \) denotes the (discrete) morphism sets in \( \mathcal{C} \). To understand this condition better, recall that the homotopy inverse limit of a functor from a groupoid to spaces can be identified with the space of sections of the projection from the homotopy direct limit to the nerve. Thus the forgetful map under investigation takes the form

\[
\Gamma \left( \frac{\text{holim}_{Y : [m] \to \mathcal{C}} FY(0)^{\text{mor}(D, \gamma Y(m))}}{\text{holim}^*_{Y : [m] \to \mathcal{C}}} \right) \to \Gamma \left( \frac{\text{holim}_{Y : [m] \to \mathcal{C}} FY(0)^{\delta \text{mor}(D, \gamma Y(m))}}{\text{holim}^*_{Y : [m] \to \mathcal{C}}} \right)
\]

or equivalently

\[
\Gamma \left( \frac{\text{holim}_{Y : [m] \to \mathcal{C}} FY(0) \times \text{mor}(D, \gamma Y(m))}{p} \right) \to \Gamma \left( \frac{\text{holim}_{Y : [m] \to \mathcal{C}} FY(0) \times \delta \text{mor}(D, \gamma Y(m))}{v^* p} \right)
\]

where \( v : \text{holim}_Y \delta \text{mor}(D, \gamma Y(m)) \to \text{holim}_Y \text{mor}(D, \gamma Y(m)) \) is obvious.

B.5. Observation. If the map \( v \) is an abelian homology equivalence (explanation follows), if all values of \( F \) (hence the fibers of \( p \)) are nilpotent spaces, and if \( u^* p \) is a trivial bundle for every nullhomologous \( u \) from \( S^1 \) to the base space of \( p \), then the map \( \Gamma(p) \to \Gamma(v^* p) \) is a homotopy equivalence.

The proof is by obstruction theory. A map \( g : X_1 \to X_2 \) between pointed connected spaces is an abelian homology equivalence \( [\text{McD}1] \) if it induces isomorphisms in homology \( H_*(X_1; \Lambda) \to H_*(X_2; \Lambda) \) for any module \( \Lambda \) over \( \mathbb{Z}[H_1(X_2)] \).
B.6. Corollary. If the conditions in B.5 are satisfied for every $D$ and $m \geq 0$, then for every $D$ the canonical map

$$\text{th–Kan}^\gamma F(D) \to \text{th–Kan}^{id} F(D)$$

is a homotopy equivalence.

Note: there is another canonical map $F(D) \to \text{th–Kan}^{id} F(D)$ which is always a homotopy equivalence, without any conditions. Note also that

$$\text{th–Kan}^\gamma F$$

is a functor from $\mathcal{D}$ to spaces. Hence, if the conditions in B.5 are satisfied, our functor $F$ does have a factorization through $\mathcal{D}$, up to equivalence.

B.7. Example. We shall use the abbreviations $Q(i) = \text{TOP}(i)$, $P(i) = \text{TOP}^\delta(i)$. Let $\mathcal{D}$ be the topological category with objects $0, 1, 2, \ldots$ where $\text{mor}(i,j)$ is the group–theoretic quotient $Q(j)/Q(j - i)$. In other words, a morphism $i \to j$ is an equivalence class of homeomorphisms $\mathbb{R}^i \times \mathbb{R}^{j-i} \to \mathbb{R}^j$, two such being equivalent if they are in the same orbit under the action of $Q(j - i)$. Warning: A morphism $i \to j$ does not determine an embedding $\mathbb{R}^m \to \mathbb{R}^n$ (because elements in $Q(j - i)$ are not required to preserve the origin of $\mathbb{R}^{j-i}$), but it does determine a fiber bundle projection $\mathbb{R}^j \to \mathbb{R}^i$.

Let $\mathcal{C}$ be the underlying discrete category. One of the conditions in B.5 (“the map $v$ is an abelian homology equivalence”) does not depend on any functor, so we can verify it right away. Fix an object $i$ in $\mathcal{C}$ (we should call it $D$ to be consistent) and fix $m \geq 0$. Each isomorphism class of contravariant functors $Y : [m] \to \mathcal{C}$ has a preferred representative

$$j_m \to j_{m-1} \to \cdots \to j_0$$

(a string in $\mathcal{C}$) where $j_0 \geq j_1 \geq \cdots \geq j_m$ and the morphism $j_k \to j_{k-1}$ is the standard one (trivial coset). The automorphism group of this representative is

$$P(j_m) \times \prod_{k=1}^m P(j_{k-1} - j_k).$$

It acts on

$$\text{mor}(i,j_m) = Q(j_m)/Q(j_m - i)$$

via projection to $P(j_m)$, and then translation of cosets. Therefore domain and codomain of $v$ in B.5 are homotopy equivalent to disjoint unions of pieces of the form

$$P(j_m)/P(j_m - i) \times \prod_{k=1}^m BP(j_{k-1} - j_k),$$

$$P(j_m)/Q(j_m - i) \times \prod_{k=1}^m BP(j_{k-1} - j_k)$$
respectively, where \( \| \) denotes a homotopy orbit construction. (Thus \( G \| X \) is the homotopy orbit space of a left action of the group \( G \) on the space \( X \).) Parentheses have been deliberately omitted to suggest associativity; this is easily justified and we therefore obtain a commutative diagram

\[
\begin{array}{ccc}
P(j_m) \| P(j_m) & \longrightarrow & P(j_m) \| Q(j_m) \\
\downarrow & & \downarrow \\
P(j_m) \| P(j_m) / P(j_m - i) & \longrightarrow & P(j_m) \| Q(j_m) / Q(j_m - i) \\
\downarrow & & \downarrow \\
BP(j_m - i) & \longrightarrow & BQ(j_m - i)
\end{array}
\]

where the columns are fibration sequences up to homotopy. The lower horizontal arrow is a homology equivalence by \([\text{McD}1]\). The space \( P(j_m) \| P(j_m) \) is contractible, and \( P(j_m) \| Q(j_m) \) is acyclic, again by \([\text{McD}1]\). Therefore the middle horizontal arrow is an abelian homology equivalence. Therefore \( v \) is an abelian homology equivalence.

**B.8. More of the same.** With \( \mathcal{C} \) and \( \mathcal{D} \) as in B.7, let \( F \) from \( \mathcal{C} \) to spaces be the functor \( F(i) = A^\%(\mathbb{R}^i) \). Induced morphisms are defined as follows: We have seen that a morphism \( i \to j \) in \( \mathcal{C} \) determines a projection \( \mathbb{R}^j \to \mathbb{R}^i \), and one uses pullback with this to define an exact functor from the category of retractive spaces over \( \mathbb{R}^i \times [0,1] \) to the category of retractive spaces over \( \mathbb{R}^j \times [0,1] \). The exact functor induces \( F(i) \to F(j) \). For us this is the “correct” map because it takes the microcharacteristic of \( \mathbb{R}^i \) to that of \( \mathbb{R}^j \).

We now check that the remaining conditions in B.5, those which depend on a functor, are satisfied for this particular functor. Clearly all values of \( F \) are nilpotent, since they are infinite loop spaces. Fix \( i \) and \( m \geq 0 \) as in B.6, and recall that the codomain of the map \( v \) discussed in B.6 is homotopy equivalent to a disjoint union of pieces of the form

\[
X = P(j_m) \| Q(j_m) / Q(j_m - i) \times \prod_{k=1}^{m} \ldots
\]

Note that \( X \) comes with a canonical principal \( P(j_m) \)-bundle, say \( \zeta \). This is relevant here because the fiber bundle \( p \) under investigation (notation of B.5) is simply the bundle with structure group \( P(j_m) \) and fiber \( F(j_m) \) associated to \( \zeta \). Elements in the commutator subgroup of \( P(j_m) \) act by automorphisms of \( F(j_m) \) which are homotopic to the identity. This follows easily from §5, especially 5.1 and 5.2. Therefore all conditions are satisfied. \( \square \)

**B.9. Uniqueness.** Suppose that \( F \) from \( \mathcal{C} \) to spaces is of the form \( \bar{F} \gamma \), where \( \bar{F} \) is from \( \mathcal{D} \) to spaces. In other words, the factorization problem is already solved. Then we would like to know that the “artificial” factorization or extension \( \text{th–Kan} \gamma F \) agrees with \( \bar{F} \) up to equivalence. In fact there is a natural transformation \( \bar{F} \to \text{th–Kan} \gamma F \), obvious from the definition of \( \text{th–Kan} \gamma F \), making the following diagram
of natural transformations commutative:

\[
\begin{array}{c}
\bar{F}\gamma \\
\downarrow = \\
F
\end{array}
\longrightarrow
\begin{array}{c}
(\text{th–Kan}^\gamma F)\gamma \\
\downarrow \\
\text{th–Kan}^{\text{id}} F
\end{array}
\]

In the situation of B.6, the diagram shows that \(\bar{F}(D) \longrightarrow \text{th–Kan}^\gamma \bar{F}\) is a homotopy equivalence for every \(D\) in \(\mathcal{D}\).

**Appendix C: Orthogonal Calculus and abstract Euler fibrations**

The goal is to prove theorem 7.2.1. A certain familiarity with [We] will be assumed. \(\mathcal{J}\) denotes the category of finite dimensional real vector spaces \(U, V, W, \ldots\) with inner product (as in 7.1, sequel of 7.1.1) and \(\mathcal{E}\) denotes the category of continuous functors from \(\mathcal{J}\) to spaces. "Spaces" are assumed to be homotopy equivalent to CW–spaces; if this raises questions, see the appendix to [We]. A morphism \(F \rightarrow E\) in \(\mathcal{E}\) is an equivalence if \(F(V) \rightarrow E(V)\) is a homotopy equivalence for all \(V\) in \(\mathcal{J}\). Objects in \(\mathcal{E}\) are equivalent if they can be related by a chain of equivalences. We use letters \(E, F\) but also \(X, Y, Z\) for objects in \(\mathcal{E}\).

**C.1. Definition.** An object \(E\) in \(\mathcal{E}\) is polynomial of degree \(\leq n\) if

\[
\rho : E(V) \longrightarrow \underleftarrow{\text{holim}}_{0 \neq U \subset \mathbb{R}^{n+1}} E(U \times V)
\]

is a homotopy equivalence for all \(V\).

Beware that the indexing category for the homotopy limit is a topological category (a partially ordered space where the order relation is closed) and the homotopy limit is really a topological homotopy limit. Compare Appendix B, but see [We, §5] for details. Here we need definition C.1 only when \(n = 0\) or \(n = 1\). A functor which is polynomial of degree 0 is of course equivalent to a constant functor.

**C.2. Definition.** An object \(E\) in \(\mathcal{E}\) is good if \(\lim b_n - n = \infty\), where \(b_n\) is the connectivity of

\[
\rho : E(V) \longrightarrow \underleftarrow{\text{holim}}_{0 \neq U \subset \mathbb{R}^2} E(U \times V)
\]

for \(n\)-dimensional \(V\).

In particular, an \(E\) which is polynomial of degree \(\leq 1\) is good. However, goodness is a much more "generic" property. By [We3] there is a fibration sequence up to homotopy

\[
E^{(i+1)}(V) \xrightarrow{u} E^{(i)}(V) \xrightarrow{\rho} \text{holim}_{0 \neq U \subset \mathbb{R}^{i+1}} E(U \times V)
\]

and another, from the introduction of [We],

\[
E^{(i+1)}(V) \rightarrow E^{(i)}(V) \rightarrow \Omega^i E^{(i)}(\mathbb{R} \times V)
\]

and \(E^{(0)} = E\). In particular \(E^{(1)}(V)\) is the homotopy fiber of \(E(V) \rightarrow E(\mathbb{R} \times V)\), and \(E^{(2)}(V)\) is the homotopy fiber of a certain stabilization map \(V^c \rightarrow \Omega(\mathbb{R} \times V)^c\).

Hence, if the connectivity of this stabilization map exceeds \(\text{dim}(V)\) by a quantity which tends to infinity with \(\text{dim}(V)\), then \(E\) is good.
C.3. Example. Let \( E(V) = BO(V) \). Then \( E \) is good. Indeed, the stabilization map turns out to be the usual one. It is \((2n-1)\)-connected by Freudenthal. Therefore \( b_n \geq 2n-1 \).

C.4. Example. Let \( Z(V) = B \text{TOP}(V) \). Here, instead of Freudenthal’s theorem, we have the estimate due to [Ig]: the connectivity of the stabilization map

\[
\text{TOP}(\mathbb{R} \times V)/\text{TOP}(V) \to \Omega (\text{TOP}(\mathbb{R} \times V \times \mathbb{R})/\text{TOP}(V \times \mathbb{R}))
\]

is \( \geq 4 \dim(V)/3 - k \) for a (small) constant \( k \). Therefore \( b_n \geq 4n/3 - k \) in this case, and we conclude that \( Z \) is good.

Recall further that each \( E \) in \( \mathcal{E} \) has a “best approximation” from the right by an object which is polynomial of degree \( \leq n \):

\[
\eta_n : E \to T_n E.
\]

This is the \( n \)-th Taylor approximation [We,§6]. It is functorial in \( E \). The objects in \( \mathcal{E} \) which are polynomial of degree \( \leq n \) can be characterized as those objects for which \( \eta_n \) is an equivalence. This motivates [We, Def. 8.1] which we also recall, even though we shall use it for \( n \leq 1 \) only.

C.5. Definition. A morphism \( g : E \to F \) in \( \mathcal{E} \) is polynomial of degree \( \leq n \) if

\[
\begin{array}{ccc}
E & \xrightarrow{g} & F \\
\downarrow \eta_n & & \downarrow \eta_n \\
T_n E & \xrightarrow{T_n(g)} & T_n F
\end{array}
\]

is a homotopy pullback square.

It is not hard to see that \( T_n \) respects fibration sequences up to homotopy, so that we have the following criterion.

C.6. Observation. Suppose that \( E \) in C.5 is a continuous functor from \( J \) to pointed connected spaces. Then \( g \) is polynomial of degree \( \leq n \) if and only if the functor \( \phi \) defined by

\[
\phi(V) := \text{hofiber}[g : F(V) \to E(V)]
\]

is polynomial of degree \( \leq n \).

There is a very satisfactory classification theory for objects and morphisms in \( \mathcal{E} \) which are polynomial of degree \( \leq n \), particularly when \( n = 1 \). See [We, §8,§9, §10]. Hence the following result is an important ingredient in the proof of 7.2.1.

C.7. Lemma. Let \( Z(V) = B \text{TOP}(V) \). If \((Y,Z^\iota,\ldots)\) is an abstract Euler fibration with Euler section on \( Z \) satisfying the hypotheses in 7.2.1, then \( p : Y \to Z \) is polynomial of degree \( \leq 1 \).

Proof. From the commutative square in 7.2.1, we see that the goodness of \( Z \) established in C.4 implies goodness of the homotopy fiber of \( \zeta \), and hence goodness of
φ, the homotopy fiber of \( p : Y \to Z \). On the other hand, stability of the abstract Euler fibration gives a chain of natural homotopy equivalences

\[
\phi(V \times \mathbb{R}) \simeq \cdots \simeq \Omega^k \phi(V \times \mathbb{R}^{k+1}).
\]

This shows that the connectivity \( b_n = b_n(\phi) \) in C.2 satisfies \( b_n \geq b_{n+k} - k \). Since also \( \lim_n b_n - n = \infty \), we must have \( b_n = \infty \) for all \( n \). \( \square \)

Using the functoriality of the construction \( T_1 \), we can now reduce 7.2.1 to the following statement. (The reduction will be explained in detail below.)

**C.8. Proposition.** Let \( Z \) in \( \mathcal{E} \) be polynomial of degree \( \leq 1 \). Up to equivalence relative to \( Z \), there exists a unique stable abstract Euler fibration with Euler section on \( Z \) such that the following is a homotopy pullback square for all \( V \) in \( \mathcal{J} \), and \( Y \) is polynomial of degree \( \leq 1 \):

\[
\begin{array}{ccc}
Z(V) & \xrightarrow{\imath^e} & Z^\sharp(V \times \mathbb{R}) \\
\downarrow & & \downarrow \zeta \\
Z(V \times \mathbb{R}) & \xrightarrow{e} & Y(V \times \mathbb{R}).
\end{array}
\]

**Reduction.** With \( Z, Y, Z^\sharp \) as in 7.2.1 let \( \bar{Z} = T_1Z, \bar{Y} = T_1Y \) and \( \bar{Z}^\sharp = T_1Z^\sharp \). Since \( T_1 \) respects homotopy pullback squares (also easy from the definition) we see that \( \bar{Z}, \bar{Y}, \bar{Z}^\sharp \) etc. form a stable abstract Euler fibration. (It is necessary to know, and easy to verify, that the endofunctor \( T_1 : \mathcal{E} \to \mathcal{E} \) commutes with the endofunctor \( \lambda : \mathcal{E} \to \mathcal{E} \) given by \( \lambda E(V) = E(V \times \mathbb{R}) \).) The square

\[
\begin{array}{ccc}
\bar{Z}(V) & \xrightarrow{\bar{\imath}^\bar{e}} & \bar{Z}^\sharp(V \times \mathbb{R}) \\
\downarrow & & \downarrow \bar{\zeta} \\
\bar{Z}(V \times \mathbb{R}) & \xrightarrow{\bar{e}} & \bar{Y}(V \times \mathbb{R}).
\end{array}
\]

is a homotopy pullback square because the map of vertical homotopy fibers which it determines is obtained from the map of vertical homotopy fibers in the square of 2.2.1 by applying \( T_1 \); but the latter is an approximation of order 1 [We, 5.16], so that \( T_1 \) of it is an equivalence [We 5.15]. Thus the Euler fibration with Euler section \( \bar{Z}, \bar{Y}, \bar{Z}^\sharp \) satisfies the hypotheses of C.8. Hence, if C.8 holds, it is unique up to suitable equivalence.

To complete the reduction we must verify that \( Y, Z^\sharp \) etc. can be recovered from (are determined by) \( \bar{Y}, \bar{Z}^\sharp \) etc. and the morphism \( \eta_1 : Z \to \bar{Z} \). In fact by C.7 we can recover \( \bar{Y}(V) \) as the homotopy pullback of

\[
Z(V) \xrightarrow{\eta_1} \bar{Z}(V) \xrightarrow{\bar{p}} \bar{Y}(V)
\]

and we can recover \( Z^\sharp(V \times \mathbb{R}) \) as the homotopy pullback of

\[
Z(V \times \mathbb{R}) \xrightarrow{\eta_1} Z(V \times \mathbb{R}) \xrightarrow{\bar{p} \cdot \bar{\zeta}} Z^\sharp(V \times \mathbb{R}).
\]
In the process we have recovered \( p \) and \( \zeta \) as well. Recovery of \( e \) and \( i \) from \( \bar{e} \) and \( \bar{i} \) is also easy. \( \square \)

Let \( \mathcal{E}(1) \subset \mathcal{E} \) be the full subcategory consisting of all objects of degree \( \leq 1 \). According to [We], the following category \( \mathcal{X} \) is a good “substitute” for \( \mathcal{E}(1) \).

An object of \( \mathcal{X} \) is a triple \((B, \Theta, s)\) where \( B \) is a space, \( \Theta \) is a parameterized \( \Omega \)-spectrum [Be], [ClPa], with involution over \( B \), and \( s \) is a possibly nontrivial section of \( \Omega_B^\infty(\Theta_{h\mathbb{Z}/2}) \), a retractive space over \( B \). Further details can be found below in C.13. Morphisms in \( \mathcal{X} \) are defined in the most obvious way. A morphism \((B, \Theta, s) \to (B', \Theta', s')\) is an equivalence if the underlying map \( f : B \to B' \) is a homotopy equivalence, and the resulting map \( \Theta \to f^*\Theta' \) is a homotopy equivalence between parameterized \( \Omega \)-spectra over \( B \). Now the “substitute” statement is as follows. There exist functors \( \alpha : \mathcal{E}(1) \to \mathcal{X} \) and \( \beta : \mathcal{X} \to \mathcal{E}(1) \) such that \( \beta \alpha \) and \( \alpha \beta \) can be related to the respective identity functors by chains of natural equivalences. We describe \( \beta \) only. Given \((B, \Theta, s)\) in \( \mathcal{X} \) let \( \beta(B, \Theta, s) \) be the functor taking \( V \) in \( \mathcal{J} \) to the homotopy pullback of

\[
B \xrightarrow{\text{zero}} \Omega_B^\infty \left( (V^c \wedge_B \Theta)_{h\mathbb{Z}/2} \right) \xleftarrow{j^*} B
\]

where \( V^c \) is the one point compactification \( V^c \) equipped with the involution \(-\text{id}\), the (fiberwise) smash product \( V^c \wedge_B \Theta \) has the diagonal involution, and \( j \) from \( \Omega_B^\infty \left( (0^c \wedge_B \Theta_{h\mathbb{Z}/2}) \right) \) to \( \Omega_B^\infty \left( (V^c \wedge_B \Theta)_{h\mathbb{Z}/2} \right) \) is the inclusion (and of course \( 0^c \cong S^0 \)).

We can say that C.8 is a statement about Euler fibrations in \( \mathcal{E}(1) \). Under the above correspondence \( \mathcal{E}(1) \leftrightarrow \mathcal{X} \), Euler fibrations in \( \mathcal{E}(1) \) correspond to what we should call Euler fibrations in \( \mathcal{X} \), and we would like to spell out what these Euler fibrations in \( \mathcal{X} \) are. Here a crucial observation is that the functor \( \lambda : \mathcal{E}(1) \to \mathcal{E}(1) \) given by \( (\lambda E)(V) = E(V \times \mathbb{R}) \) corresponds to the functor

\[
\mu : \mathcal{X} \to \mathcal{X} \quad ; \quad (B, \Theta, s) \mapsto (B, S^1_\rho \wedge_B \Theta, js)
\]

where \( S^1_\rho \) is \( S^1 \) with the reflection involution \( \rho \) having fixed point set \( S^0 \), and where \( j \) is the usual inclusion \( S^0 \wedge_B \Theta \to S^1_\rho \wedge_B \Theta \). On \( S^1_\rho \wedge \Theta \) we use the diagonal involution.

Another crucial observation is that the functor \((B, \Theta, s) \mapsto B \) on \( \mathcal{X} \) corresponds to the functor \( E \mapsto E(\mathbb{R}^\infty) \) on \( \mathcal{E}(1) \), where \( E(\mathbb{R}^\infty) = \text{hocolim}_n E(\mathbb{R}^n) \). In a situation such as C.8, all functors in sight (from \( \mathcal{J} \) to spaces) have the same value at \( \mathbb{R}^\infty \), up to canonical homotopy equivalences; therefore we may keep our \( B \) fixed. We arrive at the following.

**C.9. First reformulation.** An Euler fibration on an object \((B, \Theta, s)\) in \( \mathcal{X} \) consists of a map \( p : \Phi \to \Theta \) of parameterized spectra with involution over \( B \), and a factorization

\[
\Phi \xrightarrow{\rho} \Psi \xleftarrow{\zeta} S^1_\rho \wedge_B \Phi
\]

of the inclusion \( \Phi \to S^1_\rho \wedge_B \Phi \). In this factorization, \( \Psi \) is another parameterized \( \Omega \)-spectrum over \( B \) with involution, and the maps respect involutions. Condition:
\((S^1 \land_B p) \zeta\) from \(\Psi\) to \(S^1 \land_B \Theta\) is a homotopy equivalence. The Euler fibration is **stable** if, in addition,

\[
\begin{array}{ccc}
\Phi & \xrightarrow{\iota} & \Psi \\
\downarrow & & \downarrow \\
\Psi & \xrightarrow{(\rho \land \text{id}) \zeta} & S^1 \land_B \Phi
\end{array}
\]

is a homotopy pullback square. Here \(\rho : S^1 \to S^1\) is still the usual reflection with fixed point set \(S^0\).

Given an Euler fibration as above, not necessarily stable, an Euler section for it is a section \(e: \Theta \to \Phi\) of \(p: \Phi \to \Theta\), respecting the involutions.

**Remark.** The pair \((B, \Phi)\) can be completed in a unique way to an object \((B, \Phi, s')\) in \(X\) such that \(e: \Theta \to \Phi\) together with \(\text{id}: B \to B\) is a morphism in \(X\) from \((B, \Theta, s)\) to \((B, \Phi, s')\). Similarly \((B, \Psi)\) can be uniquely completed to an object \((B, \Psi, s'')\) such that \(e\) is a morphism. For this reason \(s'\) and \(s''\) do not appear explicitly in C.9, and as a result \(s\) is also redundant. This should explain how an Euler fibration as in C.9 gives rise to an Euler fibration as in C.8.

Note that the endofunctor \(\Theta \mapsto S^1 \land_B \Theta\) on the category of parameterized spectra with involution over \(B\) has an inverse up to equivalence. By equivalence we mean here an equivariant map which is an ordinary homotopy equivalence. This means that the zero section \(\zeta\) in C.9 can be regarded as an inverse (up to suitable equivalence) of \(p: \Phi \to \Theta\). We use this to split \(\Phi\), up to equivalence, discard the summand \(\Theta\), call the other summand \(\Phi'\), and obtain the following drastic simplification of C.9.

**C.10. Second reformulation.** An Euler fibration on \((B, \Theta, s)\) consists of a parameterized \(\Omega\)-spectrum \(\Phi'\) over \(B\), with involution, and a factorization

\[
\Phi' \to \Psi' \to S^1 \land_B \Phi'
\]

of the inclusion \(\Phi' \to S^1 \land_B \Phi'\). In this factorization, \(\Psi'\) is another parameterized spectrum over \(B\) with involution, and the maps respect involutions. Condition: \(\Psi'\) is contractible. The Euler fibration is **stable** if

\[
\begin{array}{ccc}
\Phi' & \xrightarrow{\iota} & \Psi' \\
\downarrow & & \downarrow \\
\Psi' & \xrightarrow{(\rho \land \text{id}) \zeta} & S^1 \land_B \Phi'
\end{array}
\]

is a homotopy pullback square.

Given an Euler fibration as above, not necessarily stable, an Euler section for it is a map \(e' : \Theta \to \Phi'\) between parameterized \(\Omega\)-spectra with involution over \(B\).

**Remark.** Note that according to C.10, the notion *Euler fibration on \((B, \Theta, s)\)* does not involve \(\Theta\) any more.

**Remark.** It is also important to reformulate the notion of *equivalence*. In the setting of C.10, it is clear what a morphism between Euler fibrations on \((B, \Theta, s)\) is; such
a morphism, say from $e' : \Theta \to \Phi'$ etc. to $e'' : \Theta \to \Phi''$ etc., is an equivalence if the underlying (equivariant) map from $\Phi'$ to $\Phi''$ is an ordinary homotopy equivalence. Since we are interested in a classification up to (specified) equivalence, it follows that we lose nothing by smashing $\Theta$ and $\Phi'$ fiberwise with $E\mathbb{Z}/2_+$. Then the involutions are sufficiently free, so that the equivariant factorization through a contractible object may be replaced by an equivariant nullhomotopy.

Remark. Note that C.10 implicitly mentions two commuting involutions on $S^1 \wedge \Phi'$, one that acts on both $S^1$ and $\Phi'$, and another which appears under the name $\rho_*$ in the commutative square and only acts on the first factor.

These remarks lead to the next reformulation.

C.11. Third reformulation. A stable Euler fibration on $(B, \Theta, s)$ consists of a parameterized $\Omega$–spectrum $\Phi'$ over $B$, with involution, and a homotopy equivalence

$$\kappa : S^1 \wedge_B \Phi' \longrightarrow S^1 \wedge_B \Phi'$$

which is a $\mathbb{Z}/2 \times \mathbb{Z}/2$ map for the following actions. On the domain $(a, b) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ acts by $\rho^a \wedge \tau^b$ where $\tau$ is the involution on $\Phi'$, and on the codomain $(a, b)$ acts by $\rho^{a+b} \wedge \tau^b$.

An Euler section for this stable Euler fibration is a map $e' : \Theta \to \Phi'$ between parameterized $\Omega$–spectra with involution over $B$.

The assumptions on $\Phi'$ in C.11 imply that $\Phi'$ is co–induced: there exist a parameterized $\Omega$–spectrum $\Lambda$ over $B$, without involution, and a map $f : \Phi' \to \Lambda$ such that $(f, f\tau)$ from $\Phi'$ to $\Lambda \times_B \Lambda$ is a homotopy equivalence. Prove this by thinking of $\kappa$ as a map from $(S^1/S^0) \wedge_B \Phi'$ to itself, and then taking homotopy orbits for the actions of the diagonal subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$; this shows that $S^1 \wedge_B \Phi'$ is co–induced, and it follows that $\Phi'$ itself is co–induced.

Conversely, if we only have $f : \Phi' \to \Lambda$ with the above properties, then we can recover $\kappa$. Consequently nothing is lost by composing $e'$ in C.11 with $f$, and so we get our final reformulation.

C.12. Fourth and last reformulation. A stable Euler fibration with Euler section on $(B, \Theta, s)$ is a map $\Theta \to \Lambda$, where $\Lambda$ is another parameterized $\Omega$–spectrum over $B$, without involution.

Recall now the extra condition in C.8, the one which requires that a certain square be a homotopy pullback square. In the language and notation of C.12, the corresponding requirement is that $\Theta \to \Lambda$ be a homotopy equivalence. The details of this translation are left to the reader. Clearly, up to equivalence, there is only one stable Euler fibration with section on $(B, \Theta, s)$ satisfying the extra condition. This completes the proof of C.8, and that of 7.2.1. □

Appendix D: The Waldhausen map

In §2.4, we claimed that a certain map (2–7) agreed with one constructed by Waldhausen, up to homotopy. Some comments are in order. For the map constructed by Waldhausen, we refer to [Wald2]. See also [WWG] for guidance, and [Wald1], [Wald3–5] for background.
Here is a brief summary of the constructions in [Wald2] that are relevant to us. For a compact topological manifold \(X\), and integers \(m,k > 0\), Waldhausen defines a simplicial set \(P^m_k(X)\) of certain partitions of \(X \times I\). When \(m = k = 0\), this is the space of topological \(h\)-cobordisms over \(X\). He defines a simplicial order relation on it, thereby enhancing it to a simplicial category \(hP^m_k(X)\). It turns out that approximately, i.e. up to some connectivity tending to infinity with \(m\) and \(\dim(X) - m\),

\[
\begin{align*}
\mathcal{P}^0_0(X) & \hookrightarrow \mathcal{P}^m_k(X) \hookrightarrow h\mathcal{P}^m_k(X) \\
\downarrow & \quad \downarrow \quad \downarrow \gamma \\
\text{hofiber}(\alpha) & \longrightarrow A^\%(X) \longrightarrow A(X).
\end{align*}
\]

is a fibration sequence up to homotopy. Approximately, \(P^m_k(X)\) and \(hP^m_k(X)\) can be identified with \(A^\%(X)\) and \(A(X)\), respectively, and the inclusion of \(P^m_k(X)\) in \(hP^m_k(X)\) can be identified with the assembly map. This is Theorem 1 of [Wald2]. Actually Waldhausen speaks of a homology theory rather than \(A^\%\), but it is clear from his formulation that, approximately, the inclusion \(P^m_k(X) \hookrightarrow hP^m_k(X)\) is a homotopy equivalence when \(X\) is a disk. Therefore, by the characterization of assembly maps, the homology theory in question must be \(A^\%\).

Note that the (approximate) identification of \(hP^m_k(X)\) with \(A(X)\) is the only one that needs to be specified. The other identifications follow from it, given that (D–1) is approximately a fibration sequence up to homotopy and given that \(P^m_k(\cdot)\) is approximately a homology theory. In other words, if \(\gamma : hP^m_k(X) \to A(X)\) is a natural homotopy equivalence (approximately), then it has an essentially unique completion to a morphism of fibration sequences up to homotopy (approximately)

\[
\begin{align*}
\mathcal{P}^0_0(X) & \hookrightarrow \mathcal{P}^m_k(X) \hookrightarrow h\mathcal{P}^m_k(X) \\
\downarrow (2\to7) & \quad \downarrow \beta \quad \downarrow \gamma \\
\text{hofiber}(\alpha) & \longrightarrow A^\%(X) \longrightarrow A(X),
\end{align*}
\]

This follows from the characterization of the assembly map \(\alpha\). For this reason, we can identify our map (2–7) with Waldhausen’s construction by showing that it is part of such a morphism,

\[
\begin{align*}
\mathcal{P}^0_0(X) & \hookrightarrow \mathcal{P}^m_k(X) \hookrightarrow h\mathcal{P}^m_k(X) \\
\downarrow (2\to7) & \quad \downarrow \beta \quad \downarrow \gamma \\
\text{hofiber}(\alpha) & \longrightarrow A^\%(X) \longrightarrow A(X),
\end{align*}
\]

and making sure that \(\gamma\) agrees with Waldhausen’s identification, up to homotopy. This can easily be done: the definition of \(\beta\) in terms of microcharacteristics is clear from §2.4, and the definition of \(\gamma\) in terms of Euler characteristics is clear from 2.1.

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