AUTOMORPHISMS OF MANIFOLDS AND ALGEBRAIC
K–THEORY: PART III

MICHAEL WEISS AND BRUCE WILLIAMS

Abstract. The structure space $S(M)$ of a closed topological $m$-manifold $M$ classifies bundles whose fibers are closed $m$-manifolds equipped with a homotopy equivalence to $M$. We construct a highly connected map from $S(M)$ to a concoction of algebraic $L$-theory and algebraic $K$-theory spaces associated with $M$. The construction refines the well-known surgery-theoretic analysis of the “block structure space” of $M$ in terms of $L$-theory.

1. Introduction

The structure space $S(M)$ of a closed topological $m$-manifold $M$ is the classifying space for bundles $E \to X$ with an arbitrary CW-space $X$ as base, closed topological manifolds as fibers and with a fiber homotopy trivialization $E \to M \times X$ (a homotopy equivalence and a map over $X$). The “points” of $S(M)$ should be imagined as pairs $(N, f)$ where $N$ is a closed $m$-manifold and $f : N \to M$ is a homotopy equivalence.

The main result of this paper is a calculation of the homotopy type of $S(M)$ in the so-called concordance stable range, in terms of $L$- and algebraic $K$-theory. With $m$ fixed as above, we construct a homotopy invariant functor

$$(Y, \xi) \mapsto LA_{\%}(Y, \xi, m)$$

from spaces $Y$ with spherical fibrations $\xi$ to spectra. The spectrum $LA_{\%}(Y, \xi, m)$ is a concoction of the $L$-theory and the algebraic $K$-theory of spaces [25] associated with $Y$, compounded with an assembly construction [19]. (The subscript $\%$ is for homotopy fibers of assembly maps.) In the case where $Y = M$ (connected, nonempty for simplicity) and $\xi$ is $\nu$, the normal fibration of $M$, there is a “local degree” homomorphism

$$\Omega^{\infty +m}LA_{\%}(M, \nu, m) \to 8\mathbb{Z} \subset \mathbb{Z}.$$ 

There is then a highly connected map

$$S(M) \longrightarrow \ker [\Omega^{\infty +m}LA_{\%}(M, \nu, m) \xrightarrow{\text{local deg.}} 8\mathbb{Z}]$$

where “ker” in this case means the fiber over $0 \in 8\mathbb{Z}$, an infinite loop space. The connectivity estimate is given by the concordance stable range. In practice that translates into $m/3$ approximately, but in theory it is more convoluted and the reader is referred to definition 11.5.

The result has a generalization to the case in which $M$ is compact with nonempty boundary. It looks formally the same. Points of $S(M)$ can be imagined as pairs

Date: Jan 2009.
$$(N, f)$$ where $$N$$ is a compact manifold with boundary and $$f: (N, \partial N) \to (M, \partial M)$$ is a homotopy equivalence of pairs restricting to a homeomorphism of $$\partial N$$ with $$\partial M$$.

This result has many precursors. The most fundamental and best known of these belong to surgery theory. From the surgery point of view it is very natural to introduce certain “block” structure spaces such as

$$\tilde{S}^s(M), \quad \tilde{S}^h(M).$$

These are designed in such a way that $$\pi \circ \tilde{S}^s(M)$$ and $$\pi \circ \tilde{S}^h(M)$$ are identifiable with, respectively, the subset of $$\pi \circ S(M)$$ determined by the simple homotopy equivalences, and the quotient set of $$\pi \circ S(M)$$ determined by the $$h$$-cobordism relation. In addition they have the property

$$\pi_i \circ \tilde{S}^s(M) \simeq \pi_i \circ \tilde{S}^s(M \times D^i), \quad \pi_i \circ \tilde{S}^h(M) \simeq \pi_i \circ \tilde{S}^h(M \times D^i).$$

This is obviously very useful in calculations. The surgery-theoretic calculations of these spaces are of the form

$$\tilde{S}^s(M) \simeq \ker [\Omega^{\infty + m} L^s_{\%}(M, w) \to \mathbb{Z}],$$

$$\tilde{S}^h(M) \simeq \ker [\Omega^{\infty + m} L^h_{\%}(M, w) \to \mathbb{Z}],$$

where $$L^s_{\%}$$ and $$L^h_{\%}$$ are homotopy invariant functors from spaces with double coverings to spectra. (In particular $$w$$ denotes the orientation covering of $$M$$.) The functors $$L^s_{\%}$$ and $$L^h_{\%}$$ can be defined entirely in terms of algebraic $$L$$-theory, again compounded with assembly. They are therefore fully 4-periodic:

$$\Omega^4 L^s_{\%}(X, v) \simeq L^s_{\%}(X, v), \quad \Omega^4 L^h_{\%}(X, v) \simeq L^h_{\%}(X, v).$$

This calculation of $$\tilde{S}^s(M)$$ and $$\tilde{S}^h(M)$$ is sometimes called the Casson-Sullivan-Wall-Quinn-Ranicki theorem. An earlier version of it, describing the homotopy groups of the two block structure spaces, is known as the Casson-Sullivan-Wall long exact sequence. The space level formulation was championed by Quinn. The complete and final reduction to $$L$$-theory, at the space level, is mainly due to the untiring efforts of Ranicki. This took many years.

Meanwhile Burghelea and Lashof [5, cor. D] obtained results on the homotopy type of $$S(M)$$. Localizing at odd primes, they were able to construct a highly connected map

$$\Omega S(M) \to \Omega \tilde{S}(M) \times \Omega^{\infty + 1} A_{\%}(M)_{-1}^{m+1}. $$

Here $$A(M)$$ is Waldhausen’s “algebraic $$K$$-theory of retractive spaces over $$M$$” spectrum and $$A_{\%}(M)$$ is the homotopy fiber of an assembly map

$$A(*) \wedge M \to A(M).$$

The spectrum $$A_{\%}(M)$$ comes with a standard involution related to Spanier-Whitehead duality. At odd primes, the involution splits the spectrum into “eigensummands”

$$A_{\%}(M)_{\pm 1}$$

on which the involution acts as $$+\text{id}$$ or $$-\text{id}$$, up to homotopy.

With hindsight, the Burghelea-Lashof result can be explained in terms of our calculation described above, and the surgery-theoretic calculation of the block structure space, along the following lines. The spectrum $$LA_{\%}(M, \nu, m)$$ is defined as
the homotopy fiber of a certain map

\[ L_{\mathbb{Z}/2}(M) \xrightarrow{\delta} (S^1 \wedge S^m \wedge A_{\text{even}}(M))_{\mathbb{Z}/2} . \]

Here \( h_{\mathbb{Z}/2} \) indicates a homotopy orbit construction. The group \( \mathbb{Z}/2 \) acts trivially on \( S^1 \), acts by \(-\text{id}\) on \( \mathbb{R}^m \cup \infty \cong S^m \), fixing 0 and \( \infty \), and by the standard involution on \( A_{\text{even}}(M) \). The map \( \delta \) is very interesting at the prime 2, but nullhomotopic at odd primes by construction. Moreover, at odd primes the homotopy orbit construction (for any action of \( \mathbb{Z}/2 \)) simplifies to an eigensummmand.

Our calculation of structure spaces \( S(M) \) in the concordance stable range is also in agreement with the surgery theoretic calculation of block structure spaces. That is, there is a commutative diagram

\[
\begin{array}{cccc}
S(M) & \xrightarrow{\text{incl.}} & \Omega^\infty \mathbb{L}A_{\text{even}}(M, \nu, m) & \xrightarrow{\text{forgetful}} & 8\mathbb{Z} \\
\text{incl.} & & \text{forgetful} & & \\
\tilde{S}(M) & \xrightarrow{\text{incl.}} & \Omega^\infty \mathbb{L}A_{\text{even}}(M, \nu, m) & \xrightarrow{\text{forgetful}} & 8\mathbb{Z}
\end{array}
\]

where the lower row is a homotopy fibration sequence and the upper row is a homotopy fibration sequence “in the concordance stable range”. Taking vertical homotopy fibers, we have a highly connected map

\[ \tilde{\text{TOP}}(M)/\text{TOP}(M) \longrightarrow \Omega^\infty ((S^m \wedge A_{\text{even}}(M))_{\mathbb{Z}/2}) . \]

This is reminiscent of a highly connected map

\[ \tilde{\text{TOP}}(M)/\text{TOP}(M) \longrightarrow \Omega^\infty (H^s(M)_{\mathbb{Z}/2}) \]

constructed in [31]; see also [36] for notation. Indeed these two maps are intended to be “the same”, modulo Waldhausen’s identification of the \( h \)-cobordism spectrum \( H^s(M) \) with \( A_{\text{even}}(M) \). We do not quite prove that here, but we come very close to it. It is likely to be the first order of business in the next installment.

This paper is a continuation of [31] and [32]. In another sense it is a continuation of [9], because we continue to use an index theoretic approach to manifold structures developed there. For technical support, we use a fair amount of controlled topology as in [2], the Thurston-Mather-McDuff-Segal discrete approximation theory [14] for homeomorphism groups (summarized and slightly generalized in section 10 below), and Spanier-Whitehead duality theory with its implications for algebraic \( K \)-theory as in [34].

2. Visible \( L \)-theory revisited

Mishchenko [15] and Ranicki [17], [18] introduced “symmetric structures” on certain chain complexes over rings with involution with a view to understanding signature invariants and product formulae in surgery theory. For a ring \( R \) with involution (=involutory antiautomorphism) \( r \mapsto \bar{r} \) and a bounded chain complex \( C \) of finitely generated projective left \( R \)-modules, a symmetric structure of dimension \( m \) on \( C \) is a chain map of \( \mathbb{Z}[\mathbb{Z}/2] \)-module chain complexes

\[ \varphi : \Sigma^m W \longrightarrow C^t \otimes_R C . \]
Here $C^i$ is $C$ with the right $R$–module structure defined by $xr = \bar{r}x$ and $W$ denotes the standard resolution of $\mathbb{Z}$ by free $\mathbb{Z}[\mathbb{Z}/2]$–modules:

$$Z[\mathbb{Z}/2] \xrightarrow{1-T} Z[\mathbb{Z}/2] \xrightarrow{1+T} Z[\mathbb{Z}/2] \xrightarrow{1-T} Z[\mathbb{Z}/2] \xrightarrow{1+T} \cdots.$$ 

The value of $\varphi$ on $1 \in W_0$ is an $m$–cycle in $C^i \otimes_R C$, corresponding to a degree $m$ chain map from the dual $C^{−*}$ to $C$. If this is a chain homotopy equivalence, $\varphi$ is called nondegenerate. The bordism groups of objects $(C, \varphi)$ as above, with nondegenerate $\varphi$ of dimension $m$, are the symmetric $L$–groups $L^m(R)$. (This definition of $L^m(R)$ is in agreement with [19] but in slight disagreement with [17] because we do not require that $C_k = 0$ for $k \notin \{0,1,\ldots,n\}$.) Ranicki’s analogous description of the quadratic $L$–groups $L_n(R)$, in which the homotopy fixed point construction $\text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, −)$ is replaced by a homotopy orbit construction $W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} −$, makes it easy to define multiplication operators

$$L^m(R_1) \otimes L_n(R_2) \rightarrow L_{m+n}(R_1 \otimes R_2).$$

A (connected) Poincaré duality space $X$ of formal dimension $m$ with fundamental group $\pi$ and orientation character $\sigma: \pi \rightarrow \{±1\}$ determines

- an involution on $R = \mathbb{Z}\pi$ given by $g \mapsto w(g) \cdot g^{-1}$ for $g \in \sigma \subset \mathbb{Z}\pi$,
- a chain complex $C$ over $R = \mathbb{Z}\pi$, the singular or cellular chain complex of the universal cover of $X$, and
- a nondegenerate $n$–dimensional structure $\varphi$ on $C$, obtained by evaluating the Eilenberg–Zilber diagonal chain map on the fundamental class of $X$.

The corresponding element in $L^m(R)$ is the symmetric signature $\sigma^*(X)$ of $X$. If $X$ is a closed manifold and $f: X' \rightarrow Y$ is a degree one normal map from a closed $n$–manifold to a Poincaré duality space of formal dimension $n$, then

$$(\text{id}_X \times f): X \times X' \rightarrow X \times Y$$

is also a degree one normal map. The surgery obstructions $\sigma_*(f)$ and $\sigma_*(\text{id}_X \times f)$ are related by

$$\sigma_*(\text{id}_X \times f) = \sigma^*(X) \cdot \sigma_*(f) \in L_{m+n}(\mathbb{Z}\pi \otimes \mathbb{Z}\pi'),$$

using the above product, with $\pi' = \pi_1(Y)$.

A few years later it was found [28] that the symmetric $L$–groups admit a homological description relative to the quadratic $L$–groups. That is, there is a long exact sequence

$$\cdots \rightarrow L_n(R) \rightarrow L^n(R) \rightarrow \hat{L}_n^*(R) \rightarrow L_{n-1}(R) \rightarrow \cdots$$

and the calculation of the relative terms $\hat{L}_n^*(R)$ is “only” a matter of homological algebra. Efforts to reduce the homological algebra to an absolute minimum eventually led to the visible symmetric $L$–groups $VL^m(R)$, defined for group rings $R = \mathbb{Z}\pi$. We now recall their definition, following [29].

Let $R = \mathbb{Z}\pi$ with involution $g \mapsto w(g) \cdot g^{-1}$ for some homomorphism $w: \pi \rightarrow \{±1\}$. Let $C$ be a chain complex of f.g. projective left $R$–modules, bounded as before. A symmetric structure of dimension $n$ on $C$ can be viewed as an $n$–cycle in

$$((C^i \otimes C)^{h\mathbb{Z}/2})_n \cong ((C^i \otimes C)_n)^{h\mathbb{Z}/2}$$

where the various subscripts and superscripts indicate orbit constructions and homotopy fixed point constructions for the appropriate symmetry groups, here $\pi$ and
Z/2. (Note that π acts diagonally on $C^t \otimes C$.) A visible symmetric structure of dimension $m$ on $C$ is an $m$–cycle in

$$(C^t \otimes C)^{h\mathbb{Z}/2}_{h\pi}$$

where $(-)_{h\pi}$ means $(- \otimes P)_{\pi}$ for a resolution $P$ of the trivial module $\mathbb{Z}$ by projective $\mathbb{Z}_{\pi}$–modules. In contrast to $C^t \otimes C$ and $(C^t \otimes C)^{h\mathbb{Z}/2}$, the chain complex $(C^t \otimes C)^{h\mathbb{Z}/2}$ is not bounded below if $C \neq 0$, so that there is no good reason to think that the augmentation–induced chain map

$$(C^t \otimes C)^{h\mathbb{Z}/2}_{h\pi} \longrightarrow ((C^t \otimes C)^{h\mathbb{Z}/2})_{\pi}$$

should induce an isomorphism in homology. In fact visible symmetric structures generally carry more information than symmetric structures. It is sometimes convenient to organize both types of structures into homotopy classes: the groups of such homotopy classes are denoted

$$Q^m(C) = H_m((C^t \otimes C)^{h\mathbb{Z}/2}_{h\pi}), \quad VQ^m(C) = H_m((C^t \otimes C)^{h\mathbb{Z}/2}_{h\pi}).$$

With a view to generalizations later on, we mention that there is a homotopy (co)cartesian square of chain complexes

$$
\begin{array}{ccc}
(C^t \otimes C)^{h\mathbb{Z}/2}_{h\pi} & \longrightarrow & ((C^t \otimes C)^{th\mathbb{Z}/2})_{h\pi} \\
\downarrow & & \downarrow \\
(C^t \otimes C)^{h\mathbb{Z}/2}_{\pi} & \longrightarrow & ((C^t \otimes C)^{th\mathbb{Z}/2})_{\pi}
\end{array}
$$

where the $th\mathbb{Z}/2$ superscript denotes a Tate construction $\text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\hat{W}, -)$, with

$$d_k: \hat{W}_k \to \hat{W}_{k-1} \quad z \mapsto (1 + (-1)^kT)z.$$ 

This is reflected in a long exact Mayer–Vietoris sequence

$$\cdots \to \hat{Q}^{n+1}(C) \to VQ^n(C) \to Q^n(C) \oplus \hat{Q}^n(C) \to VQ^n(C) \to \hat{Q}^{n-1}(C) \to \cdots$$

where $V\hat{Q}^n(C) = H_n(((C^t \otimes C)^{th\mathbb{Z}/2}_{h\pi})$ and $\hat{Q}^n(C) = H_n(((C^t \otimes C)^{th\mathbb{Z}/2}_{\pi})$.

The visible hyperquadratic theory has the property of being invariant under a “change of rings”. That is, for a bounded chain complex $C$ of f.g. left projective $\mathbb{Z}_{\pi}$–modules and a homomorphism $h: \pi \to \pi'$ we have

$$V\hat{Q}^*(C) \cong V\hat{Q}^*(C')$$

where $C' = \mathbb{Z}\pi' \otimes_{\mathbb{Z}_{\pi}} C$. (It is understood that $\pi$ and $\pi'$ are equipped with homomorphisms to $\{\pm 1\}$ and that $h$ respects these.) We also note that

$$V\hat{Q}^* = \hat{Q}^*$$

in the case $\pi = \{1\}$. These two properties, suitably sharpened, could be used to characterize the visible hyperquadratic theory in terms of the ordinary hyperquadratic one. But we need not go into that.

A visible symmetric structure on $C$ is considered nondegenerate if the induced symmetric structure is nondegenerate. The bordism groups of chain complexes $C$ as above with a nondegenerate $m$–dimensional visible symmetric structure are the
visible symmetric $L$–groups $VL^m(Z\pi)$. A mild improvement on the Mishchenko–Ranicki construction of the symmetric signature of a Poincaré duality space $X$ of formal dimension $m$ gives the visible symmetric signature

$$\sigma^*(X) \in VL^m(Z\pi).$$

Other useful features of the symmetric $L$–groups (such as the products and the product formula for surgery obstructions) can be transferred to the visible symmetric $L$–groups by means of the forgetful homomorphisms $VL^m(Z\pi) \to L^m(Z\pi)$.

The main result of [29] is a long exact sequence relating the quadratic $L$–groups of $Z\pi$ to the visible symmetric $L$–groups, with “easy” relative terms:

$$\cdots \to L_n(Z\pi) \to VL^n(Z\pi) \to \bigoplus_{i+j=n} H_i(B\pi; \hat{L}^j(Z)) \to L_{n-1}(Z\pi) \to \cdots.$$ 

Ranicki [19] found a generalization of this from the group ring case to the case of a simplicial group ring $Z[\Omega X]$, and used it in a revised approach to his total surgery obstruction theory, a project going back to [16]. He defines a visible symmetric $L$–theory spectrum which we (not he) denote by $VL^\bullet(Z[\Omega X])$, with homotopy groups $VL^n(Z[\Omega X])$. There is a long exact sequence

$$\cdots \to L_n(Z[\Omega X]) \to VL^n(Z[\Omega X]) \to \bigoplus_{i+j=n} H_i(X; \hat{L}^j(Z)) \to L_{n-1}(Z\pi) \to \cdots,$$

identical with the one above when $X = B\pi$. One of Ranicki’s main results in [19] states roughly that the closed manifold structures on an oriented Poincaré duality space $X$ of formal dimension $n$ are in canonical bijection with the connected components of the homotopy fiber of the assembly map

$$\Omega^\infty \Sigma^\infty (X_+ \wedge VL^\bullet(Z)) \to \Omega^\infty \Sigma^\infty VL^\bullet(Z[\Omega X])$$

over the point determined by $\sigma^*(X)$, where $VL^\bullet(\ldots)$ denotes visible $L$–theory spectra. (To be more precise, each of these connected components determines a class in $\pi_m(\Omega^\infty (X_+ \wedge VL^\bullet(Z)))$ which has a local signature $d \in 8\mathbb{Z} + 1$; the condition $d = 1$ must be added.) This is obviously relevant to our program.

We come to a definition of visible symmetric structures in the setting of retractive spaces and retractive spectra. Let $Y_1$ and $Y_2$ be finitely dominated retractive spaces over $X$, with retractions $r_1$ and $r_2$. We recall first the definition of an ‘unstable” Spanier–Whitehead product $Y_1 \smash Y_2$ from [34, 1.A.3]. This is the based space obtained by first forming the external smash product

$$Y_1 \wedge X \ Y_2 = Y_1 \times Y_2 / \sim$$

where $\sim$ identifies $(y_1, x)$ with $(r_1(y_1), x)$ and $(x, y_2)$ with $(x, r_2(y_2))$; then taking the homotopy pullback of

$$X \xrightarrow{\text{diagonal}} X \times X \xleftarrow{\text{retraction}} Y_1 \wedge X \ Y_2$$

and then dividing that by the homotopy pullback of

$$X \xrightarrow{\text{diagonal}} X \times X \xleftarrow{\text{id}} X \times X.$$

We make this unstable SW product “stable” essentially by applying $\Omega^\infty \Sigma^\infty$. More technically, however, we have to work in a stable category of retractive spaces with
objects of the form \((Y, k)\) where \(Y\) is a (finitely dominated) retractive space over \(X\) and \(k \in \mathbb{Z}\). The set of morphisms from \((Y, k)\) to \((Y', \ell)\) in the stable category is 

\[
\text{colim}_n \text{mor}_{\text{uns}}(\Sigma^{−k}Y, \Sigma^{−\ell}Y')
\]

where \(\Sigma\) is short for \(\Sigma_X\) and \(\text{mor}_{\text{uns}}\) refers to morphisms in the ordinary category of retractive spaces over \(X\). It is worth noting that \(\Sigma^k(Y, k)\) is isomorphic to \((Y, 0)\) in the stable category, so that \((Y, k)\) can be regarded as a formal \(k\)-fold desuspension of \(Y\) alias \((Y, 0)\).

**Definition 2.1.** We let 

\[
(Y_1, k) \circ (Y_2, \ell) = \text{colim}_n \Omega^{2n}(\Sigma^{n-k}Y_1 \wedge \Sigma^{n-\ell}Y_2).
\]

More generally, we let 

\[
(Y_1, k) \circ_j (Y_2, \ell) = \text{colim}_n \Omega^{2n}\Sigma^j(\Sigma^{n-k}Y_1 \wedge \Sigma^{n-\ell}Y_2),
\]

so that \((Y_1, k) \circ (Y_2, \ell) = (Y_1, k) \circ_0 (Y_2, \ell)\), and denote the \(\Omega\)-spectrum with \(j\)-th term \((Y_1, k) \circ_j (Y_2, \ell)\) by 

\[
(Y_1, k) \circ_j (Y_2, \ell).
\]

(By convention \(\Omega^nZ\), for a based space \(Z\) and an integer \(m \geq 0\), is the geometric realization of the simplicial set whose \(n\)-simplices are the based maps from the one-point compactification of \(\Delta^n \times \mathbb{R}^m\) to \(Z\). Hence all the spaces and spectra in definition 2.1 are \(CW\) spaces and \(CW\) spectra.)

Note that \(\circ_j\) comes with a structural symmetry \((Y_1, k) \circ_j (Y_2, \ell) \cong (Y_2, \ell) \circ_j (Y_1, k)\) determined by the obvious symmetry of \(\wedge\). For \(Y_1 = Y_2 = Y\) and \(k = \ell\) we obtain an \(\Omega\)-spectrum \((Y, k) \circ_j (Y, k)\) with an action of \(\mathbb{Z}/2^j\).

**Definition 2.2.** An \(n\)-dimensional symmetric structure on \((Y, k)\) is an element of \(\Omega^n((Y, k) \circ (Y, k))^h\mathbb{Z}/2\). An \(n\)-dimensional visible symmetric structure on \((Y, k)\) is an element of \(\Omega^n((Y, k) \circ (Y, k))^{\mathbb{Z}/2}\).

The first part of this definition comes from [34], but the second part is new. (We are extremely grateful to John Klein for suggesting it as an improvement on some earlier attempts of ours.) It is best understood from the point of view of equivariant homotopy theory. The \(\Omega\)-spectrum \((Y, k) \circ_j (Y, k)\), with the action of \(\mathbb{Z}/2\), turns out to be the “underlying spectrum” of a \(\mathbb{Z}/2\)-spectrum in the sense of the equivariant theory [7], [13]. (See also [1].) We shall explain this using the following (conservative) language.

**Conventions 2.3.** Let \(G\) be a finite group, \(W\) the regular representation of \(G\). Let \(nW\) be the direct sum of \(n\) copies of \(W\), with one-point compactification \(S^{nW}\). A \(G\)-spectrum \(C\) is a family of well–based \(G\)-spaces \(C_{nW}\), defined for all sufficiently large positive integers \(n\), together with based \(G\)-maps 

\[
S^W \wedge C_{nW} \rightarrow C_{(n+1)W},
\]

with the diagonal action of \(G\) on \(S^W \wedge C_{nW}\). The underlying spectrum \(UC\) of \(C\) is the ordinary spectrum whose \(j\)-th space is 

\[
\text{colim}_n \Omega^{nW}\Sigma^j C_{nW}.
\]

It is an ordinary \(\Omega\)-spectrum with a degreewise action of \(G\) (coming from the conjugation action of \(G\) on \(\Omega^{nW}\Sigma^j C_{nW}\), for each \(n\) and \(j\)). The subspectrum of fixed points, \((UC)^G\), is often called the fixed point spectrum of \(C\). It is again an \(\Omega\)-spectrum.
Remark. The above definition of a $G$–spectrum is economical in that we only use the representations $nW$ for bookkeeping. The price for that is a mildly under-motivated definition of the “underlying spectrum”. As before, the loop spaces $\Omega^{nW}$ which appear in the definition of the underlying spectrum are to be constructed as geometric realizations of certain simplicial sets, so that the passage to the (co)limit is safe from the point of view of homotopy theory. Note that if $G$ is trivial, $G = \{1\}$, then $uC$ is simply a CW–substitute for $C$.

Beware that the expressions spectrum and $\Omega$–spectrum, as used here, correspond roughly to prespectrum and spectrum, respectively, in the language of [7] and [13] for example.

Returning to definition 2.2 now, we have that $(Y, k) \circ_* (Y, k)$ is the underlying spectrum of the $\mathbb{Z}/2$–spectrum given by $nW \mapsto S^{(n-k)W} \wedge Y^2$, where $Y^2$ is short for $Y \wedge Y$. (For $G = \mathbb{Z}/2$ we like to identify the regular representation $W$ with the permutation representation on $\mathbb{R}^2$.) Clearly this $\mathbb{Z}/2$–spectrum (not its underlying spectrum) can be described as

$$S^{-kW}_{\mathbb{Z}/2} \wedge Y^2$$

where $S^{-kW}_{\mathbb{Z}/2}$ is a shifted $\mathbb{Z}/2$–sphere spectrum, given by $nW \mapsto S^{(n-k)W}$.

**Proposition 2.4.** For any finite group $G$, any well–based $G$–space $Z$ which is free away from the base point and any $G$–spectrum $C$, the fixed point spectrum $(u(C \wedge Z))^G$ is homotopy equivalent to the homotopy orbit spectrum $(u(C \wedge Z))_{hG}$. Under this identification, the inclusion of $(u(C \wedge Z))^G$ in the homotopy fixed point spectrum $(u(C \wedge Z))^{hG}$ corresponds to the norm map.

**Corollary 2.5.** There is a natural homotopy fiber sequence of spectra

$$((Y, k) \circ_* (Y, k))_{h\mathbb{Z}/2} \longrightarrow (Y, k) \circ_* (Y, k)_{\mathbb{Z}/2} \longrightarrow J \Sigma^{-k}(Y/X)$$

where $\Sigma^{-k}(Y/X)$ means a CW–substitute for the suspension spectrum of $(Y/X)$.

**Proof of the corollary.** Let $T = Y^2$ and $T' = (E\mathbb{Z}/2)_+ \wedge T$. Let $f: T' \rightarrow T$ be the projection. The homotopy cofiber sequence of $\mathbb{Z}/2$–spaces $T' \longrightarrow T \longrightarrow \text{cone}(f)$ determines a homotopy fiber sequence of spectra

$$(u(C \wedge T'))_{\mathbb{Z}/2} \longrightarrow (u(C \wedge T))_{\mathbb{Z}/2} \longrightarrow (u(C \wedge \text{cone}(f)))_{\mathbb{Z}/2}$$

with $C = S^{-kW}_{\mathbb{Z}/2}$. The middle term in the sequence is

$$((Y, k) \circ_* (Y, k))_{\mathbb{Z}/2}$$

by construction. Proposition 2.4 identifies the left–hand term with

$$(u(C \wedge T'))_{h\mathbb{Z}/2} \simeq (u(C \wedge T))_{h\mathbb{Z}/2} = ((Y, k) \circ_* (Y, k))_{h\mathbb{Z}/2}.$$ 

The expression $(u(C \wedge \text{cone}(f)))_{\mathbb{Z}/2}$ can be identified with $\Sigma^{-k}(Y/X)$ as follows. It is an $\Omega$–spectrum whose $j$–th space is

$$\text{colim}_n \text{map}_{\mathbb{Z}/2}(S^{nW}, S^j \mathbb{R} \wedge (n-k)W \wedge \text{cone}(f))$$

where $\text{map}_{\mathbb{Z}/2}(\ldots)$ denotes a space of equivariant based maps and $j\mathbb{R}$ denotes a $j$–dimensional trivial representation of $\mathbb{Z}/2$. Because $\text{cone}(f)$ is non–equivariantly
contractible, equivariant based maps from $S_n W$ to $S^{2n \oplus (n-k)} W \wedge \text{cone}(f)$ are essentially determined by their restrictions to the fixed point sets. Hence the above expression for the $j$–th space of $(u(C \wedge \text{cone}(f)))^{\mathbb{Z}/2}$ simplifies to
\[
\operatorname{colim}_n \text{map}_\ast (S^n, S^j + n - k \wedge \text{cone}(f))^{\mathbb{Z}/2}.
\]
Since $\text{cone}(f)^{\mathbb{Z}/2}$ is $T^\mathbb{Z}/2 \simeq Y/X$, this simplifies even more to $\Omega^\infty \Sigma^\infty Y/X$, which is the $j$–th space in the $\Omega$–spectrification of $\Sigma^\infty Y/X$. □

Proof of the proposition. This is a standard fact from equivariant homotopy theory. We begin with a preliminary remark about pathologies. Choose a based $G$–CW–space $Z'$ which is $G$–free away from the base point and a $G$–map $e: Z' \to Z$ which is a weak equivalence. Because $Z$ is well–based, $v$ induces weak equivalences $id \wedge e: C \wedge Z' \to C \wedge Z$ for any well–based $G$–space $C$, in particular for $C = C_n W$. It follows that $e$ induces a homotopy equivalence of underlying spectra,
\[
u(C \wedge Z') \to \nu(C \wedge Z),
\]
and a homotopy equivalence of the fixed point spectra, $(\nu(C \wedge Z'))^G \to (\nu(C \wedge Z))^G$. Therefore, without loss of generality, $Z$ is a based $G$–CW–space which is $G$–free away from the base point.

We shall prove the two parts of the proposition together using a characterization of the norm map as an “assembly” transformation. (For this idea we are again indebted to John Klein.) Let $\mathcal{C}$ be the category of all based $G$–CW–spaces which are $G$–free away from the base point, with based $G$–maps as morphisms. Let $F$ be a functor from $\mathcal{C}$ to spectra which takes homotopy equivalences to weak equivalences and takes $\ast$ to a weakly contractible spectrum. Then there exists a natural transformation $\alpha: F^G \to F$ where $F^G$ is another functor from $\mathcal{C}$ to spectra, and

- $F^G$ respects weak equivalences,
- $F^G$ respects (weak) homotopy pushout squares,
- $F^G$ respects arbitrary wedges up to weak equivalence,
- $\alpha: F^G(Z) \to F(Z)$ is a weak equivalence when $Z = G_+$.

The pair $(F^G, \alpha)$ is essentially determined by $F$ and $\alpha$ is called the assembly transformation for $F$. For the case $G = \{1\}$, the proof (and a more detailed statement) can be found in [33] and the general case follows the same lines. (One possible definition of $F^G(Z)$ for arbitrary $Z$ in $\mathcal{C}$ is as follows. Take the geometric realization of the simplicial spectrum
\[
Z(n) \wedge \Delta^n_+ \wedge G_+
\]
where $Z(n)$ is the based set of singular $n$–simplices of $Z$ ; then divide out by the diagonal action of $G$.)

Now put $F(Z) := (\nu(C \wedge Z))^{hG}$. This $F$ clearly takes homotopy equivalences to weak equivalences and takes the trivial space $\ast$ to a trivial spectrum. It also respects (weak) homotopy pushout squares, but it does not satisfy the wedge axiom for infinite wedges. The norm transformation
\[
(\nu(C \wedge Z))^{hG} \to F(Z)
\]
satisfies all the properties which characterize the assembly for $F$. Therefore it is the assembly for $F$. We can now give our proof by showing that the natural transformation

$$(u(C \wedge Z))^G \longrightarrow F(Z)$$

given by the inclusion of fixed point spectra in homotopy fixed point spectra also satisfies all the properties which characterize the assembly for the functor $F$.

Of the four properties listed, three hold by inspection. So it only remains to check that the inclusion of $(u(C \wedge Z))^G$ in $(u(C \wedge Z))_0G$ is a homotopy equivalence when $Z = G_+$. From the definitions, an element of $\pi_n((u(C \wedge G_+))^G)$ is represented by a $G$-map

$$f: S^{(n+j)\mathbb{R} \oplus iW} \longrightarrow S^{\mathbb{R}} \wedge C_{iW} \wedge G_+$$

with “large” $i$ and $j$ (where $j \mathbb{R}$ for example denotes a trivial $j$-dimensional representation). We may assume that $f$ is transverse to $0 \times C_{iW} \times G$. The inverse image of $0 \times C_{iW} \times 1$ is then a framed smooth closed $(n + iG)$-dimensional submanifold $M$ of $(n + j)\mathbb{R} \oplus iW$. Clearly $M \cap gM = \emptyset$ for $g \in G \setminus \{1\}$ and we have a map $f|_M$ from $M$ to $C_{iW}$. Conversely, given any framed smooth closed $(n + iG)$-dimensional submanifold $M$ of $(n + j)\mathbb{R} \oplus iW$ with $M \cap gM = \emptyset$ for $g \in G \setminus \{1\}$, and a map $q$ from $M$ to $C_{iW}$, the Pontryagin–Thom construction gives us an appropriate $f$ for which $M = f^{-1}(0 \times C_{iW} \times 1)$ and $f|_M = q$. In the limit, when $j$ and $i$ tend to infinity, the condition $M \cap gM = \emptyset$ for $G \setminus \{1\}$ becomes irrelevant and so the $n$-th homotopy group under consideration is identified with the $n$-dimensional framed bordism group of $C$, i.e., with $\pi_n(u(C))$. Moreover, this identification clearly agrees with the homomorphism induced by the composition

$$(u(C \wedge G_+))^G \longrightarrow (u(C \wedge G_+))^G \longrightarrow u(C)$$

where the second arrow is forgetful and the third is induced by the (non-equivariant) map $C \wedge G_+ \rightarrow C$ which isolates the summand $C \wedge \{1\}_+$. Since the composition of the last two arrows is a homotopy equivalence, the first arrow is a homotopy equivalence.

The following continuation of definition 2.2 is suggested by corollary 2.5.

**Definition 2.6.** An $n$-dimensional visible hyperquadratic structure on $(Y, k)$ is an element in $\Omega^n\Omega^\infty\Sigma^\infty-^k(Y/X)$. An $n$-dimensional quadratic structure on $(Y, k)$ is an element of $\Omega^n\Omega^\infty(((Y, k) \circ \bullet (Y, k))_{h\mathbb{Z}/2})$. Alternatively, an $n$-dimensional quadratic structure on $(Y, k)$ can be defined as an element of $\Omega^n\Omega^\infty$ of the homotopy fiber of the natural map $J: ((Y, k) \circ \bullet (Y, k))_{h\mathbb{Z}/2} \rightarrow \Sigma^\infty-^k(Y/X)$.

An $n$-dimensional visible symmetric structure on $(Y, k)$ is considered nondegenerate if the underlying $n$-dimensional symmetric structure is nondegenerate. Writing $sR(X)$ for the stable category of finitely dominated retractive spaces over $X$, we obtain the definition of a visible symmetric $L$–theory spectrum

$$VL^*(sR(X)) = VL^*(X)$$

by substituting nondegenerate visible symmetric structures for nondegenerate symmetric structures throughout in the construction of the symmetric $L$–theory spectrum $L^*(sR(X)) = L^*(X)$. See [34]. The standard map

$$L_*(X) \rightarrow L^*(X)$$

can be factorized as $L_*(X) \rightarrow VL^*(X) \rightarrow L^*(X)$. This is clear from definition 2.6.
We write $\mathcal{V}\mathcal{L}^\ast(X)$ for the mapping cone (in the category of spectra) of the above map $L\ast(X) \to \mathcal{V}\mathcal{L}^\ast(X)$.

**Theorem 2.7.** The functor $X \mapsto \mathcal{V}\mathcal{L}^\ast(X)$ is homotopy invariant and excisive.

"Homotopy invariance" is intended to mean that the functor takes weak equivalence to homotopy equivalences, and this is clear. (An equivalent formulation says that, for each $X$, the maps $\mathcal{V}\mathcal{L}^\ast(X) \to \mathcal{V}\mathcal{L}^\ast(X \times [0,1])$ induced by $x \mapsto (x,0)$ and $x \mapsto (x,1)$ are homotopic. They are indeed homotopic because the exact functors which they induce are related by a chain of natural weak equivalences.) The excision property means that the functor takes empty space to a contractible spectrum and takes weak homotopy pushout squares (also known as cocartesian squares) of spaces to homotopy pushout squares (equivalently, homotopy pullback squares) of spectra. Our proof of the excision property relies on three decomposition lemmas.

For the first of these, suppose that $X$ is the union of two closed subspaces $X_a$ and $X_b$ with intersection $X_{ab}$, such that the inclusions $X_{ab} \to X_a$ and $X_{ab} \to X_b$ are cofibrations. Let $r: E \to X$ be a fibration with section $s$ making $E$ into a homotopy finite retractive space over $X$. Let $Y$ be a finite retractive space over $X$ with a morphism $f: Y \to E$ of retractive spaces over $X$. We assume that $Y$ is decomposed as $Y := Y_a \cup Y_b$ where $Y_a$, $Y_b$ and $Y_{ab} = Y_a \cap Y_b$ are finite retractive spaces over $X_a$, $X_b$ and $X_{ab}$, respectively, with cofibrations $Y_{ab} \cup X_a \to Y_a$ and $Y_{ab} \cup X_b \to Y_b$.

**Lemma 2.8.** The morphism $f: Y \to E$ has a factorization of the form $Y \xrightarrow{f_1} Z \xrightarrow{g} E$ where

(i) $Z$ is a finite retractive space over $X$
(ii) $f_1$ is a cofibration
(iii) $g$ is a weak equivalence
(iv) the decomposition of $Y$ extends to a similar decomposition $Z := Z_a \cup Z_b$ where $Z_a$, $Z_b$ and $Z_{ab} = Z_a \cap Z_b$ are finite retractive spaces over $X_a$, $X_b$ and $X_{ab}$, respectively, with cofibrations $Z_{ab} \cup X_a \to Z_a$ and $Z_{ab} \cup X_b \to Z_b$.

**Proof.** Since the inclusions $X_{ab} \to X_a$ and $X_{ab} \to X_b$ are cofibrations, we can easily reduce to the situation where $X_{ab}$ has collar neighborhoods $X_{ab} \times [-1,0]$ in $X_a$ and $X_{ab} \times [0,1]$ in $X_b$. Ignoring condition (iv), we can easily produce a factorization $f = gf_1$ with properties (i), (ii) and (iii); then $Z$ has a filtration $X = Z^{-1} \subset Z^0 \subset Z^1 \subset \cdots \subset Z^k = Z$ where $Z^i$ is the relative $i$-skeleton. Suppose now that $Z_i^{-1}$ is already decomposed as in (iv). We can assume that the attaching data for the cells we must attach to obtain $Z_i$ have the form of a commutative diagram

$$
\begin{array}{ccc}
\coprod_a S^{i-1} & \longrightarrow & \coprod_a D^i \\
\downarrow & & \downarrow \\
Z^{i-1} & \longrightarrow & E
\end{array}
$$
such that the composition $\coprod D^i \to E \to X$ is transverse to the subspace $X_{ab} \times \{0\}$ of $X_{ab} \times [-1,+1] \subset X$. Triangulating the pair $(\coprod D^i, \coprod S^{i-1})$ in such a way that the inverse image of $X_{ab} \times \{0\}$ is a subcomplex, we can also arrange that the attaching map $S^{i-1} \to Z^{i-1}$ is cellular for the chosen triangulation. (Here we are using the assumption that $E$ is fibered over $X$.) Using the triangulation cell structure on the attached $\coprod D^i$, we obtain a “new” relative CW structure on $Z'$ which extends the CW structure on $Z^{i-1}$ and in which $Z^i$ decomposes as in (iv). We continue inductively. \[\blacksquare\]

Keeping the notation of lemma 2.8, we define an $n$–dimensional quadratic structure on $(Z_a, Z_b, k)$, for $n \geq 0$, to be an element in $\Omega^n \Omega\infty$ of the homotopy pushout of the diagram

$$
\begin{array}{c}
((Z_{ab}, k) \circ \bullet (Z_{ab}, k))_{hZ/2} \\
\downarrow
\end{array}
\begin{array}{c}
((Z_a, k) \circ \bullet (Z_a, k))_{hZ/2}
\end{array}
$$

Here the $\circ\bullet$ products are taken with respect to the base spaces $X_a$, $X_{ab}$ and $X_b$, respectively. By a similar generalization process, we arrive at the notions of a visible symmetric structure on $(Z_a, Z_b, k)$, and the notion of a visible hyperquadratic structure on $(Z_a, Z_b, k)$. The corresponding abelian groups of path classes are denoted

$$Q_n(Z_a, Z_b, k), VQ^n(Z_a, Z_b, k), \hat{V}Q^n(Z_a, Z_b, k)$$

respectively. They are actually defined for all $n \in \mathbb{Z}$ as $n$–th homotopy groups of the appropriate homotopy pushout spectra. Clearly the map $g: Z \to E$ in lemma 2.8 induces homomorphisms from $Q_n(Z_a, Z_b, k)$ to $Q_n(E, k)$ and from $VQ^n(Z_a, Z_b, k)$ to $VQ^n(E, k)$. We ask whether these homomorphisms become isomorphisms “in the (co)limit”. To speak of a colimit we need an indexing category $\mathcal{C}$, and in our case this should clearly have objects of the form

$$g: Z \to E$$

where $Z$ is finite, $g$ is a weak equivalence of retractive spaces over $X$ and $Z$ is decomposed into $Z_a$ and $Z_b$ with intersection $Z_{ab}$ as before. Morphisms in the category; say from $(g', Z', Z'_a, Z'_b)$ to $(g, Z, Z_a, Z_b)$, are cofibrations $u: Z \to Z'$ with $g' u = g$ and $u(Z_a) \subset Z'_a$, $u(Z_b) \subset Z'_b$. Lemma 2.8 implies that $\mathcal{C}$ is directed. That is, for any two objects $Z$ and $Z'$ in $\mathcal{C}$ (in shorthand notation) there exists an object $Z''$ and morphisms $Z \to Z''$, $Z' \to Z''$; and given any two morphisms $u, v: Z \to Z'$ there exists a morphism $w: Z'' \to Z''$ such that $wu = vw$.

Lemma 2.9. In the above notation, we have

$$\begin{align*}
\colim_{(g, Z, Z_a, Z_b)} Q_u(Z_a, Z_b, k) & \cong Q_u(E, k), \\
\colim_{(g, Z, Z_a, Z_b)} VQ^n(Z_a, Z_b, k) & \cong VQ^n(E, k), \\
\colim_{(g, Z, Z_a, Z_b)} \hat{V}Q^n(Z_a, Z_b, k) & \cong \hat{V}Q^n(E, k)
\end{align*}$$

where the direct limits are taken over $\mathcal{C}$.
Proof. The second isomorphism is a consequence of the first and the third (and the “five lemma”), because by corollary 2.5 there are exact sequences of type
\[ \cdots \to V\hat{\mathbb{Q}}^{n+1} \to Q_n \to V\mathbb{Q}^n \to V\hat{\mathbb{Q}}^n \to Q_{n-1} \to \cdots. \]
The third isomorphism is obvious from definition 2.6. It remains to establish the first isomorphism. Let \( E_a := E|X_a \), \( E_b := E|X_b \) and \( E_{ab} = E|X_{ab} \). These are all fibered retractive spaces over the appropriate base spaces: \( X_a \), \( X_a \) and \( X_{ab} \), respectively. Therefore, in forming the \( \wedge \) product \( E_a \wedge E_a \), for example, we can proceed more directly than otherwise by forming the fiberwise smash product of \( E_a \) with \( E_a \) over \( X_a \), and then dividing out by the zero section \( X_a \). This leads immediately to a homotopy pushout square consisting of the four spaces \( E_a \wedge E_a \), \( E_b \wedge E_a \), \( E_{ab} \wedge E_b \) and \( E \wedge E \), and consequently a homotopy pushout square of spectra
\[
\begin{array}{ccc}
((E_a, k) \circ \bullet (E_{ab}, k))_{h\mathbb{Z}/2} & \longrightarrow & ((E_a, k) \circ \bullet (E_b, k))_{h\mathbb{Z}/2} \\
\downarrow & & \downarrow \\
((E_b, k) \circ \bullet (E_{b}, k))_{h\mathbb{Z}/2} & \longrightarrow & ((E_b, k) \circ \bullet (E_b, k))_{h\mathbb{Z}/2}.
\end{array}
\]
where the \( \circ \bullet \) products are taken with respect to the appropriate base spaces: \( X_{ab} \), \( X_a \), \( X_b \) and \( X \). (Note that, strictly speaking, the \( \wedge \) product and the \( \circ \bullet \) product have only been defined for finitely dominated retractive spaces. We have no reason to think that \( E_a \), \( E_b \) and \( E_{ab} \) are all finitely dominated, but the definition of \( \wedge \) extends without difficulties.) We have therefore
\[ Q_n(E_a, E_b, k) \xrightarrow{\sim} Q_n(E, k) \]
for all \( n \), with the obvious interpretation of \( Q_n(E_a, E_b, k) \). This reduces our task to showing that
\[ \colim_{(g, Z, Z_a, Z_b)} Q_n(Z_a, Z_b, k) \xrightarrow{\sim} Q_n(E_a, E_b, k). \]
By a Mayer–Vietoris and five lemma argument, this reduces further to showing that the homomorphisms
\[
\begin{align*}
\colim_{(g, Z, Z_a, Z_b)} Q_n(Z_a, k) & \xrightarrow{\sim} Q_n(E_a, k) \\
\colim_{(g, Z, Z_a, Z_b)} Q_n(Z_b, k) & \xrightarrow{\sim} Q_n(E_b, k) \\
\colim_{(g, Z, Z_a, Z_b)} Q_n(Z_{ab}, k) & \xrightarrow{\sim} Q_n(E_{ab}, k)
\end{align*}
\]
are all isomorphisms. By lemma 2.8, we may now enlarge the indexing categories to allow objects \((f, Y, Y_a, Y_b)\) where \( Y \) is a finite retractive space over \( X \) with a decomposition \( Y = Y_a \cup Y_b \) etc., and where \( f: Y \to E \) is any map (not necessarily a weak equivalence) of retractive spaces over \( X \). Then \( E_a \) for example can easily be identified with the homotopy direct limit of the \( Y_a \), etc., and \( Q_n \) takes the homotopy direct limits to direct limits, so that the isomorphisms become obvious. \( \square \)

Lemma 2.10. Let \( X = X_a \cup X_b \) as in lemma 2.8. Let \( Z \) be a finite retractive space over \( X \) with a decomposition \( Z = Z_a \cup Z_b \) as in lemma 2.8, so that \( Z_a \) is retractive over \( X_a \) and \( Z_b \) is retractive over \( X_b \). Let \( k, n \in \mathbb{Z} \). Then there exist a
finite retractive space \( V \) over \( X \) with a decomposition \( V = V_a \cup V_b \) as in lemma 2.8, an integer \( \ell \) and a nondegenerate element \( \eta \) in \( \pi_n \) of the homotopy pushout of

\[
(V_a, \ell) \circ \bullet (Z_a, k) \longrightarrow (V_{ab}, \ell) \circ \bullet (Z_{ab}, k) \longrightarrow (V_b, \ell) \circ \bullet (Z_b, k)
\]

such that the images of \( \eta \) in

\[
\pi_n((V_a, \ell) \circ \bullet (Z, k)), \quad \pi_n((V_a, \ell) \circ \bullet (Z_a/Z_{ab}, k)), \quad \pi_n-1((V_{ab}, \ell) \circ \bullet (Z_{ab}, k))
\]

are all nondegenerate.

**Proof.** The guiding principle here is the fact that passage from finite retractive spaces to cellular chain complexes over the appropriate group(oid) rings respects and detects nondegenerate pairings. This is due to [24]. The relevant group(oid) rings here are \( \mathbb{Z}\pi_1(X_a) \), \( \mathbb{Z}\pi_1(X_b) \), \( \mathbb{Z}\pi_1(X_{ab}) \) and \( \mathbb{Z}\pi_1(X) \). Note also that a change of rings, such as the passage from chain complexes over \( \mathbb{Z}\pi_1(X_{ab}) \) to chain complexes over \( \mathbb{Z}\pi_1(X_a) \) by means of

\[
\mathbb{Z}\pi_1(X_a) \circ \mathbb{Z}\pi_1(X_{ab})
\]

respects nondegenerate pairings. It follows that cobase change, such as the passage from retractive spaces over \( X_{ab} \) to retractive spaces over \( X_a \) by means of

\[
X_a \sqcup X_{ab}
\]

respects nondegenerate pairings. Consequently we can construct \( V \) and \( V_a, V_b \) in the following way. We first find an \((n-1)\)-dual for \((Z_{ab}, k)\) as a stable retractive space over \( X_{ab} \). This amounts to finding a retractive space \( V_{ab} \) over \( X_{ab} \), an integer \( \ell \) and a nondegenerate element \( \eta_{ab} \) in \( \pi_{n-1}((V_{ab}, \ell) \circ \bullet (Z_{ab}, k)) \). Next we find an \( n \)-dual for the pair \((Z_a, k), (Z_{ab}, k)\) which extends our chosen \((n-1)\)-dual for \((Z_{ab}, k)\). This amounts to finding \( V_a \), a cofibration \( V_{ab} \) and an element \( \eta_a \) in \( \pi_n \) of the mapping cone of

\[
(V_{ab}, \ell) \circ \bullet (Z_{ab}, k) \longrightarrow (V_a, \ell) \circ \bullet (Z_a, k)
\]

whose image in \( \pi_n((V_a, \ell) \circ \bullet (Z_a/Z_{ab}, k)) \) is nondegenerate and whose image in \( \pi_{n-1}((V_{ab}, \ell) \circ \bullet (Z_{ab}, k)) \) is \( \eta_{ab} \). (It may be necessary to increase \( \ell \).) We proceed similarly with the pair \((Z_b, k), (Z_{ab}, k)\) to obtain \( V_b \) and \( \eta_b \). Then we define

\[
V := V_a \sqcup V_{ab} \sqcup V_b
\]

and find \( \eta \) in \( \pi_n \) of the homotopy pushout of

\[
(V_a, \ell) \circ \bullet (Z_a, k) \longrightarrow (V_{ab}, \ell) \circ \bullet (Z_{ab}, k) \longrightarrow (V_b, \ell) \circ \bullet (Z_b, k)
\]

mapping to \( \eta_a \) and \(-\eta_b \) under the appropriate projections. The existence of such an \( \eta \) follows from a suitable Mayer–Vietoris sequence. The image of \( \eta \) in the homotopy group \( \pi_n(V, \ell) \circ \bullet (Z, k) \) will automatically be nondegenerate, by Vogell’s chain complex criterion.

\[\square\]

As another preliminary for the proof of theorem 2.7, we offer a lengthy discussion of how elements in the homotopy group

\[
\hat{V}(X) = \pi_n \mathcal{V}(X)
\]

can be represented. For that discussion we return briefly to the case of the \( L \)-theory of a group ring \( \mathbb{Z}\pi \). Let \( D \) be the category of bounded (above and below) chain complexes of f.g. left projective \( \mathbb{Z}\pi \)-modules. We regard \( D \) as a Waldhausen category in which the cofibrations are the chain maps which are split injective in
each dimension. Cofibrations \( C \to D \) in \( D \) are often regarded as pairs \((D, C)\).

Let \( E \) be an object of \( D \) with an \( n \)–dimensional symmetric structure \( \varphi \). The inclusion of \( E \) in the algebraic mapping cone of \( \varphi_0: E^{n-*} \rightarrow E \) classifies an “extension” with base \( E \) in the shape of a short exact sequence

\[
0 \to C \to D \to E \to 0
\]

where \( C \simeq \Sigma^{-1}\text{cone}(\varphi_0) \) and \( D \simeq E^{n-*} \). According to Ranicki [17], the symmetric structure \( \varphi \) on \( E \) has a preferred lift to an \( n \)–dimensional nondegenerate symmetric structure \( (\varphi, \partial \varphi) \) on the pair \((D, C)\), so that \( \varphi/\partial \varphi = \varphi \) under the identification \( D/C \cong E \). (Warning: \( \varphi \) is an \( n \)–chain in \( \text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, D^t \otimes_{\mathbb{Z}} D) \) with boundary \( \partial \varphi \) in the image of \( \text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, C^t \otimes_{\mathbb{Z}} C) \). Our notation for symmetric and quadratic structures on pairs deviates from Ranicki’s.) This resolution procedure of Ranicki’s leads to a bijective correspondence between homotopy types of chain complexes \( E \) with an \( n \)–dimensional symmetric structure, and homotopy types of chain complex pairs \((D, C)\) with an \( n \)–dimensional nondegenerate symmetric structure. There is a similar correspondence for visible symmetric and quadratic structures.

Of particular interest to us is the mixed case, i.e. the case of nondegenerate symmetric pairs \((D, C)\) with quadratic boundary. For a pair \((D, C)\) with an \( n \)–dimensional nondegenerate symmetric structure \( (\varphi, \partial \varphi) \), improving the \((n-1)\)–dimensional symmetric structure \( \partial \varphi \) on \( C \) to a quadratic structure on \( C \) amounts to “trivializing” the induced \((n-1)\)–dimensional hyperquadratic structure

\[
J(\partial \varphi) \in \text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\bar{W}, C^t \otimes C)
\]

on \( C \), in other words, finding an \( n \)–chain with boundary \( J(\partial \varphi) \). But since the functor

\[
C \mapsto \text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\bar{W}, C^t \otimes C)
\]

respects homotopy cofiber sequences, finding such a trivialization is equivalent to finding a trivialization for the suspended hyperquadratic structure on \( \Sigma C \). By Ranicki’s correspondence, if we write \( E = D/C \) and \( \varphi = \varphi/\partial \varphi \), the suspension of \( J(\partial \varphi) \) can be identified with the image of \( J \varphi \) under the inclusion

\[
E \to \text{cone}(\varphi_0).
\]

Summarizing, there is a bijective correspondence between homotopy types of \( n \)–dimensional nondegenerate symmetric pairs in \( D \) with quadratic boundary, and homotopy types of single objects \( E \) in \( D \) with an \( n \)–dimensional symmetric structure \( \varphi \) and a trivialization \( \tau \) of the image of \( J \varphi \) under the inclusion \( E \to \text{cone}(\varphi_0) \). This remains correct if symmetric and hyperquadratic structures are replaced with visible symmetric and visible hyperquadratic structures throughout.

The correspondence extends to nullbordisms. For more precision we suppose again that \((D, C), (\varphi, \partial \varphi)\) is an \( n \)–dimensional nondegenerate symmetric pair with quadratic boundary. Let \((E, \varphi) = (D/C, \varphi/\partial \varphi)\) be its Thom complex. Let \( \tau \) be the trivialization of the image of \( J \varphi \) under \( E \to \text{cone}(\xi_0) \) determined by the preferred trivialization of \( J(\partial \varphi) \). Then a (nondegenerate) nullbordism of the nondegenerate pair \((D, C), (\varphi, \partial \varphi)\) determines a nullbordism of \((E, \varphi, \tau)\), by which is meant:

- a chain complex pair \((F, \partial F)\) with \((n+1)\)–dimensional symmetric structure \((\xi, \partial \xi)\), where \( \partial F = E \) and \( \partial \xi = \varphi \);
- a trivialization \((\kappa, \partial \kappa)\) with \( \partial \kappa = \tau \) of the image of \( J(\xi, \partial \xi) \) under the inclusion \((F, \partial F) \to \text{cone}(\xi_0), \text{cone}(\partial \xi_0)) \). Here \( \xi_0 \) is regarded as a chain map from \( \text{cone}(F^{n-*} \rightarrow E^{n-*}) \) to \( F \).
Conversely, a nullbordism of \((E, \varphi, \tau)\) determines a nondegenerate nullbordism of the nondegenerate pair \(((D, C), (\bar{\varphi}, \partial \varphi))\). Again, this remains correct if symmetric and hyperquadratic structures are replaced with visible symmetric and visible hyperquadratic structures throughout.

Returning now to the stabilization \(sR(X)\) of the category of finitely dominated retractive spaces over a fixed space \(X\), we remark that these correspondences apply, mutatis mutandis, in \(sR(X)\). The fact that there are strictly speaking no “canonical” \(n\)-duals in \(sR(X)\) does complicate matters slightly (but only at first). For an object \((Y, k)\) in \(sR(X)\) with an \(n\)-dimensional symmetric or visible symmetric structure \(\varphi\), the correct way to determine an \(n\)-dual \((Y, k - n)^*\) with \(n \geq 0\) is to find an object \((Z, \ell)\) in \(sR(X)\) and a nondegenerate element \(\eta \in \Omega^n((Z, \ell) \circ (Y, k))\).

Modulo a “trivial” enlargement of \((Y, k)\), a morphism \(f : (Z, \ell) \to (Y, k)\) and a path from \(f_\ell(\eta)\) to \(\varphi_0 \in \Omega^n((Y, k) \circ (Y, k))\) can then be found. (The trivial enlargement is an object of \(sR(X)\) related to \((Y, k)\) by a morphism which is both a cofibration and a weak equivalence.) The morphism \(f\) then deserves to be regarded as the adjoint of \(\varphi_0\). Therefore, when we write

\[
\varphi_0 : (Y, k - n)^* \longrightarrow (Y, k),
\]

we mean \(f : (Z, \ell) \to (Y, k)\).

**Proof of theorem 2.7.** We assume that \(X = X_a \cup X_b\) and \(X_a \cap X_b = X_{ab}\) as in lemma 2.8. We need to show that the gluing homomorphism

\[
\alpha_n : V\tilde{L}^n(X_a, X_b) \to V\tilde{L}^n(X)
\]

is an isomorphism, where \(V\tilde{L}^n(X_a, X_b)\) is the \(n\)-th homotopy group of the homotopy pushout of

\[
V\tilde{L}^\bullet(X_a) \leftarrow V\tilde{L}^\bullet(X_{ab}) \to V\tilde{L}^\bullet(X_b).
\]

We establish this only when \(n > 0\). The case \(n < 0\) can be handled in the same way. (Replace \(\circ\) by \(\circ_j\) for some \(j\) with \(j + n \geq 0\) in the argument below.)

Starting with the surjectivity part and assuming \(n > 0\), we represent an element of \(V\tilde{L}^n(X)\) by an object \((Z, k)\) in \(sR(X)\), an \(n\)-dimensional visible symmetric structure \(\varphi\) on \((Z, k)\) and a “trivialization” \(\tau\) of the \(n\)-dimensional visible hyperquadratic structure on \(\text{cone}(\varphi_0)\) obtained by pushing \(J\varphi\) forward along the inclusion \((Z, k) \to \text{cone}(\varphi_0)\). Here we view \(\varphi_0\) as a morphism from an \(n\)-dual of \((Z, k)\) to \((Z, k)\). We may assume that \(Z\) is not only finitely dominated, but finite. (Otherwise replace \(Z\) by \(Z \wedge \Sigma Z\), which has zero finiteness obstruction; also, replace \(\varphi\) and \(\tau\) by their images under appropriate maps induced by the inclusion \(Z \to Z \vee \Sigma Z\).)

By lemma 2.8 and lemma 2.9 we may then assume that

\[
Z = Z_a \cup Z_b,
\]

\[
\varphi = \varphi' + \varphi''
\]

where \(Z_a\) and \(Z_b\) are as in lemma 2.8 and \((\varphi', \partial \varphi'), (\varphi'', \partial \varphi'')\) are visible symmetric structures on the pairs \(((Z_a, k), (Z_{ab}, k))\) and \(((Z_b, k), (Z_{ab}, k))\), respectively, with \(\partial \varphi' = - \partial \varphi''\). (In more detail, if \(Z\) and \(\varphi\) do not come equipped with such a decomposition, then we first use the Serre construction to enlarge \(Z\) to a fibered retractive space \(E\) over \(X\). The fibered retractive space \(E\) can in turn be approximated as in lemma 2.8 by another retractive space \(Z'\) over \(X\) which is decomposed into \(Z'_a\) and \(Z'_b\). Then lemma 2.9 can be applied, etc.) The equations \(\varphi = \varphi' + \varphi''\) and \(\partial \varphi' = - \partial \varphi''\) can be more accurately expressed by saying that \(\varphi\) is parametrized
by $S^n$ and that its restrictions to the “upper” and “lower” hemispheres of $S^n$ define $n$–dimensional visible symmetric structures on the pairs $((Z_a, k), (Z_{ab}, k))$ and $((Z_b, k), (Z_{ab}, k))$, respectively. Under these conditions, $\varphi$ and $\varphi', \varphi''$ represent an element of what we have called $VQ^n(Z_a, Z_b, k)$ in lemma 2.9. Now by lemma 2.10 we may assume that we have an $n$–dual $(V, \ell)$ for $(Z, k)$ which is also decomposed, $V = V_a \cup V_b$. Hence we have a decomposition in the shape of a pushout square

$$
\begin{array}{ccc}
\text{cone}((V_{ab}, \ell) \xrightarrow{\partial \varphi''} (Z_{ab}, k)) & \longrightarrow & \text{cone}((V_b, \ell) \xrightarrow{\varphi''} (Z_b, k)) \\
\downarrow & & \downarrow \\
\text{cone}((V_a, \ell) \xrightarrow{\varphi'} (Z_a, k)) & \longrightarrow & \text{cone}((V, \ell) \xrightarrow{\varphi_0} (Z, k))
\end{array}
$$

and the trivialization $\tau$ automatically decomposes in the same manner. This completes the solution of our decomposition problem and so establishes the surjectivity part of the proof.

A relative version (which we will not write out in detail) of the argument shows that, if the original representative $((Z, k), \varphi, \tau)$ is nullbordant, in the sense which we gave to the word “nullbordant” earlier, then the lift across $\alpha_n: V\hat{L}^n(X_a, X_b) \to V\hat{L}^n(X)$ which we have constructed is also nullbordant. Hence our surjectivity proof amounts to a homomorphism of bordism groups which is right inverse to $\alpha_n$. By a straightforward inspection, it is also left inverse to $\alpha_n$. □

3. THE HYPERQUADRATIC $L$–THEORY OF A POINT

The $L$–theory of a point is, in our terminology, the $L$–theory of the (stabilization of) the category of finite based $CW$–spaces with the standard notion of Spanier–Whitehead duality. In this chapter we “calculate” the homotopy types of the spectra

$$
\hat{L}^\bullet(\ast), V\hat{L}^\bullet(\ast).
$$

The calculations will not be used for anything else in this paper, but they are interesting for a number of reasons. In particular we shall see that the inclusion

$$
V\hat{L}^\bullet(\ast) \to \hat{L}^\bullet(\ast)
$$

is not a homotopy equivalence (which spoils the analogy with the linear version of visible symmetric $L$–theory, outlined above). But in fact it deviates very little from being a homotopy equivalence, and the source turns out to be nothing more or less than a cleaned–up version of the target.

Understanding $V\hat{L}^\bullet(\ast)$ and $\hat{L}^\bullet(\ast)$ has a lot to do with understanding the “homology theories”

$$
(Y, k) \mapsto \Sigma^{\infty - k}Y, \quad (Y, k) \mapsto ((Y, k) \odot (Y, k))^{thZ/2},
$$

and the natural transformation from the first to the second which is implicit in corollary 2.5. This natural transformation can be made explicit by (re)defining $\Sigma^{\infty - k}Y$ as the homotopy cofiber of the improved norm map

$$
((Y, k) \odot (Y, k))_{htZ/2} \to ((Y, k) \odot (Y, k))^{Z/2}
$$
of proposition 2.4 and corollary 2.5, and (re)defining \(((Y,k) \odot_{\bullet} (Y,k))^{thZ/2}\) as the homotopy cofiber of the ordinary norm map

\[
((Y,k) \odot_{\bullet} (Y,k))_{hZ/2} \longrightarrow ((Y,k) \odot_{\bullet} (Y,k))^{hZ/2}.
\]

Because we are dealing with homology theories, we can simplify through a chain of natural weak homotopy equivalences,

\[
((Y,k) \odot_{\bullet} (Y,k))_{hZ/2} \simeq \cdots \simeq (-k)\text{-fold shift of } Y \wedge (S^0 \odot_{\bullet} S^0)^{hZ/2}.
\]

and more obviously \(\Sigma \infty -k Y \simeq Y \wedge \Sigma \infty -k S^0\). This is explained in [33]. Now it is easy to identify \(S^0 \odot_{\bullet} S^0\) with the sphere spectrum \(S\) through a chain of equivariant homotopy equivalences (using the flip action of \(Z/2\) on \(S^0 \odot_{\bullet} S^0\), and the trivial action of \(Z/2\) on \(S\)). Hence what we need to understand is \(S^{thZ/2}\).

The Segal conjecture [6] for a single point with the (trivial) action of \(Z/2\) means that \(S^{hZ/2}\) is homotopy equivalent to a certain completion of the fixed point spectrum of the equivariant sphere spectrum \(S_{Z/2}\). The fixed point spectrum of the equivariant sphere spectrum can be identified with the \(K\)-theory of the symmetric monoidal category of finite \(Z/2\)-sets [7] and therefore breaks up as

\[S \vee S_{hZ/2}\]

where the summands correspond to the isomorphism types of irreducible \(Z/2\)-sets. If we identify the Burnside ring \(\pi_0(S \vee S_{hZ/2})\) with \(\mathbb{Z} \otimes \mathbb{Z}\), then the augmentation ideal \(I\) consists of the elements of the form \((2z, -z)\). We have to complete at \(I\). It is therefore to our advantage to reconsider the splitting of the equivariant fixed point spectrum: write

\[
(u(S_{Z/2}))^{Z/2} \simeq S \vee \Gamma(-\text{transfer})
\]

where \(\Gamma(-\text{transfer})\) is the graph of the negative of the transfer from \(S_{hZ/2}\) to \(S\). Then the \(I\) is the \(\pi_0\) of the second summand. Its powers are the ideals \(2^n I\). Therefore, indicating completion at 2 by a left–hand superscript \(c\), we have

\[S^{hZ/2} \simeq S \vee \Gamma(-\text{transfer})^c.\]

In this decomposition, the norm map

\[S_{hZ/2} \rightarrow S^{hZ/2}\]

has first component equal to the transfer and second component equal to the identity (followed by completion). In calculating the homotopy cofiber, we may replace the source by its 2–completion and 2–complete the first summand of the target as well; the homotopy cofiber remains the same. We summarize the result in the following

**Lemma 3.1.** The homology theory \((Y, k) \mapsto ((Y, k) \odot_{\bullet} (Y, k))^{thZ/2}\) has coefficient spectrum \(\ast S\). □

Evaluating the natural transformation \(\Sigma \infty -k Y \rightarrow ((Y, k) \odot_{\bullet} (Y, k))^{thZ/2}\) just constructed on the object \((Y, k) = (S^0, 0)\), we have a map from \(\Sigma \infty -k Y = S\) to \(((Y, k) \odot_{\bullet} (Y, k))^{thZ/2} \simeq \ast S\).

**Lemma 3.2.** The map under consideration is the inclusion \(S \rightarrow \ast S\).
Proof. The homotopy fiber sequence of corollary 2.5 splits when \((Y,k) = (S^0,0)\). Our map can therefore be obtained from the composition
\[
S \vee S_{h\mathbb{Z}/2} \simeq (u(S_{\mathbb{Z}/2}))^{\mathbb{Z}/2} \to S^{h\mathbb{Z}/2} \to S^{th\mathbb{Z}/2},
\]
which we analyzed earlier, by restricting to the summand \(S\).

We now recall, following [28], how the chain bundle method for determining hyperquadratic \(L\)-theory (and certain variations on that) works in the linear case, and then transport the technology to the nonlinear situation.

Let \(R\) be a ring with involution \(R\), let \(B\) be a bounded (below and above) chain complex of f.g. projective left \(R\)-modules and let \(\gamma\) be a 0-dimensional cycle in
\[
\text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\tilde{W}, (B^{-*})^t \otimes_R B^{-*}) = ((B^{-*})^t \otimes_R B^{-*})^{th\mathbb{Z}/2}.
\]
Such a thing is called a chain bundle on \(B\) and will be treated as a chain complex analogue of a spherical fibration.

In particular, let \((C,\varphi)\) be a symmetric Poincaré chain complex over \(R\), of formal dimension \(n\). Then \(C\) has a normal chain bundle \(\nu\), which comes together with an \((n+1)\)-chain \(\gamma\) in
\[
(C^t \otimes C)^{th\mathbb{Z}/2}
\]
whose boundary is the difference between \(J\varphi\) and \((\varphi_0)_*(\Sigma^n \nu)\). Here \(\Sigma^n \nu\) is the \(n\)-fold homological suspension of \(\nu\), an \(n\)-cycle in \((C^{n-*})^t \otimes C^{n-*})^{th\mathbb{Z}/2}\). Because \(\varphi_0\) is invertible up to chain homotopy, the pair consisting of \(\nu\) and \(\gamma\) is sufficiently unique.

Given \(B\) and a chain bundle \(\gamma\) on \(B\), a \((B,\gamma)\)-structure on a symmetric Poincaré chain complex \((C,\varphi)\) of formal dimension \(n\) consists of a chain map \(f:C \to B\) and an \((n+1)\)-chain \(\tau\) in \((C^t \otimes_R C)^{th\mathbb{Z}/2}\) whose boundary is the difference of \(J\varphi\) and \((\varphi_0)_*\Sigma^n(f^*\gamma)\). The chain \(\tau\) gives an identification of \(f^*\gamma\) with the normal chain bundle of \((C,\varphi)\).

Let \(\textbf{L}^*(R;B,\gamma)\) be the algebraic bordism spectrum constructed from the bordism theory of symmetric algebraic Poincaré complexes \((C,\varphi)\) over \(R\) with a \((B,\gamma)\)-structure. In the case where \(B = 0\) this is the quadratic \(L\)-theory of \(R\), and in the general case there is a comparison map
\[
\textbf{L}^*(R) \to \textbf{L}_*(R;B,\gamma)
\]
with homotopy cofiber \(\textbf{L}_*(R;B,\gamma)\). The main result of [28] is a long exact sequence
\[
\cdots \to \tilde{L}^n(R;B,\gamma) \to Q^n(B) \xrightarrow{J_{\gamma}} \tilde{Q}^n(B) \to \tilde{L}^{n-1}(R;B,\gamma) \to \cdots
\]
where \(J_{\gamma}[\varphi] := J[\varphi] - (\varphi_0)_*(\Sigma^n[\gamma])\) with \(\Sigma^n[\gamma] \in \tilde{Q}(B^{n-*})\). There is also a stunted version of this. To get that we make the changes
\[
W \sim W_{\leq 0}, \quad \tilde{W} \sim \tilde{W}_{\leq 0}
\]
in the above (passing to 0-skeletons). More practically we define
\[
Q^n_\gamma(B) := H_n(W^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (B^t \otimes B))
\]
where \(W^-\) is the dual of \(\tilde{W}_{\leq 0}\), or alternatively, the mapping cone of the (chain) map \(W \to \mathbb{Z}[\mathbb{Z}/2]\) which takes \(1 \in W_0\) to \(1 + T\). This gives an obvious inclusion-induced map \(\iota: H_n(B^t \otimes_R B) \to Q^n_\gamma(\Sigma B)\). The stunted version of the above long exact sequence is another long exact sequence
\[
\cdots \to \tilde{L}^n(R;B,\gamma) \to H_n(B^t \otimes B) \xrightarrow{J_{\gamma}} Q^n_\gamma(B) \to \tilde{L}^{n-1}(R;B,\gamma) \to \cdots
\]
The above long exact sequence for the universal inverse limit of the chain bundle groups associated with these (sub)complexes. \( \gamma \) satisfying these finiteness assumptions, and \( B \) can no longer be chosen to be bounded above and below and f.g. in each degree. It must be said that this is a little harder to justify and use, because in most cases \( B \) can always be constructed as a direct limit of chain complexes \( B' \) satisfying these finiteness assumptions, and \( \gamma \) can be constructed as an element in the inverse limit of the chain bundle groups associated with these (sub)complexes. The above long exact sequence for the universal \( B \) and \( \gamma \) is then obtained as a direct limit for the long exact sequences associated with the subcomplexes \( B' \) and chain bundles \( \gamma|B' \).

In the case where \( B \) is not universal, we always have a comparison chain map to the universal specimen. We may think of \( B \) as a classifying object for another cohomology theory which comes with a natural transformation to ordinary chain bundle theory.

Returning to finite spectra, we see that to calculate \( \tilde{L}^\bullet(*) \) and \( \tilde{V}_*^\bullet(*) \), we must replace \( B \) by \( S^{h\mathbb{Z}/2} \simeq S^\mathbb{S} \) and by \( S \), respectively, in the above. Since \( S \) has better finiteness properties than \( \mathbb{S} \), the visible case is easier and we begin with that. By analogy with one of the long exact sequences just described (the “stunted version”), we obtain a homotopy fiber sequence of spectra

\[
\begin{align*}
\text{V} \tilde{L}^\bullet(*) & \longrightarrow S \wedge S \longrightarrow (S \wedge S)_{h\mathbb{Z}/2}^-.
\end{align*}
\]

Here \( (S \wedge S)_{h\mathbb{Z}/2}^- \) is the homotopy cofiber of the transfer from \( (S \wedge S)_{h\mathbb{Z}/2} \) to \( S \wedge S \).

(Aside. For the present purposes, the “right” notion of smash product of two spectra \( E \) and \( F \) would be the spectrum with \( i \)-th term \( \Omega^i(E_i \wedge F_i) \), where the loop coordinates are associated with the antidiagonal of \( \mathbb{R}^i \times \mathbb{R}^i \). This is “commutative” but neither associative nor unital, so it is one of many naive smash products.)

The spectrum \( \text{V} \tilde{L}^\bullet(*) \) is a ring spectrum. There is no need to know what it is to see that the following extremal case is particularly easy, since all we need is the unit map for the ring structure \( S \longrightarrow \text{V} \tilde{L}^\bullet(*) \).

Evaluation on \( \pi_0 \) shows that this unit map is a homotopy right inverse for the map \( \text{V} \tilde{L}^\bullet(*) \longrightarrow S \wedge S \) in the homotopy fiber sequence just above. Therefore \( \iota_\gamma \) has a preferred nullhomotopy and we have

**Theorem 3.3.** \( \text{V} \tilde{L}^\bullet(*) \simeq S \vee \Omega(S_{h\mathbb{Z}/2}^-) = S \vee \mathbb{RP}^\infty \).\n
This is surprising. We have shown that the unit map for \( \text{V} \tilde{L}^\bullet(*) \) is the injection of a wedge summand \( S \), up to homotopy equivalence. In particular, multiplication
by 8 does not annihilate its homotopy class. It follows immediately that the standard homomorphism \( L_0(\ast) \to VL^0(\ast) \) does not send the (signature 8) generator to 8 times the unit of \( VL^0(\ast) \).

**Corollary 3.4.** For a space \( X \) with CW–approximation \( X' \to X \), we have

\[
VL^\bullet(X) \simeq X'_+ \wedge (S \lor \mathbb{RP}^\infty_1).
\]

**Comment.** This is a formal consequence of theorem 3.3 and the excision theorem 2.7. Beware that the definition of of \( VL^\bullet(X) \) which we use here depends on a specific \( SW \) product in \( s\mathbb{R}(X) \). There are “twisted” versions which will be considered later. \( \square \)

Next we calculate \( \hat{L}^\bullet(\ast) \). The only new aspect in this calculation is that our basic “homology theory” is now \( \left( Y, k \right) \to \left( \left( Y, k \right) \wedge (Y, k) \right)^{h\mathbb{Z}/2} \) and the representing object is \( ^\ast S \). From the point of view of chain bundle theory, generalized to the nonlinear setting, this means that our calculation of the hyperquadratic \( L \)–theory of a point is going to be almost identical with that of the visible hyperquadratic \( L \)–theory of a point. The difference is that \( S \) has to be replaced by \( ^\ast S \) where applicable. Noting that \( \left( ^\ast S \wedge ^\ast S \right)^{-h\mathbb{Z}/2} \simeq \left( S \wedge S \right)^{-h\mathbb{Z}/2} \), we obtain a homotopy fiber sequence of spectra

\[
\hat{L}^\bullet(\ast) \to \left( ^\ast S \wedge ^\ast S \right) \to \left( S \wedge S \right)^{-h\mathbb{Z}/2}.
\]

The map \( \gamma \) in this case can immediately be understood by comparison with the visible case. It must be zero because its restriction to \( S \wedge S \) is zero and the homotopy groups of the target are all 2–torsion. Therefore:

**Theorem 3.5.** \( \hat{L}^\bullet(\ast) \simeq \left( ^\ast S \wedge ^\ast S \right) \vee \Omega \left( S^{-h\mathbb{Z}/2} \right) = \left( ^\ast S \wedge ^\ast S \right) \vee \mathbb{RP}^\infty_1. \) \( \square \)

### 4. Excision and restriction in controlled \( L \)–theory

We start with the Waldhausen category \( R^{id}(\hat{Q}, Q) \) of \([9, \text{dfn.7.1}]\). Here \( \hat{Q} \) is locally compact Hausdorff, \( Q \) is open in \( \hat{Q} \) and we add the assumption that \( \hat{Q} \) has a countable base. Often we stabilize with respect to the suspension functor \( \Sigma \) and write the result as \( sR^{id}(\hat{Q}, Q) \). Objects in the stable category can be written as \( \left( Y, k \right) \) for some \( Y \) in \( R^{id}(\hat{Q}, Q) \) and \( k \in \mathbb{Z} \). In the stabilized category, we want to introduce a Spanier–Whitehead (external) product in the sense of \([34, \text{dfn.1.1}]\). (This has been done in \([34, \text{1.A.7}]\), but it will not hurt to present it from a slightly different angle.)

**Definition 4.1.** Let \( \tilde{Q}^\bullet \) be the one-point compactification of \( \tilde{Q} \). Let \( Y \) be a retractive space over \( Q \), with retraction \( r: Y \to Q \). We write \( Y \cup_Q \tilde{Q}^\bullet \) for the union of \( Y \) and \( \tilde{Q}^\bullet \) along \( Q \), equipped with the coarsest topology such that the inclusion \( Y \to Y \cup_Q \tilde{Q}^\bullet \) embeds \( Y \) as an open subset, and the retraction \( r \cup \text{id}: Y \cup_Q \tilde{Q}^\bullet \to \tilde{Q}^\bullet \) is continuous. (This means that a subset \( V \) of \( Y \cup_Q \tilde{Q}^\bullet \) is a neighborhood of some \( z \in \tilde{Q}^\bullet \setminus Q \) in \( Y \cup_Q \tilde{Q}^\bullet \)) if \( V \) contains \( r \cup \text{id}^{-1}(W) \) for some neighborhood \( W \) of \( z \) in \( \tilde{Q}^\bullet \).) Let \( Y/Q \) be the topological quotient of \( Y \cup_Q \tilde{Q}^\bullet \) by the subspace \( \tilde{Q}^\bullet \),

\[
Y/Q = \frac{Y \cup_Q \tilde{Q}^\bullet}{\tilde{Q}^\bullet}.
\]
Remark. In the important special case where \( Y \) has a locally finite controlled CW-structure relative to \( Q \), the “special” quotient \( Y/Q \) can be described directly in terms of the ordinary quotient \( Y/Q \), which is a based CW-space. Namely, \( Y/Q \) is the topological inverse limit of the based CW-spaces \( Y/Y' \) where \( Y' \) runs through the cofinite based CW-subspaces of \( Y \). (Here “cofinite” means that \( Y \setminus Y' \) is a union of finitely many cells.) In general, the homotopy groups of \( Y/Q \) should be regarded as “locally finite” variants of the homotopy groups of \( Y/Q \).

Example. Let \( Q = [0, 1] \) and \( Q = [0, 1] \). Let \( T = \{1 - 2^{-i} \mid i = 0, 1, 2, 3, \ldots \} \) and \( Y = \Omega T \). With the inclusion of the first copy of \( Q \) as the zero section, \( Y \) becomes a retractive space over \( Q \). It has an obvious locally finite controlled CW-structure relative to \( Q \). The ordinary quotient \( Y/Q \) is a wedge of infinitely many circles. Its fundamental group is free on generators \( g_1, g_2, g_3, \ldots \) corresponding to the 1-cells of \( Y/Q \). In particular it is countably infinite. But \( Y/Q \) is homeomorphic to the Hawaiian earring. Its fundamental group is an inverse limit of finitely generated free groups, and it is uncountable. Similarly, for the suspension \( \Sigma Q Y \) (taken in the category of retractive spaces over \( Q \)), we have

\[
\pi_2(\Sigma Q Y/Q) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}, \quad \pi_2(\Sigma Q Y/Q) \cong \prod_{i=1}^{\infty} \mathbb{Z}.
\]

Remark. Because of [9] we are stuck with the notation \((\bar{Q}, Q)\) for control spaces, even though we do not require that \( Q \) be dense in \( \bar{Q} \). We will consequently try to avoid the overline notation for topological closures. (The overline notation is also used in section 10 for something completely unrelated.)

Definition 4.2. Let \( Y \) and \( Z \) be objects of \( \mathcal{R}^{fd}(\bar{Q}, Q) \). To define their \( \Omega \) product \( Y \odot Z \), we introduce first an unstable form \( Y \odot Z \) of it. We define it as the geometric realization of a based simplicial set. An \( n \)-simplex of this simplicial set is a pair \((f, \gamma)\) where

(i) \( f \) is a continuous map from the standard \( n \)-simplex \( \Delta^n \) to \( Y/Q \wedge Z/Q \);

(ii) \( \gamma \) is a continuous assignment \( c \mapsto \gamma_c \) of paths in \( Q \), defined for \( c \in \Delta^n \) with \( f(c) \) not equal to the base point \( \star \).

The paths \( \gamma_c \) are to be parametrized by \([-1, +1]\) and must satisfy \( \gamma_c(-1) = r_Y f_Y(c) \) and \( \gamma_c(+1) = r_Z f_Z(c) \), where \( r_Y, r_Z \) are the retractions and \( f_Y, f_Z \) are the coordinates of \( f(c) \). Finally there is a control condition:

For \( z \in \bar{Q}^* \smallsetminus Q \) and any neighborhood \( V \) of \( z \) in \( \bar{Q}^* \), there exists a smaller neighborhood \( W \) of \( z \) in \( Q^* \) such that, for any \( c \in \Delta^n \) with \( f(c) \neq \star \), the path \( \gamma_c \) either avoids \( W \) or runs entirely in \( V \).

Definition 4.3. For \( Y \) and \( Z \) in \( \mathcal{R}^{fd}(\bar{Q}, Q) \) and integers \( k, \ell \in \mathbb{Z} \), let

\[
(Y, k) \odot (Z, \ell) = \text{colim}_n \Omega^{2n}(\Sigma^{n-k} Y \wedge \Sigma^{n-\ell} Z).
\]

More generally let \((Y, k) \odot (Z, \ell)\) be the \( \Omega \)–spectrum with \( j \)–th space

\[
(Y, k) \odot_j (Z, \ell) = \text{colim}_n \Omega^{2n} \Sigma^j (\Sigma^{j-k} Y \wedge \Sigma^{j-\ell} Z).
\]

Remark. We have \( \mathcal{R}^{fd}(\bar{Q}, Q) = \mathcal{R}^{fd}(Q^*, Q) \) (an equality of Waldhausen categories). The meaning of \( Y \odot Z \) is the same in both categories. But there is a difference
between passage to germs near \( Q \setminus Q \) (which we consider next) and passage to germs near \( Q^\bullet \setminus Q \) (which we are not interested in).

Next we work in the germ category \( R^\mathcal{G}_{\text{id}}(Q, Q) \) of [9, dfn.7.1] and its stable form, \( sR^\mathcal{G}_{\text{id}}(Q, Q) \). Let \( Y \) and \( Z \) be objects of \( R^\mathcal{G}_{\text{id}}(Q, Q) \). Note that \( Y \) and \( Z \) are honest retractive spaces over \( Q \). Again, to define their \( SW \) product \( Y \odot Z \) in the germwise setting (recycled notation), we begin with an unstable form \( Y \wedge Z \) (also recycled notation) which is the geometric realization of a simplicial set. An \( n \)-simplex in this simplicial set is a germ of triples \( (U, f, \gamma) \) where

\[
\begin{align*}
(\text{i}) & \quad U = \hat{U} \cap Q \text{ for an open neighborhood } \hat{U} \text{ of } Q \setminus Q \text{ in } Q; \\
(\text{ii}) & \quad f \text{ is a continuous map from } \Delta^n \text{ to } (Y_\hat{U} \vee U) \wedge (Z_\hat{U} \vee U), \text{ where } Y_\hat{U} = r_{Y^{-1}}(U) \text{ and } Z_\hat{U} = r_{Z^{-1}}(U); \\
(\text{iii}) & \quad \gamma \text{ is a continuous assignment of paths } \ldots \text{ (as before).}
\end{align*}
\]

We impose the same control condition on \( \gamma \) as before. In (ii), we regard \( Y_\hat{U} \) and \( Z_\hat{U} = r_{Z^{-1}}(U) \) as retractive spaces over \( U \), and \( U \) is the nonsingular part of the control space \((\hat{U}, U)\). Note that \( \hat{U} \) is the union of \( U \) and the singular set \( Q \setminus Q \). (It is not defined as the closure of \( U \) in \( Q \).) Passage to germs is achieved by taking the direct limit over all possible \( U \). (It is a direct limit but the indexing is contravariant, i.e., we approach it by making \( U \) smaller and smaller.)

**Definition 4.4.** Put \((Y, k) \odot (Z, \ell) := \colim_n \Omega^{2n}(\Sigma^{n-k}Y \wedge \Sigma^{j-\ell}Z)\). More generally let \((Y, k) \odot_j (Z, \ell) \) be the \( \Omega \)-spectrum with \( j \)-th space

\[
(Y, k) \odot_j (Z, \ell) = \colim_n \Omega^{2n}(\Sigma^{n-k}Y \wedge \Sigma^{j-\ell}Z).
\]

**Remark.** Later we will have to consider twisted versions of the above, depending on a spherical fibration on \( Q \).

It is straightforward to verify that the above definitions of \( \odot \) and \( \odot_j \) in the stable categories \( sR^\mathcal{G}_{\text{id}}(Q, Q) \) and \( sR^\mathcal{G}_{\text{id}}(Q, Q) \) satisfy the conditions of [34, §1] for \( SW \) products. It is less straightforward to verify that they also satisfy the axioms of [34, §2], which are about existence and uniqueness of “duals”. But this has been verified in [34, §2.4]. Hence there are associated quadratic \( L \)-theory spectra [34] which we denote by \( L_\bullet((Q, Q)) \) and \( L_\bullet((Q, Q)_\infty) \), respectively. Also, visible symmetric structures on objects of \( sR^\mathcal{G}_{\text{id}}(Q, Q) \) and \( sR^\mathcal{G}_{\text{id}}(Q, Q) \) can be defined by analogy with definition 2.2. Hence there are visible symmetric \( L \)-theory spectra denoted by \( VL_\bullet((Q, Q)) \) and \( VL_\bullet((Q, Q)_\infty) \), respectively.

We now specialize to the case \((Q, Q) = JX = (X \times [0, 1], X \times [0, 1]) \) where \( X \) is an ENR. In fact we think of \( X \mapsto L^\bullet(JX_\infty) \) and \( X \mapsto L_\bullet(JX_\infty) \) as covariant functors on the category \( \mathcal{E} \) whose objects are the ENR’s and where a morphism from \( X_1 \) to \( X_2 \) is a based map \( X_1^\bullet \rightarrow X_2^\bullet \) of the one-point compactifications. (This is the same thing as a proper map from an open subset of \( X_1 \) to \( X_2 \).)

**Theorem 4.5.** The spectrum valued functor \( X \mapsto E(X) \), where \( E(X) \) means \( L_\bullet(JX_\infty) \), is homotopy invariant and excisive. In detail:

- The projection from \( X \times [0, 1] \) to \( X \) induces a homotopy equivalence of \( E(X \times [0, 1]) \) with \( E(X) \).
- For an open subset \( V \) of \( X \), the collapse map \( j: X^\bullet \rightarrow V^\bullet \) and the inclusion \( i: X \setminus V \rightarrow X \) determine a homotopy fiber sequence of spectra

\[
E(X \setminus V) \xrightarrow{j_*} E(X) \xrightarrow{j^*} E(V).
\]
We replace the categories $RG$ dimensionwise locally finite.

The cells of $Y$ any compact region of $Q$ such that the closure of any neighborhood $V_n$ map must satisfy the usual control condition: given $e ≥ e'$ if the smallest based $CW$-subspace containing $e$ also contains $e'$. We say that $Y$ is dimensionwise locally finite [30, Dfn.6.1] if, for every cell $e$ in $Y$ (not allowing *) and every $j ≥ 0$ there are only finitely many $j$-cells in $Y$ which are $≥ e$. For example, a wedge of infinitely many based compact $CW$-spaces is dimensionwise locally finite.

We will work with based $CW$-spaces, which we generally view as $CW$-spaces relative to $*$. On the set of cells (not including $*$) of such a $Y$, there is a partial ordering: $e ≥ e'$ if the smallest based $CW$-subspace containing $e$ also contains $e'$. We say that $Y$ is dimensionwise locally finite [30, Dfn.6.1] if, for every cell $e$ in $Y$ (not allowing $*$) and every $j ≥ 0$ there are only finitely many $j$-cells in $Y$ which are $≥ e$. For example, a wedge of infinitely many based compact $CW$-spaces is dimensionwise locally finite.

We replace the categories $RG^{id}(Q, Q)$ by more tractable ones, denoted $R(*) : Q, Q)_0$ in [30, §6]. An object of $R(*) : Q, Q)_0$ is a dimensionwise locally finite based $CW$-space $Y$ where the set of cells (excluding $*$) is equipped with a map to $Q$. This map must satisfy the usual control condition: given $n ≥ 0$ and $z ∈ Q \setminus Q$ and a neighborhood $V$ of $z$ in $Q$, there exists a smaller neighborhood $W$ of $z$ in $Q$ such that the closure of any $n$-cell with label in $W$ is contained in a compact based $CW$-subspace for which the cell labels are all in $V$. In addition, for any $n ≥ 0$ and any compact region of $Q$, the set of $n$-cells of $Y$ with labels in that compact region is required to be finite. (There is also a finite domination condition to which we return in a moment.) A morphism $Y → Z$ is a sequence $(f_n)$ of compatible cellular map germs between the skeletons, $f_n : Y^n_U → Z^n$, where $f_n$ need only be defined on the cells of $Y^n$ with labels in some open $U ⊂ Q$, where $U = U \cap Q$ for some open neighborhood $U$ of the singular set. The maps $f_n$ are subject to a straightforward control condition formulated in terms the of cell labels. There is a good notion of “controlled homotopy” in the category $R(*) : Q, Q)_0$, so that the weak equivalences in $R(*) : Q, Q)_0$ can simply be defined as the morphisms which are invertible up to controlled homotopy. The cofibrations are, by definition, those morphisms $Y → Z$ whose underlying $CW$ map germ is a composition of $CW$ isomorphisms and $CW$ subspace inclusions. It remains to make the finite domination condition on objects $Y$ explicit. This is automatically satisfied if $Y = Y^n$ for some $n$. In general it means that for some $n$ and all $m ≥ n$, the inclusion $Y^n → Y^m$ admits a (controlled) homotopy right inverse, so that $Y^m$ is a homotopy retract of $Y^n$ (in the “germ” sense). See [30, §6] for more details.

We come to the definition of $Y \wedge Z$ (again recycled notation) for objects $Y$ and $Z$ of
R(∗; ¯Q, Q)∞. Again this is defined as the geometric realization of a simplicial set. An n-simplicial in this simplicial set corresponds to a germ of certain pairs (U, f). Here

- U = ¯U ∩ Q for an open neighborhood ¯U of the singular set in ¯Q;
- f is a continuous map from ∆n to (Y_U/∥U) ∩ (Z_U/∥U).

Here Y_U and Z_U are the largest based CW-subspaces of Y and Z, respectively, containing only cells with labels in U. We impose the usual control condition:

For z ∈ ¯Q ∩ Q and any neighborhood V of z in ¯Q, there exists a smaller neighborhood W of z in Q such that, for any c ∈ ∆n with f(c) ̸= ∗, either both fy(c) and fz(c) are in cells with labels in V, or both are in cells with labels outside W.

(Note the absence of “paths”.) We pass to germs by taking the direct limit over all possible R. Let

\[(Y, k) ∩ (Z, ℓ) := \text{colim}_n \Omega^2n(\Sigma^{n-k}Y \wedge \Sigma^{n-ℓ}Z),\]

\[(Y, k) ⊠_j (Z, ℓ) := \text{colim}_n \Omega^2n\Sigma^j(\Sigma^{n-k}Y \wedge \Sigma^{n-ℓ}Z).\]

Then ⊠ and ⊠_j satisfy the axioms for an SW-product listed in [34].

Now we specialize to the situation(s) where (Q, Q) = ∥X for some X in E∗. We abbreviate E′(X) = L(∗; Q, Q)∞. Again we want to view the assignment

\[X \mapsto E′(X)\]

as a covariant functor on E∗. Indeed, every morphism X_1 → X_2 in E∗, alias based map f: X_1^* → X_2^*, has a factorization

\[X_1^* → V^* → X_2^*\]

where V = X_1 \setminus f^{-1}(∞). In this factorization, the second morphism V^* → X_2^* is induced by a proper map V → X_2 and this determines in a straightforward way a map E′(V) → E′(X_2). The first morphism X_1^* → V^* induces an exact functor from R(∗; ∥X)∞ to R(∗; ∥V)∞, hence a map E′(X_1) → E′(V), roughly as follows. For an object Y of R(∗; ∥X)∞, the largest based CW-subspace of Y having all its cell labels in V × [0, 1] is an object of R(∗; ∥V)∞. Later we will show, following [30, §9], that E′(X) is related to E(X) in theorem 4.5 by a chain of natural weak equivalences. But first we will prove the analogue of theorem 4.5 for the functor E′.

The main ingredients in this proof are certain approximation statements, related to Waldhausen’s approximation theorem [25]. To state these we fix X (an ENR) and an open V ⊂ X. On the category R(∗; ∥X)_∞ we have, in addition to the standard notion of weak equivalence, a coarser one denoted by ω. Namely, a morphism is regarded as a weak ω-equivalence if the induced morphism in \(R(∗; ∥V)_∞\) is a weak equivalence. We write \(R_ω(∗; ∥X)_∞\) for \(R(∗; ∥X)_∞\) equipped with the coarse notion of weak equivalence. We write \(R^ω(∗; ∥X)_∞\) for the full subcategory of \(R(∗; ∥X)_∞\) consisting of the objects which are weakly ω-equivalent to the zero object, and this is equipped with the standard notion of weak equivalence inherited from \(R(∗; ∥X)_∞\).

**Lemma 4.6.** The functors of stable categories determined by the inclusion functor from \(R(∗; ∥X)_∞\) to \(R_ω(∗; ∥X)_∞\) and the restriction from \(Rω(∗; ∥X)_∞\) to \(R(∗; ∥V)_∞\) satisfy the hypotheses of Waldhausen’s approximation theorem.
Proof. In the first case, the hypotheses are verified in [30, §3, §7], and this works even without stabilization. In the second case, a closely related statement is also proved in [30, §3, §7], with more general assumptions. Specialized to our situation this says that the induced functor

\[ R_\omega(\ast ; \mathbb{I}X)_\infty \rightarrow R(\ast ; \mathbb{I}V)_\infty \]

between the full subcategories of finite dimensional objects satisfies the hypotheses of the approximation theorem. Given that all weak equivalences in the categories \( R(\ast ; \mathbb{I}X)_\infty \) and \( R(\ast ; \mathbb{I}V)_\infty \) are invertible up to homotopy, it is easy to extend this result from the full subcategories of finite dimensional objects to the ambient categories, at the price of stabilizing, by means of the next lemma. □

**Lemma 4.7.** Every object of \( R(\ast ; \mathbb{I}X)_\infty \) becomes weakly equivalent to a finite dimensional object after at most two suspensions.

**Proof.** The excision and homotopy invariance theorem for the algebraic K–theory functor \( X \mapsto K(R(\ast ; \mathbb{I}X)_\infty) \) is proved in [30, 7.1, 7.2]. The coefficient spectrum is analyzed in [30, 8.2, 8.3] and it is found to have a vanishing \( \pi_0 \). In particular, the \( K_0 \) group of \( R(\ast ; \mathbb{I}X)_\infty \) is zero. Therefore, by standard finiteness obstruction theory, all objects of \( R(\ast ; \mathbb{I}X)_\infty \) are weakly equivalent to finite dimensional ones after two suspensions. (This is more fully explained in the proof of [30, 9.5], especially in the statement labelled (**)). The point is that the general case can be reduced to the situation where an object is a homotopy retract of another object whose cells are all concentrated in one dimension. If that dimension is at least two, the homotopy retraction can be linearized without any loss of information.) □

For an object \( Y \) of \( R(\ast ; \mathbb{I}X)_\infty \) let \( \mu(Y) \) be the monoid of endomorphisms of \( Y \) which are mapped to the identity by the restriction functor from \( R(\ast ; \mathbb{I}X)_\infty \) to \( R(\ast ; \mathbb{I}V)_\infty \). We note that, for objects \( Y \) and \( Z \) of \( R(\ast ; \mathbb{I}X)_\infty \), the product monoid \( \mu(Y) \times \mu(Z) \) acts on the SW–product \( Y \odot Z \). We like to think of \( \mu(Y) \times \mu(Z) \) as a category with one object. The action is a functor on that category. Hence there is a canonical map

\[ \text{hocolim}_{\mu(Y) \times \mu(Z)} Y \odot Z \rightarrow j_\ast(Y) \odot j_\ast(Z) \]

where \( j_\ast : R(\ast ; \mathbb{I}X)_\infty \rightarrow R(\ast ; \mathbb{I}V)_\infty \) is the restriction functor.

The next approximation lemma about SW–products and its corollary (about quadratic structures) are adaptations of [29, 2.7, 14.1].

**Lemma 4.8.** For a finite dimensional object \( Y \) of \( R(\ast ; \mathbb{I}X)_\infty \), the monoid \( \mu(Y) \) is directed in the following sense: given \( f_1, f_2 \in \mu(Y) \), there is \( f_3 \in \mu(Y) \) such that \( f_3 f_1 = f_3 = f_3 f_2 \). For two finite dimensional objects \( Y \) and \( Z \) in \( R(\ast ; \mathbb{I}X)_\infty \), the canonical map of unstable SW products

\[ \text{hocolim}_{\mu(Y) \times \mu(Z)} Y \wr Z \rightarrow j_\ast(Y) \wr j_\ast(Z) \]

is a homotopy equivalence.

**Proof.** The statement about directedness is a consequence of the following observations. For every \( f \in \mu(Y) \), there exist a neighborhood \( U(f) \) of \( X \times \{1\} \) in \( X \times [0, 1] \) and a neighborhood \( W(f) \) of \( V \times \{1\} \) in \( V \times [0, 1] \), with \( U(f) \supset W(f) \), such that
Conversely, given any neighborhood \( W \subseteq \mathcal{Y} \) such that some representative of \( F \) is zero on cells with labels outside \( W \). Assume \( Y = Y^n \). Choose a representative of an endomorphism of \( Y^n/Y^{n-1} \) which is zero on cells with labels outside \( W \), and which belongs to \( \mu(Y^n/Y^{n-1}) \). There is a smaller neighborhood \( W' \subseteq V \times \{1\} \subseteq V \times [0,1] \) such that this representative is the identity on all cells with labels in \( W' \). Next, choose a representative of an endomorphism of \( Y^{n-1} \) which belongs to \( \mu(Y^{n-1}) \) and is zero on cells with labels outside \( W' \). The two representatives then combine to give an endomorphism of \( Y^n = Y \) with the required property.) Combining these two observations, we can choose \( f_3 \in \mu(Y) \) in such a way that it vanishes on all cells with labels outside \( W(f_1) \cap W(f_2) \), and then clearly \( f_3 f_1 = f_3 = f_3 f_2 \).

Now for the statement about \( SW \)-products: it is already clear from the foregoing that we have an identification of (geometric realizations of) simplicial sets

\[
\colim_{\mu(Y) \times \mu(Z)} Y \wedge Z \cong j_*(Y) \wedge j_*(Z)
\]

where \( \wedge \) denotes the unstable form of the \( SW \)-product. As the colimit is a colimit of based \( CW \)-spaces and based cellular maps over a directed category, we may replace it by a homotopy colimit.

**Corollary 4.9.** For a finite dimensional object \( Y \) of \( \mathcal{R}(\ast ; \mathbb{J}X)_{\infty} \) and \( k \in \mathbb{Z} \), there is a canonical homotopy equivalence of spectra

\[
\colim_{\mu(Y)_{\ast}} ((Y, k) \odot_{\ast} (Y, k))_{h\mathbb{Z}/2} \longrightarrow ((j_*(Y, k) \odot_{\ast} (j_*(Y, k)))_{h\mathbb{Z}/2}.
\]

**Proof of Theorem 4.5, excision part, with \( E' \) instead of \( E \).** Writing \( i_* \) for the inclusion functor

\[
\mathcal{R}(\ast ; \mathbb{J}(X \setminus V))_{\infty} \longrightarrow \mathcal{R}(\ast ; \mathbb{J}X)_{\infty}
\]

we have natural homotopy equivalences \( i_* \). Consequently the homotopy classification of (nondegenerate) quadratic structures is the same for an object of \( s\mathcal{R}(\ast ; \mathbb{J}(X \setminus V))_{\infty} \) and its image in \( s\mathcal{R}(\ast ; \mathbb{J}X)_{\infty} \). Therefore and by the first part of Lemma 4.6, the map

\[
i_* : L_\ast(\mathcal{R}(\ast ; \mathbb{J}(X \setminus V))_{\infty}) \longrightarrow L_\ast(\mathcal{R}(\ast ; \mathbb{J}X)_{\infty})
\]

is a homotopy equivalence. For the rest of the argument, we use an \( L \)-theoretic precursor, due to Ranicki, of Waldhausen’s fibration theorem in algebraic \( K \)-theory [25]. Applied to our situation this gives a homotopy fiber sequence of spectra

\[
L_\ast(\mathcal{R}(\ast ; \mathbb{J}X)_{\infty}) \longrightarrow L_\ast(\mathcal{R}(\ast ; \mathbb{J}X)_{\infty}) \longrightarrow L_\ast(\mathcal{R}(\ast ; \mathbb{J}X)_{\infty}, \mathcal{R}(\ast ; \mathbb{J}X)_{\infty})
\]

where \( L_\ast(\mathcal{R}(\ast ; \mathbb{J}X)_{\infty}, \mathcal{R}(\ast ; \mathbb{J}X)_{\infty}) \) denotes the bordism theory of objects in the (stable category of) \( \mathcal{R}(\ast ; \mathbb{J}X)_{\infty} \) equipped with a quadratic structure which is non-degenerate modulo \( \mathcal{R}(\ast ; \mathbb{J}X)_{\infty} \). See e.g. [19, §3] and [23]; see also remark 14.2 below. Therefore it only remains to show that the map

\[
L_\ast(\mathcal{R}(\ast ; \mathbb{J}X)_{\infty}, \mathcal{R}(\ast ; \mathbb{J}X)_{\infty}) \longrightarrow L_\ast(\mathcal{R}(\ast ; \mathbb{J}V)_{\infty})
\]
induced by $j$ is a homotopy equivalence. We verify that the induced maps of homotopy groups $L_n(\ldots)$ are isomorphisms for all $n \in \mathbb{Z}$. For the surjectivity part, fix an object $(Y', k)$ in $s\mathcal{R}(\ast ; J\mathbb{X})_\infty$ and an $n$-dimensional nondegenerate quadratic structure $\psi'$ on it. By lemma 4.6, we may assume that $Y' = j_* Y$ for some $Y$ in $\mathcal{R}(\ast ; \mathbb{X})_\infty$. By lemma 4.8, there is an $n$-dimensional quadratic structure $\psi$ on $(Y, k)$ such that $j_* \psi$ is homotopic to $\psi'$. Then $\psi$ is automatically nondegenerate modulo $s\mathcal{R}^\omega(\ast ; \mathbb{X})_\infty$, so that $((Y, k), \psi)$ represents a class in $L_n(\mathcal{R}(\ast ; \mathbb{X})_\infty, \mathcal{R}^\omega(\ast ; \mathbb{X})_\infty)$ which maps to the class of $((Y', k), \psi')$ in $L_n(\mathcal{R}(\ast ; \mathbb{X})_\infty)$. For the injectivity part, fix an object $(Z, k)$ in $s\mathcal{R}(\ast ; \mathbb{X})_\infty$ with an appropriate endomorphism of $T$, and assume that $((Z', k), \psi') := ((j_* Z, k), j_* \varphi)$ is nullbordant. Then there exist a cofibration $u' : (Z', k) \to (T', \ell)$ in $s\mathcal{R}(\ast ; \mathbb{X})_\infty$ and a nullhomotopy $\tau'$ of $u'_* \varphi'$ such that $((Z', k) \to (T', \ell), (\tau', \partial \tau'))$ with $\partial \tau' = \varphi'$ is a nondegenerate quadratic pair in $s\mathcal{R}(\ast ; \mathbb{X})_\infty$. By lemma 4.6, we may assume that $u'$ is obtained from a cofibration $u : (Z, k) \to (T, \ell)$ in $s\mathcal{R}^\omega(\ast ; \mathbb{X})_\infty$ by applying $j_*$. By lemma 4.8, on composing $u$ with an appropriate endomorphism of $T$ (and restoring the cofibration property by means of a mapping cylinder construction), we may also assume that $\tau'$ is obtained from a nullhomotopy $\tau$ for $u_\ast \varphi$ by applying $j_*$. Then $((Z, k) \to (T, \ell), (\tau, \partial \tau))$ is a quadratic pair in $s\mathcal{R}(\ast ; \mathbb{X})_\infty$ which is nondegenerate modulo $s\mathcal{R}^\omega(\ast ; \mathbb{X})_\infty$. Hence $((Z, k), \varphi)$ represents the zero class.

**Proof of theorem 4.5, homotopy invariance part, with $E'$ instead of $E$.** It is enough to show that the inclusion $i : X \times \{0\} \to X \times [0, 1]$ induces a homotopy equivalence $i_* : E'(X) \to E'(X \times [0, 1])$. By the excision property which we just established, it is also enough to show that $E'(X \times [0, 1])$ is contractible. This uses an Eilenberg swindle. The details are as in [30, §4], except for a correction to [30, §4] in remark 14.3 below.

**Proof of theorem 4.5, disjoint union axiom, with $E'$ instead of $E$.** We leave this to the reader as a matter of inspection.

**Proof of theorem 4.5, coefficient spectrum part, with $E'$ instead of $E$.** This is similar to the excision part. We take $X = \ast$. We introduce a Waldhausen category $\mathcal{R}(\ast ; J\ast)$, defined like $\mathcal{R}(\ast ; J\ast)_\infty$ but without the germ relation. Thus an object of $\mathcal{R}(\ast ; J\ast)$ is a based CW-space with a map from the set of cells (excluding the base point) to $[0, 1]$. A morphism $Y \to Z$ in $\mathcal{R}(\ast ; J\ast)$ is a based cellular map (not a germ of such maps) from $Y$ to $Z$, subject to the usual control condition. There is a finite domination condition on objects $Y$, which says that for some $n$, each $Y^n$ with $m \geq n$ is a homotopy retract of $Y^n$ in the appropriate controlled homotopy category. The weak equivalences are defined as the morphisms which are invertible in the controlled homotopy category.

In addition to the standard notion of weak equivalence in $\mathcal{R}(\ast ; J\ast)$, we have a coarse notion $\omega$ of weak equivalence. Namely, a morphism in $\mathcal{R}(\ast ; J\ast)$ is a weak $\omega$-equivalence if the induced morphism in $\mathcal{R}(\ast ; J\ast)_\infty$ is a weak equivalence. As in the excision part, we obtain from general principles a homotopy fiber sequence of spectra

$$L_* (\mathcal{R}(\ast ; J\ast)) \to L_* (\mathcal{R}(\ast ; J\ast)) \to L_* (\mathcal{R}(\ast ; J\ast), \mathcal{R}(\ast ; J\ast)).$$
The Waldhausen category $\mathcal{R}^\omega(\star;\mathbb{I}\mathbb{I})$ has an exact subcategory consisting of those objects $Y$ which have only finitely many cells. This is equivalent to the category of based finite $CW$–spaces, so that its $L$–theory spectrum is $L_\omega(\star)$. The inclusion of this exact subcategory in $\mathcal{R}^\omega(\star;\mathbb{I}\mathbb{I})$ satisfies the conditions of the approximation theorem; for the proof, see [30, 8.3]. The homotopy classification of (nondegenerate) quadratic structures on an object in the (stabilized) subcategory is the same whether we classify in the subcategory or in the ambient category. Consequently we have

$$L_\omega(\mathcal{R}^\omega(\star;\mathbb{I}\mathbb{I})) \simeq L_\omega(\star).$$

It remains to show that the “passage to germs” functor from $\mathcal{R}(\star;\mathbb{I}\mathbb{I})$ to $\mathcal{R}(\star;\mathbb{I}\mathbb{I})_\infty$ induces a homotopy equivalence of spectra

$$L_\omega(\mathcal{R}(\star;\mathbb{I}\mathbb{I}), \mathcal{R}^\omega(\star;\mathbb{I}\mathbb{I})) \to L_\omega(\mathcal{R}(\star;\mathbb{I}\mathbb{I})_\infty),$$

and this can be done by considering the homotopy groups. We need to know that the functor of stable categories determined by $\mathcal{R}^\omega(\star;\mathbb{I}\mathbb{I}) \to \mathcal{R}(\star;\mathbb{I}\mathbb{I})_\infty$ satisfies the conditions of the approximation theorem; for this, see again [30, 8.3] and make use of lemma 4.7 above. The other ingredient is an approximation lemma for quadratic structures analogous to lemma 4.8, but applicable to the “passage to germs” functor from $\mathcal{R}(\star;\mathbb{I}\mathbb{I})$ to $\mathcal{R}(\star;\mathbb{I}\mathbb{I})_\infty$. We leave the remaining details to the reader. □

Proof of theorem 4.5: comparing $E'$ and $E$. Recall that $E(X) = L_\omega(\mathcal{R}^\text{id}(\bar{Q}, Q))$ and $E'(X) = L_\omega(\mathcal{R}(\star;Q, Q)_\infty)$ where $(\bar{Q}, Q) = JX$. The Waldhausen categories $\mathcal{R}^\text{id}(\bar{Q}, Q)$ and $\mathcal{R}(\star;Q, Q)_\infty$ are related, for a general control space $(\bar{Q}, Q)$, by exact functors

$$\mathcal{R}^\text{id}(\bar{Q}, Q) \xrightarrow{\text{inclusion}} \mathcal{R}^\text{id}(Q, Q) \xrightarrow{v} \mathcal{R}(\star;Q, Q)_\infty.$$

Here $\mathcal{R}^\text{id}(\bar{Q}, Q)$ is defined very much like $\mathcal{R}^\text{id}(Q, Q)$, but the objects $Y$ come equipped with a finite dimensional controlled $CW$–structure relative to $Q$ and morphisms are required to be cellular relative to $Q$. See the proof of [30, 9.5] for details. (Except for a homotopy finiteness condition, which is unimportant in our setting thanks to lemma 4.7, the category $\mathcal{R}^\text{id}(\bar{Q}, Q)$ is identical with something denoted $\mathcal{R}(\star;Q, Q)_\infty$ in that proof, and $\mathcal{R}^\text{id}(\bar{Q}, Q)$ is denoted $\mathcal{B}$ there.) It is also proved in [30, §9] that the two exact functors in the chain, viewed as functors of the associated stable categories, satisfy the conditions of the approximation theorem (again modulo lemma 4.7). Finally $v$ respects the $SW$–products, in the strong sense that we have a binatural homotopy equivalence $Y \odot Z \to v(Y) \odot v(Z)$, for $Y$ and $Z$ in $\mathcal{R}^\text{id}(\bar{Q}, Q)$. It follows that $u$ and $v$ induce homotopy equivalences of the associated quadratic $L$–theory spectra. □

Our next goal is to formulate an excision theorem similar to theorem 4.5 for the visible symmetric $L$–theory. (We do not have, and we do not need, an analogue of the excision theorem for a controlled version of ordinary symmetric $L$–theory.) This is straightforward modulo chapter 2.

**Definition 4.10.** An $n$–dimensional visible symmetric structure on an object $(Y, k)$ in $\mathcal{R}^\text{id}(\bar{Q}, Q)$ is an element of $\Omega^n(\langle (Y, k) \odot (Y, k) \rangle)^{\mathbb{I}\mathbb{I}/2}$, with $\langle (Y, k) \odot (Y, k) \rangle$ as in definition 4.4. An $n$–dimensional visible symmetric structure on an object $(Y, k)$ in $\mathcal{R}(\star;Q, Q)_\infty$ is an element of $\Omega^n(\langle (Y, k) \odot (Y, k) \rangle)^{\mathbb{I}\mathbb{I}/2}$, with the appropriate definition of $\mathbb{I}\mathbb{I}$ for the category $\mathcal{R}(\star;Q, Q)_\infty$. 

\[WWII\]
Again \((Y, k) \circ \bullet (Y, k)\) turns out to be the underlying spectrum of a \(\mathbb{Z}/2\)-spectrum which we can describe as a shifted suspension spectrum
\[
\Sigma_{\mathbb{Z}/2}^{-k} Y^{\wedge 2}.
\]
(The meaning of \(Y^{\wedge 2} = Y \wedge Y\) depends on the category, which may be \(\mathcal{R}^{\text{id}}(\bar{Q}, Q)\) or \(\mathcal{R}(\bullet ; \bar{Q}, Q)_\infty\).) The analogues of corollary 2.5 hold in both categories, and they are still corollaries of proposition 2.4.

Let \(Y\) be an object of \(\mathcal{R}^{\text{id}}(\bar{Q}, Q)\) or \(\mathcal{R}(\bullet ; \bar{Q}, Q)_\infty\). Let \( \bar{U} \) be an open neighborhood of the singular set in \( \bar{Q} \) and put \( U = \bar{U} \cap Q \). Recall that \( Y_U \) means \( r^{-1}(U) \) for \( Y \) in \( \mathcal{R}^{\text{id}}(\bar{Q}, Q) \); for \( Y \) in \( \mathcal{R}(\bullet ; \bar{Q}, Q)_\infty \) it means the largest based \( CW\)-subspace of \( Y \) having all its cell labels in \( U \). In the second case we also introduce the notation \( Y_U/\bullet/\bullet \) for the topological inverse limit of the based spaces \( Y_U/Y'_U \) where \( Y'_U \) runs though all cofinite based \( CW\)-subspaces of \( Y_U \).

**Corollary 4.11.** In the setting of definition 4.10, there is a natural homotopy fiber sequence of spectra
\[
((Y, k) \circ \bullet (Y, k))_{h\mathbb{Z}/2} \longrightarrow ((Y, k) \circ \bullet (Y, k))_{\mathbb{Z}/2} \overset{J}{\longrightarrow} \text{hocolim}_U \Sigma^\infty_{-k}(Y_U/\bullet/U).
\]

**Corollary 4.12.** For any object \( Y \) of \( \mathcal{R}(\bullet ; \bar{Q}, Q)_\infty \) and \( k \in \mathbb{Z} \), there is a homotopy fiber sequence of spectra
\[
((Y, k) \circ \bullet (Y, k))_{h\mathbb{Z}/2} \longrightarrow ((Y, k) \circ \bullet (Y, k))_{\mathbb{Z}/2} \overset{J}{\longrightarrow} \text{hocolim}_U \Sigma^\infty_{-k}(Y_U/\bullet/\bullet).
\]

In these two corollaries, \( U \) runs through the open subsets of \( Q \) of the form \( U = \bar{U} \cap Q \) where \( \bar{U} \) is an open neighborhood of the singular set in \( \bar{Q} \). The homotopy colimits are reduced (taken in the based category) and the suspension spectrum construction \( \Sigma^\infty_{-k} \) is meant to include a \( CW\)-approximation mechanism.

There is also an analogue of corollary 4.9. We keep the assumptions and notation of that corollary to state the analogue:

**Corollary 4.13.** For a finite dimensional object \( Y \) of \( \mathcal{R}(\bullet ; \bar{X})_\infty \) and \( k \in \mathbb{Z} \), there is a canonical homotopy equivalence of spectra
\[
\text{hocolim}_{\mu(Y)} ( (Y, k) \circ \bullet (Y, k) )_{\mathbb{Z}/2} \longrightarrow ( (j_*Y, k) \circ \bullet (j_*Y, k) )_{\mathbb{Z}/2}.
\]

*Proof.* By corollaries 4.9 and 4.12, and a five lemma argument, it is enough to verify that the canonical map
\[
\text{hocolim}_{\mu(Y)} \text{hocolim}_U \Sigma^\infty_{-k}(Y_U/\bullet/U) \longrightarrow \text{hocolim}_W \Sigma^\infty_{-k}(j_*Y_W/\bullet/\bullet)
\]
is a homotopy equivalence. But this is obvious. \( \square \)

With these tools available, the visible symmetric \( L\)-theory version of theorem 4.5, which we are about to state, can be proved in strict analogy with the original (quadratic \( L\)-theory) version.

**Theorem 4.14.** The spectrum valued functor \( X \mapsto E(X) \), where \( E(X) \) means \( V\mathcal{L}^\bullet(\bar{X}_\infty) \), is homotopy invariant and excisive. The coefficient spectrum \( E(\bullet) \) is homotopy equivalent to \( \Sigma V\mathcal{L}^\bullet(\bullet) \). \( \square \)
5. Control and visible $L$-theory

In this section our goal is to generalize theorem 2.7 to a controlled setting, as far as possible.

Fix a compact Hausdorff space $S$. For most of this section the only control spaces we shall be interested in are of the form $(\bar{X}, X)$, with compact $\bar{X}$ and an identification $\bar{X} \setminus X \cong S$. The only morphisms $f : (\bar{X}_1, X_1) \to (\bar{X}_2, X_2)$ between such control spaces that we shall be interested in are those which are relative to $S$. These objects and morphisms form a category $\mathcal{K}^S$.

We can also speak of homotopies between morphisms in $\mathcal{K}^S$. These will also be relative to $S$, and they allow us to define a homotopy category $\mathcal{H}\mathcal{K}^S$. A morphism in $\mathcal{K}^S$ is a cofibration if it is an embedding which has the homotopy extension property, for such homotopies.

**Theorem 5.1.** On $\mathcal{K}^S$, the functor $(\bar{X}, X) \mapsto \mathcal{V}\hat{L}^\bullet((\bar{X}, X))$ is homotopy invariant, excisive and satisfies a strong "wedge" axiom.

This needs a few explanations. The homotopy invariance property means that the functor takes homotopy equivalences in $\mathcal{K}^S$ to homotopy equivalences of spectra. Here homotopy equivalences in $\mathcal{K}^S$ refers to morphisms in $\mathcal{K}^S$ which become invertible in $\mathcal{H}\mathcal{K}^S$.

For the excision property, suppose given a pushout diagram

\[
\begin{array}{ccc}
(\bar{X}_{ab}, X_{ab}) & \longrightarrow & (\bar{X}_a, X_a) \\
\downarrow & & \downarrow \\
(\bar{X}_b, X_b) & \longrightarrow & (\bar{X}, X)
\end{array}
\]

in $\mathcal{K}^S$ where all the arrows are cofibrations (and without loss of generality, all are inclusions). It is being claimed that in such a case

\[
\begin{array}{ccc}
\mathcal{V}\hat{L}^\bullet((\bar{X}_{ab}, X_{ab})) & \longrightarrow & \mathcal{V}\hat{L}^\bullet((\bar{X}_a, X_a)) \\
\downarrow & & \downarrow \\
\mathcal{V}\hat{L}^\bullet((\bar{X}_b, X_b)) & \longrightarrow & \mathcal{V}\hat{L}^\bullet((\bar{X}, X))
\end{array}
\]

is homotopy cocartesian, and also that $\mathcal{V}\hat{L}^\bullet$ applied to the initial object $(S, \emptyset)$ of $\mathcal{K}^S$ is a weakly contractible spectrum.

For the strong wedge axiom, suppose that $(\bar{X}, X)$ is in $\mathcal{K}^S$ and $X$ is a topological disjoint union of subspaces $X_\alpha$, where $\alpha$ runs through some (countable) set. Let $\bar{X}_\alpha$ be the union of $X_\alpha$ and the singular set $\bar{X} \setminus X$. Then the embedding

\[(\bar{X}_\alpha, X_\alpha) \to (\bar{X}, X)\]

is a morphism in $\mathcal{K}^S$, for every $\alpha$. It follows from the ordinary excision property (just above) that the induced homomorphisms

\[\mathcal{V}\mathcal{L}^\alpha((\bar{X}_\alpha, X_\alpha)) \longrightarrow \mathcal{V}\mathcal{L}^\alpha((\bar{X}, X))\]

are split injective with a preferred splitting, since $(\bar{X}, X)$ is the coproduct in $\mathcal{K}^S$ of $(\bar{X}_\alpha, X_\alpha)$ and $(\bar{X}, X \setminus X_\alpha)$. The decomposition of $(\bar{X}, X)$ into the $(\bar{X}_\alpha, X_\alpha)$ could be regarded as a generalized wedge decomposition (because it is when $S$ is a point).
It is not in general a coproduct decomposition. Nevertheless, it is being claimed that the projections $V\hat{L}^n((\bar{X},X)) \to V\hat{L}^n((\bar{X}_\alpha, X_\alpha))$ induce an isomorphism

$$V\hat{L}^n((\bar{X}, X)) \to \prod_\alpha V\hat{L}^n((\bar{X}_\alpha, X_\alpha)).$$

We turn to the proofs. The homotopy invariance property in theorem 5.1 can be proved by the same argument as the homotopy invariance property in theorem 2.7. The excision property for the special case of a coproduct (that is, $(\bar{X}, X)$ in $\mathcal{K}^S$ with $X = X_1 \amalg X_2$) and the strong “wedge axiom” are valid by inspection. This leaves the general excision property. It is correct to say that the proof of the excision property in theorem 2.7 carries over, but some clarifications are nevertheless in order. The difficulty is that in lemma 2.8, lemma 2.9 etc., which were part of the proof of theorem 2.7, we made essential use of the concept of fibration. Here we will need a corresponding concept of controlled fibration and this is not completely obvious.

Fix a control space $(\bar{X}, X)$ with compact $\bar{X}$ for simplicity, but not necessarily in $\mathcal{K}^S$. Let $p: E \to X$ be any map. Suppose that a collection of open subspaces $E_\lambda \subseteq E^\lambda$ has been specified, where $\lambda$ runs through a directed set; suppose also that the indexing is monotone, so that $\lambda_1 < \lambda_2$ implies $E_{\lambda_1} \subset E_{\lambda_2}$. We assume that $E$ is the union of the $E_\lambda$.  

**Definition 5.2.** The map $p: E \to X$ together with the directed system of open subspaces $\{E_\lambda\}$ such that $E = \bigcup E_\lambda$ is a controlled Serre fibration system if the following holds. For every controlled finite dimensional locally finite CW-space $Y$ over $X$, every controlled map $f: Y \to E_\lambda$ and controlled homotopy $h: Y \times [0,1] \to X$ starting with $p \circ f$, there exists $\kappa > \lambda$ and a homotopy $Y \times [0,1] \to E_\kappa$ which lifts $h$ and starts with $f$.

**Remark.** Controlled fibration systems can be pulled back along maps of control spaces. More precisely, if $f: (\bar{X}_1, X_1) \to (\bar{X}_2, X_2)$ is a map of control spaces (compact $X_1$ and $\bar{X}_2$), and $(p: E \to X_1, \{E_\lambda\})$ is a controlled Serre fibration system over $X_2$, then the pullback $f^* E$ with the subspaces $f^* E_\lambda$ is a controlled Serre fibration system over $X_1$.

**Lemma 5.3.** Let $Z$ be a retractive space over $X$, with retraction $r: Z \to X$. Then there exists a controlled Serre fibration system $(p: E \to X, \{E_\lambda\})$ and an embedding $Z \to \bigcap E_\lambda$ over $X$, inducing controlled homotopy equivalences $Z \to E_\lambda$ for all $\lambda$.

**Proof.** This is supposed to be a controlled version of the Serre construction in ordinary fibration theory. Let $W$ be an open neighborhood of the diagonal in $X \times X$, invariant under permutation of the two factors $X$. We say that $W$ is controlled if its closure in $\bar{X} \times \bar{X}$ is disjoint from $X \times (\bar{X} \setminus X)$. (Equivalently, $W$ is controlled if, for every $x \in \bar{X} \setminus X$ and every neighborhood $V$ of $x$ in $\bar{X}$, there exists a smaller neighborhood $V'$ of $z$ in $\bar{X}$ such that for any $(y_1, y_2) \in W$ with $y_1 \in V'$ we have $y_2 \in V$.) Ordered by inclusion, these controlled neighborhoods form a directed system $\Lambda$. For $W$ in $\Lambda$ let $E_W$ be the space of pairs $(z, \omega)$ where $z \in Z$ and $\omega: [0,1] \to X$ is a path in $X$ with $\omega(0) = r(z)$, subject to the control condition determined by $W$. (Namely, all points of the form $(\omega(s), \omega(t))$ are in $W$.) Let $p_W: E_W \to X$ be defined by $p_W(z, \omega) = \omega(1)$. The inclusion $Z \to E_W$ is clear. It is a map over $X$ and as such it is a controlled homotopy equivalence. The
inclusions $E_W \rightarrow E_{W'}$ for $W' > W$ in $\Lambda$ are also clear. We let $E = \bigcup E_W$. The remaining details are left to the reader. 

Returning to $K^S$ and the proof of theorem 5.1, it only remains to say that lemma 2.10 carries over to the controlled situation without essential changes. In the controlled version of lemma 2.8, the fibration $E \rightarrow X$ should be replaced by a controlled Serre fibration system as in definition 5.2; the morphisms $f$ should land in some $E_{\lambda_1}$ by assumption and the morphism $g$ should be constructed to land in some $E_{\lambda_2}$ where $\lambda_2 \geq \lambda_1$. In the controlled version of lemma 2.9, the groups $Q_\alpha(E;k), VQ^n(E;k)$ and $V\hat{Q}^n(E,k)$ should be replaced by the direct limits over $\lambda$ of $Q_\alpha(E_{\lambda};k), VQ^n(E_{\lambda};k)$ and $V\hat{Q}^n(E_{\lambda},k)$, respectively. Constructions like $E_n = E|X_n$ should be read as restrictions or pullbacks of controlled fibration systems. These two lemmas (in the controlled version) feed into the proof of theorem 5.1 via the controlled Serre construction, lemma 5.3. The proof then proceeds as in the situation without control. 

6. Control and suspension

The results of this section are mainly variations of well-established facts in controlled algebraic $K$-theory. We have suppressed some of the more mechanical details in the proofs.

Let $X$ be a compact Hausdorff space. Fix an integer $i \geq 0$ and form the control space $(X \times S^{i-1}, X \times S^{i-1} \setminus S^{i-1})$. We usually identify $X \times S^{i-1} \setminus S^{i-1}$ with $X \times \mathbb{R}^i$ and write $(X \times S^{i-1}, X \times \mathbb{R}^i)$.

Let $L^h_i((X \times S^{i-1}, X \times \mathbb{R}^i))$ be the controlled $L$-theory spectrum of $(X \times S^{i-1}, X \times \mathbb{R}^i)$, constructed using locally finite and finite dimensional retractive spaces (with a controlled relative CW-structure) over $X \times \mathbb{R}^i$ rather than locally finitely dominated ones. Similarly, $A^h_i((X \times S^{i-1}, X \times \mathbb{R}^i))$ is the controlled $A$-theory constructed using locally finite and finite dimensional retractive spaces (with a controlled relative CW-structure) over $X \times \mathbb{R}^i$.

**Theorem 6.1.** We have

\begin{align*}
L_i((X \times S^{i-1}, X \times \mathbb{R}^i)) & \simeq \Omega L^h_i((X \times S^i, X \times \mathbb{R}^{i+1})), \\
VL^*_i((X \times S^{i-1}, X \times \mathbb{R}^i)) & \simeq \Omega VL^*_i((X \times S^i, X \times \mathbb{R}^{i+1})), \\
A((X \times S^{i-1}, X \times \mathbb{R}^i)) & \simeq \Omega A^h_i((X \times S^i, X \times \mathbb{R}^{i+1})).
\end{align*}

**Proof.** We concentrate on the quadratic $L$-theory case, the other two cases being very similar. The first step is to replace the control conditions by stronger ones. Let $p: X \times \mathbb{R}^i \rightarrow \mathbb{R}^i$ be the projection. Given retractive spaces $Y_1$ and $Y_2$ over $X \times \mathbb{R}^i$ and a controlled map $f: Y_1 \rightarrow Y_2$ (which is understood to be relative to $X \times \mathbb{R}^i$ but need not respect the reections to $X \times \mathbb{R}^i$), we say that $f$ is bounded if there exists a real number $a \geq 0$ such that $\|p(y) - pf(y)\| \leq a$ for all $y \in Y_1$. Similarly, there is a notion of bounded and controlled homotopy between bounded and controlled maps (between retractive spaces over $X \times \mathbb{R}^i$). A morphism $f: Y_1 \rightarrow Y_2$ between retractive spaces over $X \times \mathbb{R}^i$ (i.e., a map which respects both the zero sections and the reections) is a bounded map for trivial reasons; we call it a weak equivalence (in the bounded sense) if it is invertible up to bounded homotopies relative to $X \times \mathbb{R}^i$. A finite dimensional controlled CW-structure on a retractive space $Y$ over $X \times \mathbb{R}^i$, relative to $X \times \mathbb{R}^i$, is bounded if there exists $a > 0$ such that the image...
of each cell in $\mathbb{R}^i$ has diameter $\leq a$. We use all that to introduce a Waldhausen category

$$\mathcal{R}^{id}(\langle X \times S^{i-1}, X \times \mathbb{R}^i \rangle; b)$$

similar to $\mathcal{R}^{id}(\langle X \times S^{i-1}, X \times \mathbb{R}^i \rangle)$, but with all control conditions replaced by the corresponding “boundedness” conditions. There is also a stable version

$$s\mathcal{R}^{id}(\langle X \times S^{i-1}, X \times \mathbb{R}^i \rangle; b)$$

obtained by adjoining formal desuspensions. There is also a preferred induced map in two objects which have all their cells in a single dimension.) Hence we have an induced map in $L$-theory

$$L_*\left(s\mathcal{R}^{id}(\langle X \times S^{i-1}, X \times \mathbb{R}^i \rangle; b)\right) \longrightarrow L_*\left(s\mathcal{R}^{id}(\langle X \times S^{i-1}, X \times \mathbb{R}^i \rangle)\right).$$

It is a key fact, and one whose proof we want to skip, that this map is a homotopy equivalence. See [2]. Similarly there is an inclusion map

$$L_*\left(s\mathcal{R}^{id}(\langle X \times S^i, X \times \mathbb{R}^{i+1} \rangle; b)\right) \longrightarrow L_*\left(s\mathcal{R}^{id}(\langle X \times S^i, X \times \mathbb{R}^{i+1} \rangle)\right).$$

of $L$-theory spectra, where the superscript “lf” refers to retractive spaces with a locally finite, finite dimensional and bounded $CW$-structure relative to $X \times \mathbb{R}^{i+1}$. This is again a homotopy equivalence.

For the remainder of this proof we work in the bounded setting. We use the Waldhausen fibration theorem [25] or rather its analogue in $L$-theory due to Ranicki/Vogel. The category to which we will apply it is

$$s\mathcal{R}^{lf}(\langle X \times S^i, X \times \mathbb{R}^{i+1} \rangle; b)$$

and for the purposes of this proof we abbreviate this to $s\mathcal{R}$. In addition to the default notion of weak equivalence in $s\mathcal{R}$, we introduce two coarser notions involving passage to germs. Let $Y_1$ and $Y_2$ be retractive spaces over $X \times \mathbb{R}^{i+1}$. A $u$-germ of bounded and controlled maps from $Y_1$ to $Y_2$ is represented by a bounded and controlled map from the portion of $Y_1$ lying over $X \times \mathbb{R}^i \times [a, \infty[ \,$ for some $a \in \mathbb{R}$, to $Y_2$. Two such representatives define the same $u$-germ if they are both defined on $X \times \mathbb{R}^i \times [a, \infty[ \,$ for some $a' \in \mathbb{R}$ and agree there. Similarly, a $v$-germ of bounded and controlled maps from $Y_1$ to $Y_2$ is represented by a bounded and controlled map from the portion of $Y_1$ lying over $X \times \mathbb{R}^i \times [a, \infty[ \,$ for some $a \in \mathbb{R}$, to $Y_2$. Call a morphism in $s\mathcal{R}$ a $u$-equivalence if its mapping cone is weakly equivalent to the zero object in the $u$-germ sense. Call it a $v$-equivalence if its mapping cone is weakly equivalent to zero in the $v$-germ sense. Using the notation which Waldhausen uses in his formulation of the fibration theorem, we obtain a commutative diagram of
Waldhausen categories with SW-duality
\[ sR^u \cap sR^v \longrightarrow sR^u \]
\[ \downarrow \quad \downarrow \]
\[ sR^v \longrightarrow sR. \]
In the resulting commutative diagram of \( L \)-theory spectra
\[ L_\bullet(sR^u \cap sR^v) \longrightarrow L_\bullet(sR^u) \longrightarrow L_\bullet(sR^u, sR^u \cap sR^v) \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ L_\bullet(sR^v) \longrightarrow L_\bullet(sR) \longrightarrow L_\bullet(sR, sR^v) \]
the rows are homotopy fibration sequences by the fibration theorem. The terms \( L_\bullet(sR^u) \) and \( L_\bullet(sR^v) \) are contractible because the Waldhausen categories involved are “flasque”. (More precisely there is an Eilenberg swindle argument for contractibility, as follows. The translation \( x \mapsto x - n \) acting on \( \mathbb{R} \) induces an endofunctor \( \kappa_n \) of \( sR^u \). The sum
\[ \tau = \bigvee_{n \geq 0} \kappa_n \]
is again an endofunctor of \( sR^u \). While it is not strictly true that \( \tau \cong \text{id} \lor \tau \), it is easy to relate \( \tau \) and \( \text{id} \lor \tau \) by a chain of natural equivalences, preserving SW-duality. Hence the identity map of \( L_\bullet(sR^u) \) is contractible.) Next, the right-hand vertical arrow in the diagram is a homotopy equivalence by an easy application of the approximation theorem. (What we have in mind here is an \( L \)-theoretic cousin of Waldhausen’s approximation theorem in algebraic \( K \)-theory. This is very neatly formulated in [23].) Hence the left-hand square in the diagram is homotopy cartesian and so
\[ L_\bullet(sR^u \cap sR^v) \cong \Omega L_\bullet(sR). \]
Finally the projection \( X \times \mathbb{R}^i \times \mathbb{R} \to X \times \mathbb{R}^i \) induces a map from \( L_\bullet(sR^u \cap sR^v) \) to \( L_\bullet((X \times S^{i-1}, X \times \mathbb{R}^i)) \) which is a homotopy equivalence by another (but easier) application of the approximation theorem.

**Corollary 6.2.** We have
\[ A((X \times S^{i-1}, X \times \mathbb{R}^i))^{th\mathbb{Z}/2} \simeq \Omega A^b((X \times S^i, X \times \mathbb{R}^{i+1}))^{th\mathbb{Z}/2}. \]

**Proof.** One proof consist in deducing this from the \( A \)-theory case of theorem 6.1 by applying “Tatification” to both sides; the outcome is
\[ A((X \times S^{i-1}, X \times \mathbb{R}^i))^{th\mathbb{Z}/2} \simeq (\Omega A^b((X \times S^i, X \times \mathbb{R}^{i+1})))^{th\mathbb{Z}/2}. \]

Another possibility: repeat the proof of theorem 6.1 using \( A^{th\mathbb{Z}/2} \) in place of \( A \), \( L_\bullet \) or \( VL^\bullet \) throughout. The outcome is then
\[ A((X \times S^{i-1}, X \times \mathbb{R}^i))^{th\mathbb{Z}/2} \simeq (\Omega (A^b((X \times S^i, X \times \mathbb{R}^{i+1})))^{th\mathbb{Z}/2}). \]
But the operators \( \Omega \) and “Tatification” are interchangeable, due to the linear behaviour of the latter. The two proofs and the two identifications agree in that sense. \( \square \)

There is of course another way to relate the \( L \)-theory (quadratic or visible symmetric) of \((X \times S^{i-1}, X \times \mathbb{R}^i)\) to that of \((X \times S^i, X \times \mathbb{R}^{i+1})\), and that is by taking product with \( \mathbb{R} \). In more detail, a retractive space \( Y \) over \( X \times \mathbb{R}^i \) determines a
retractive space \( Y \times \mathbb{R} \) over \( X \times \mathbb{R}^{i+1} \). The \( SW \)-products are related by a homotopy equivalence
\[
(Y_1, k) \circ (Y_2, \ell) \sim \Omega((Y_1 \times \mathbb{R}, k) \circ (Y_2 \times \mathbb{R}, \ell))
\]
for retractive spaces \( Y_1 \) and \( Y_2 \) over \( X \times \mathbb{R}^{i} \). This respects the involutions and respects nondegenerate pairings. (See also remark 6.4 below.) Hence an \( n \)-dimensional quadratic or visible symmetric structure (in the controlled sense) on a retractive space \( Y \) over \( X \times \mathbb{R}^{i} \) determines an \((n + 1)\)-dimensional quadratic or visible symmetric structure on \( Y \times \mathbb{R} \) as a retractive space over \( X \times \mathbb{R}^{i+1} \). In this way, product with \( \mathbb{R} \) becomes a map from \( L_*((X \times S^{i-1}, X \times \mathbb{R}^{i})) \) to \( \Omega L_*((X \times S^i, X \times \mathbb{R}^{i+1})) \), and from \( \text{VL}^*((X \times S^{i-1}, X \times \mathbb{R}^{i})) \) to \( \Omega \text{VL}^*((X \times S^i, X \times \mathbb{R}^{i+1})) \).

**Proposition 6.3.** With the identifications of theorem 6.1, the maps \( \times \mathbb{R} \) from \( L_*((X \times S^{i-1}, X \times \mathbb{R}^{i})) \) to \( \Omega L_*((X \times S^i, X \times \mathbb{R}^{i+1})) \) and from \( \text{VL}^*((X \times S^{i-1}, X \times \mathbb{R}^{i})) \) to \( \Omega \text{VL}^*((X \times S^i, X \times \mathbb{R}^{i+1})) \) correspond to the inclusions
\[
\Omega L_*^b((X \times S^i, X \times \mathbb{R}^{i+1})) \longrightarrow \Omega L_*((X \times S^i, X \times \mathbb{R}^{i+1})),
\]
\[
\Omega \text{VL}^*((X \times S^i, X \times \mathbb{R}^{i+1})) \longrightarrow \Omega \text{VL}^*((X \times S^i, X \times \mathbb{R}^{i+1})).
\]

**Proof.** We concentrate on the visible \( L \)-theory case. Unravelling the proof of theorem 6.1, we find that the composite map
\[
\text{VL}^*((X \times S^{i-1}, X \times \mathbb{R}^{i})) \xrightarrow{6.1} \Omega \text{VL}^* b((X \times S^i, X \times \mathbb{R}^{i+1})) \xrightarrow{\text{inclusion}} \Omega \text{VL}^*((X \times S^i, X \times \mathbb{R}^{i+1}))
\]
comes out of a commutative square of Waldhausen categories with \( SW \)-product and exact functors preserving the \( SW \)-product,
\[
s R^{id}((X \times S^{i-1}, X \times \mathbb{R}^{i}); b) \longrightarrow s R^{id}((X \times S^i, X \times \mathbb{R}^{i+1}); b)^u
\]
\[
\downarrow
\]
\[
s R^{id}((X \times S^i, X \times \mathbb{R}^{i+1}); b)^v \longrightarrow s R^{id}((X \times S^i, X \times \mathbb{R}^{i+1}); b).
\]
Here the meaning of the superscripts \( u \) and \( v \) is carried over from the proof of theorem 6.1. We abbreviate the names of the categories involved like this:
\[
A \longrightarrow C^u
\]
\[
\downarrow
\]
\[
C^v \longrightarrow C.
\]
The main point was the contractibility of the symmetric \( VL \)-spectra of \( C^u \) and \( C^v \). This leads to nullhomotopies for the maps
\[
\text{VL}^*(A) \rightarrow \text{VL}^*(C^u), \quad \text{VL}^*(A) \rightarrow \text{VL}^*(C^v).
\]
These two nullhomotopies, suitably concatenated and pushed forward to \( \text{VL}^*(C) \), determine our map \( \text{VL}^*(A) \rightarrow \Omega \text{VL}^*(C) \). However, the two nullhomotopies for the two maps
\[
\text{VL}^*(A) \rightarrow \text{VL}^*(C^u), \quad \text{VL}^*(A) \rightarrow \text{VL}^*(C^v)
\]
can also be described explicitly. (Because of the contractibility of the target spectra, any explicit nullhomotopies that we can think of are “the right ones”. ) One explicit
choice of a nullhomotopy for the map
\[ \text{VL}^*(A) \to \text{VL}^*(C^v) \]
is given by product with \([0, \infty]\). In somewhat more detail, a retractive space or spectrum \(Y\) over \(X \times \mathbb{R}^i\) gives rise to a pair of retractive spaces or spectra
\[ (Y \times [0, \infty], Y \times 0 \cup X \times \mathbb{R}^i \times [0, \infty]) \]
over \(X \times \mathbb{R}^i \times [0, \infty]\). Cobase change along the inclusion
\[ X \times \mathbb{R}^i \times [0, \infty] \to X \times \mathbb{R}^i \times \mathbb{R} \]
makes this into a pair of retractive spaces or spectra over \(X \times \mathbb{R}^i \times [0, \infty] + 1\) which we denote rather informally by \(Y \times (0, \infty)\). If \(Y\) was in \(A\), then \(Y \times (0, \infty)\) is a pair in \(C^v\). If \(Y\) came with a nondegenerate visible symmetric structure of dimension \(n\), then the pair \(Y \times (0, \infty)\) comes with a nondegenerate visible symmetric structure of dimension \(n + 1\). More generally, if \(Y\) was a “cofibrant” functor from the poset of faces of \(\Delta^m\) to \(A\), i.e., an object of \(A^*(m)\) in the notation of \([34]\), then \(Y \times [0, \infty]\) can be interpreted as a cofibrant functor from the poset of faces of \(\Delta^m \times \Delta^1\) to \(C^v\), trivial on the faces contained in \(\Delta^m \times d_0 \Delta^1\). (See remark 6.4 below.) And again, if \(Y\) came with a nondegenerate visible symmetric structure of dimension \(n\), then \(Y \times [0, \infty]\) comes with a nondegenerate visible symmetric structure of dimension \(n + 1\). (In other words the nullhomotopy that we are describing here is given by “nullbordisms”.)

There is an analogous choice of nullhomotopy for the map \(\text{VL}^*(A) \to \text{VL}^*(C^v)\). It follows easily that the concatenation of the two nullhomotopies results in a map from \(\text{VL}^*(A)\) to \(\Omega \text{VL}^*(C)\) which is given by product with \(\mathbb{R}\), as claimed.\(\square\)

**Remark 6.4.** This proof assumes that the reader has an explicit definition of the map \(\times \mathbb{R}\) in mind. If that is not the case, we have a suggestion: introduce extra simplicial directions to accommodate products, and afterwards show that they are redundant. This is the (admittedly pedestrian) method used in \([35]\). For an example, suppose that the task is to describe or define “the” product
\[ \Omega^\infty \text{VL}^*(X) \land \Omega^\infty \text{VL}^*(Y) \to \Omega^\infty \text{VL}^*(X \times Y). \]
Use the standard models of \(\Omega^\infty \text{VL}^*(X)\) and \(\Omega^\infty \text{VL}^*(Y)\). They are geometric realizations of certain incomplete simplicial sets, alias \(\Delta\)-sets. Use a slightly nonstandard model of \(\Omega^\infty \text{VL}^*(X \times Y)\) : this is the geometric realization of an incomplete bisimplicial set, alias bi-\(\Delta\)-set. (Then it is easy to define the product.) The nonstandard model of \(\Omega^\infty \text{VL}^*(X \times Y)\) contains two copies of the standard model of \(\Omega^\infty \text{VL}^*(X \times Y)\) as its “horizontal 0-skeleton” and “vertical 0-skeleton”, respectively. The inclusions of these two 0-skeleta are (weak) homotopy equivalences. See \([35]\). These remarks apply not only to \(L\)-theory, but also to the “Tatified” algebraic \(K\)-theory, as in proposition 6.6 below. All that and more is explained in \([35]\).

There is no straightforward analogue of proposition 6.3 for the functor \(A\). Instead we have the following degenerate version.

**Lemma 6.5.** The map
\[ \times \mathbb{R} : A((X \ast S^{i-1}, X \times \mathbb{R}^i)) \to A((X \ast S^i, X \times \mathbb{R}^{i+1})) \]
is nullhomotopic.
Proof. As in the proof of theorem 6.1, we replace the control conditions by bound-
edness conditions. For each retractive space \( Y \) over \( X \times \mathbb{R}^t \) we have a cofibration sequence of retractive spaces over \( X \times \mathbb{R}^{t+1} \), as follows:

\[
Y' \longrightarrow Y \times \mathbb{R} \longrightarrow (Y \times \mathbb{R})/Y'
\]

where \( Y' \) is the union of \( Y \times 0 \) and \( X \times \mathbb{R}^t \times \mathbb{R} = X \times \mathbb{R}^{t+1} \). The quotient \( (Y \times \mathbb{R})/Y' \) breaks up as a coproduct of two retractive spaces over \( X \times \mathbb{R}^{t+1} \), one which is trivial over \( X \times \mathbb{R}^t \times [0, \infty] \) and another which is trivial over \( X \times \mathbb{R}^t \times ]-\infty, 0] \). Hence by the additivity theorem, the map \( \times \mathbb{R} \) is homotopic to the sum of three maps induced by the exact functors taking \( Y \) to \( Y' \) and to the left-hand and right-hand summand of \( (Y \times \mathbb{R})/Y' \). Each of these three maps is nullhomotopic because the exact functor which induces it factors through a flasque subcategory, such as the full subcategory defined by the objects which are trivial over \( X \times \mathbb{R}^t \times [0, \infty[ \). \( \square \)

By contrast, the proof of proposition 6.3 works perfectly well for \( \mathbb{A}_{th\mathbb{Z}/2} \).

Proposition 6.6. With the identification of corollary 6.2, the map

\[
\times \mathbb{R} : \mathbb{A}((X \ast S^{i-1}, X \times \mathbb{R}^i))^{th\mathbb{Z}/2} \longrightarrow \Omega \mathbb{A}((X \ast S^{i}, X \times \mathbb{R}^{i+1}))^{th\mathbb{Z}/2}
\]

corresponds to the inclusion

\[
\Omega \mathbb{A}^h((X \ast S^{i}, X \times \mathbb{R}^{i+1}))^{th\mathbb{Z}/2} \longrightarrow \Omega \mathbb{A}((X \ast S^{i}, X \times \mathbb{R}^{i+1}))^{th\mathbb{Z}/2}.
\]

Proof. For a Waldhausen category \( \mathcal{D} \) with an \( SW \)-product satisfying the conditions of [34, §2], the spectrum \( K(\mathcal{D})^{th\mathbb{Z}/2} \) can be described as the geometric realization of the simplicial spectrum without degeneracy operators

\[
[m] \mapsto K(\mathcal{D}^p(m))^{h\mathbb{Z}/2}.
\]

Taking \( \mathcal{D} = \mathcal{A} \) or \( \mathcal{D} = \mathcal{C}^v \) from the proof of proposition 6.3, product with \([0, \infty[\) gives us maps

\[
K(\mathcal{A}^h(m))^{h\mathbb{Z}/2} \longrightarrow K((\mathcal{C}^v)^h(m, 1))^{h\mathbb{Z}/2}
\]

for \( m \geq 0 \), respecting face operators (in the \( m \)-variable). Here \((\mathcal{C}^v)^h(m, 1)\) consists of certain functors from the poset of faces of \( \Delta^m \times \Delta^1 \) to \( \mathcal{C}^v \). By passage to geometric realizations these maps determine a single map

\[
K(\mathcal{A})^{th\mathbb{Z}/2} \land \Delta^1_+ \rightarrow K(\mathcal{C}^v)^{th\mathbb{Z}/2}
\]

which can be regarded as a homotopy from the inclusion-induced map to zero. \( \square \)

For the following lemma, let \( \mathcal{R} \) be any Waldhausen category with an \( SW \)-product satisfying the axioms of [34, §2]. Let \( H \) be a subgroup of the group \( K_0(\mathcal{R}) \) which is invariant under the involution determined by the \( SW \)-product. Let \( \mathcal{R}^H \subset \mathcal{R} \) be the full Waldhausen subcategory consisting of all objects \( C \) whose class \([C] \in K_0(\mathcal{R})\) belongs to \( H \). The \( SW \)-product on \( \mathcal{R} \) restricts to one on \( \mathcal{R}^H \) and this satisfies the axioms of [34, §2], too.

Lemma 6.7. The following square is homotopy cartesian:

\[
\begin{array}{ccc}
\mathbb{L}_*(\mathcal{R}^H) & \xrightarrow{\Xi} & \mathbb{A}(\mathcal{R}^H)^{th\mathbb{Z}/2} \\
\downarrow \text{inclusion} & & \downarrow \text{inclusion} \\
\mathbb{L}_*(\mathcal{R}) & \xrightarrow{\Xi} & \mathbb{A}(\mathcal{R})^{th\mathbb{Z}/2}.
\end{array}
\]
Proof. It is well known [21] that the relative $n$-th homotopy group of the inclusion map $L_\bullet(\mathcal{R}^H) \to L_\bullet(\mathcal{R})$ is isomorphic to the Tate cohomology group

$$\tilde{H}^{-n}(\mathbb{Z}/2; K_0/H)$$

where $K_0 = K_0(\mathcal{R})$. More precisely, let an element in this relative homotopy group be represented by a pair $C \to D$ in $\mathcal{R}$ with a nondegenerate quadratic structure of formal dimension $n$, and with $C$ in $\mathcal{R}^H$. Poincaré duality implies that $[C] \in K_0(\mathcal{R})/H$ is in the $(-1)^n$ eigensubgroup of the standard involution. Hence $[C]$ determines an element in the above-mentioned Tate cohomology group. This describes the isomorphism. — But now it is also clear that the relative $n$-th homotopy group of the inclusion map $A(\mathcal{R}^H)^{th/2} \to A(\mathcal{R})^{th/2}$ is isomorphic to $\tilde{H}^{-n}(\mathbb{Z}/2; K_0/H)$.

Indeed this relative homotopy group can be identified with $\pi_n$ of the spectrum $(A(\mathcal{R})/A(\mathcal{R}^H))^{th/2}$, and here $A(\mathcal{R})/A(\mathcal{R}^H)$ is an Eilenberg-MacLane spectrum with (at most) one nonzero homotopy group, $\pi_0(A(\mathcal{R})/A(\mathcal{R}^H)) = K_0/H$.

Using these identifications, the relative $n$-th homotopy group for the two columns in the above commutative square are “the same” and we want to show that $\Xi$ induces the identity map. This is clear in the case $n = 0$ by inspection. The cases where $n > 0$ can be reduced to the case $n = 0$ by the following trick. The $SW$-product on $\mathcal{R}$ is (or can be made) $\Omega$-spectrum valued and as such can be looped down. Let $\mathcal{R}^{(1)}$ be the same Waldhausen category as $\mathcal{R}$, but with the $SW$ product obtained from the original one by a single looping. In other words, $C_1 \odot C_2$ in the $\mathcal{R}^{(1)}$ setting is the loop space of $C_1 \odot C_2$ in the $\mathcal{R}$ setting, for objects $C_1$ and $C_2$ of $\mathcal{R}$. Then it is elementary to verify

$$L_\bullet(\mathcal{R}^{(1)}) \simeq \Omega L_\bullet(\mathcal{R}) , \quad A(\mathcal{R}^{(1)})^{th/2} \simeq \Omega A(\mathcal{R})^{th/2} ,$$

and these identifications are compatible under $\Xi$. Therefore replacing $\mathcal{R}$ by $\mathcal{R}^{(1)}$ throughout has the effect of looping all the spaces in our commutative diagram, and relabelling relative $n$-th homotopy groups of the columns as relative $(n-1)$-th homotopy groups. Finally, to reduce the cases $n < 0$ to $n = 0$, we use deloopings of the $SW$-product. (This is probably best done by smash with $S^1$ followed by $\Omega$-spectrification.)

□

Corollary 6.8. The following commutative squares are homotopy cartesian:

$$\begin{align*}
L_\bullet((X \ast S^{i-1}, X \times \mathbb{R}^i)) & \xrightarrow{\Xi} A((X \ast S^{i-1}, X \times \mathbb{R}^i))^{th/2} , \\
\downarrow_{\times R} & \downarrow_{\times R} \\
\Omega L_\bullet((X \ast S^{i}, X \times \mathbb{R}^{i+1})) & \xrightarrow{\Xi} \Omega A((X \ast S^{i}, X \times \mathbb{R}^{i+1}))^{th/2} , \\
\downarrow_{\times R} & \\
\Omega V L_\bullet((X \ast S^{i-1}, X \times \mathbb{R}^i)) & \xrightarrow{\Xi} A((X \ast S^{i-1}, X \times \mathbb{R}^i))^{th/2} , \\
\downarrow_{\times R} & \downarrow_{\times R} \\
\Omega V L_\bullet((X \ast S^{i}, X \times \mathbb{R}^{i+1})) & \xrightarrow{\Xi} \Omega A((X \ast S^{i}, X \times \mathbb{R}^{i+1}))^{th/2} .
\end{align*}$$
Proof. By propositions 6.3 and 6.6, the first square can be identified with
\[
\Omega L^0((X \ast S, X \times \mathbb{R}^i)) \xrightarrow{\Xi} \Omega A^h((X \ast S, X \times \mathbb{R}^i))^{th/2}
\]
which is indeed homotopy cartesian by lemma 6.7. For the second square, it is enough to show that the square
\[
L^0((X \ast S, X \times \mathbb{R}^i)) \xrightarrow{\text{symmetrization}} VL^0((X \ast S, X \times \mathbb{R}^i))
\]
is homotopy cartesian. This boils down to showing that
\[
\times \mathbb{R} : VL^0((X \ast S, X \times \mathbb{R}^i)) \rightarrow V\hat{\Omega}L^0((X \ast S, X \times \mathbb{R}^i))
\]
is a homotopy equivalence, which is indeed the case by theorem 5.1. □

We turn to a slightly different but related theme: the homotopy invariance properties of constructions such as $L^0$, $VL^0$ and $A$ when applied to control spaces $S^1 \times \mathbb{R}$ where $X$ is compact Hausdorff. It is easy to show that these are indeed homotopy invariant in the variable $X$, but we are also interested in the other variable, the sphere.

Proposition 6.9. The constructions $L^0$, $VL^0$ and $A$ applied to control spaces of the form $(X \ast S^i, X \times \mathbb{R})$, with compact Hausdorff $X$, are homotopy invariant as functors of the sphere variable.

Proof. We concentrate on the $A$-theory case. There are many interpretations of “homotopy invariant”. What we mean here is that homotopic maps $f, g$ from $S^i$ to $S^j$ induce homotopic maps
\[
f_* : g_* : A((X \ast S^i, X \times \mathbb{R})) \longrightarrow A((X \ast S^j, X \times \mathbb{R})).
\]
To facilitate the proof of this, we apply theorem 6.1; this increases $i$ and $j$ by 1, which is not a problem, and replaces $A$ by $A^h$.

Let $I = [0, 1]$. We shall now prove that the projection $(X \ast S^i) \times I \rightarrow X \ast S^i$ induces a homotopy equivalence
\[
A^h((X \ast S^i \times I, X \times \mathbb{R}^i)) \longrightarrow A^h((X \ast S^i, X \times \mathbb{R}^i)).
\]
(This implies most other forms of “homotopy invariance”. ) Now we have several choices of Waldhausen category with an algebraic $K$-theory spectrum that deserves to be called $A^h((X \ast S^i, X \times \mathbb{R}^i))$. Among these we choose one which is more “algebraic” than the one which we normally prefer. This is described in [30, §6]. The objects of this Waldhausen category, call it $\mathcal{R}$ for now, are certain retractive CW-spaces $Y$ over $X$ with a finite dimensional relative CW-structure and a map from the set of cells to $\mathbb{R}^i$. This labelling map must satisfy certain local finiteness and control conditions. (The control conditions are expressed in terms of the compactification $D^i$ of $\mathbb{R}^i$.) The morphisms in $\mathcal{R}$ are retractive cellular maps (over and under $X$) which, again, satisfy certain control conditions (expressed in terms of the cell labels). The relationship between this Waldhausen category $\mathcal{R}$ and the
usual one, $\mathcal{R}^H((X \times S^{t-1}, X \times \mathbb{R}^i))$, is essentially given by cobase change along the projection $X \times \mathbb{R}^i \to X$, regarded as a functor

$$\mathcal{R}^H((X \times S^{t-1}, X \times \mathbb{R}^i)) \to \mathcal{R}.$$  

Note that the objects in $\mathcal{R}^H((X \times S^{t-1}, X \times \mathbb{R}^i))$ are retractive spaces over $X \times \mathbb{R}^i$ with a relative $CW$-structure. See [30, §9] for related ideas.

We introduce a similar Waldhausen category $\mathcal{Q}$ with an algebraic $K$-theory spectrum that deserves to be called $\mathcal{A}^H(((X \times S^{t-1}) \times I, X \times \mathbb{R}^i \times I))$. Its objects are retractive spaces over $X$ with a relative $CW$-structure and a map from the set of cells to $\mathbb{R}^i \times I$, subject to certain local finiteness and control conditions (which are formulated using the compactification $D^i \times I$ of $\mathbb{R}^i \times I$). It is convenient to stabilize the two categories by introducing formal desuspensions; then we have $s\mathcal{Q}$ and $s\mathcal{R}$.

Now our task is to show that the functor $s\mathcal{Q} \to s\mathcal{R}$

induced by the projection $\mathbb{R}^i \times I \to \mathbb{R}^i$ induces a homotopy equivalence of the algebraic $K$-theory spectra. Note that this functor does not satisfy the first hypothesis of the approximation theorem (which is about “detection” of weak equivalences). We use the fibration theorem instead. In addition to the standard notion of weak equivalence in $s\mathcal{Q}$, we therefore introduce a coarser notion. Let $I' = [0, 1]$ and call a morphism in $s\mathcal{Q}$ a $\sim$-equivalence if the germ near $S^{t-1} \times I'$ of its mapping cone is weakly equivalent to zero in the usual controlled sense. Note here that $S^{t-1} \times I'$ is part of the singular set $S^{t-1} \times I$ of the control space $((X \times S^{t-1}) \times I, X \times \mathbb{R}^i \times I))$. By the fibration theorem there is a homotopy fibration sequence of $K$-theory spectra,

$$K(s\mathcal{Q}^w) \to K(s\mathcal{Q}) \to K(s\mathcal{Q}_{v}).$$

Hence it is enough to verify that

(i) $s\mathcal{Q}_{v}$ is “flasque” enough so that $K(s\mathcal{Q}_{v})$ is contractible;

(ii) the composition of exact functors $s\mathcal{Q}^\circ \to s\mathcal{Q} \to s\mathcal{R}$ satisfies the hypotheses of the approximation theorem, so that $K(s\mathcal{Q}^w) \simeq K(s\mathcal{R})$.

As regards (i), the situation is really quite similar to the one which we have considered in the “homotopy invariance” part of the proof of theorem 4.5. To make this connection clearer we note that $s\mathcal{Q}_{v}$ can be replaced by a simpler category $s\mathcal{Q}_{(v)}$ with the same objects, where the morphisms themselves are germs of retractive maps over $X \times \mathbb{R}^i \times I$ defined near (i.e., over an arbitrarily small neighborhood of) $S^{t-1} \times I'$. Indeed, another application of the approximation theorem shows that the forgetful functor $s\mathcal{Q}_{v} \to s\mathcal{Q}_{(v)}$ induces a homotopy equivalence of the $K$-theory spectra. Up to an equivalence of categories, $s\mathcal{Q}_{(v)}$ however depends only on

$$(W, W \setminus (S^{t-1} \times I'))$$

where $W$ can be any neighborhood of $S^{t-1} \times I'$ in $(X \times S^{t-1}) \times I$. We can take $W$ to be homeomorphic to the mapping cylinder of the projection

$$X \times S^{t-1} \times I' \to S^{t-1} \times I'.$$

In particular when $X$ is a point, the control pair $(W, W \setminus (S^{t-1} \times I'))$ is what we have called $\mathcal{P}(S^{t-1} \times I')$. The same arguments for “flasqueness” as in the homotopy invariance part of the proof of theorem 4.5 apply to the category $s\mathcal{Q}_{(v)}$; this includes remark 14.3 below.
As regards (ii), it is hard to verify directly that the hypotheses of the approximation theorem are satisfied by the functor $sQ^v \to sR$. It seems wiser therefore to introduce a full Waldhausen subcategory $sQ^{(v)}$ of $sQ^v$ consisting of the objects $Y$ whose set of cell labels avoids a neighborhood (depending on $Y$) of $S^{i-1} \times I'$ in $(X * S^{i-1}) \times I$. Then it is clear that the composition

$$sQ^{(v)} \to sQ^v \to sQ \to sR$$

is an equivalence of (Waldhausen) categories. Hence it remains only to verify that the inclusion $sQ^{(v)} \to sQ^v$ satisfies the hypotheses of the approximation theorem. The first condition (about detection of weak equivalences) is clearly satisfied. For the second condition, suppose given a cofibration $f: Y_1 \to Y_2$ in $sQ^v$, with $Y_1$ in $sQ^{(v)}$. From the definitions, it is easy to construct a factorization

$$Y_1 \to Y_3 \to Y_2$$

of $f$, where $Y_1 \to Y_3$ is still a cofibration, $Y_3$ is also in $sQ^{(v)}$ and the morphism from $Y_3$ to $Y_2$ is a “domination” (i.e. there exists a controlled map $Y_2 \to Y_3$, not necessarily a morphism, which is right inverse to $Y_3 \to Y_2$ up to a controlled homotopy). Attaching additional cells to $Y_3$ where necessary, one can then easily improve $Y_3 \to Y_2$ to a weak equivalence in $sQ^v$. (Solve the corresponding problem in $sR$ first and use that solution as a model.) □

7. Spherical fibrations and twisted duality

It is well known [24] that a spherical fibration $\xi$ on a space $X$ determines a twisted SW product in the category of finitely dominated retractive spaces on $X$. This is compatible with the so-called $w$–twisted involution on $\mathbb{Z}[\pi_1 X]$ where $w: \pi_1(X) \to \mathbb{Z}$ is $w_1(\xi)$. We recall some of the details, following [34, 1.A.9] rather more than [24]. We assume that $\xi: E \to X$ is a fibration with fibers homotopy equivalent to $S^d$ and with a preferred section which is a fiberwise cofibration [11]. (What we have in mind is, for example, the fiberwise one–point compactification of a vector bundle on $X$, with the preferred section which picks out the point at infinity in each fiber.) Let $Y_1$ and $Y_2$ be finitely dominated retractive spaces over $X$, with retractions $r_1$ and $r_2$. Again we start by defining an unstable Spanier–Whitehead product $Y_1 \wedge Y_2$. This is the based space obtained by first forming the external smash product

$$Y_1 \wedge_X Y_2 = Y_1 \times Y_2 / \sim$$

where $\sim$ identifies $(y_1, x)$ with $(r_1(y_1), x)$ and $(x, y_2)$ with $(x, r_2(y_2))$; then forming the homotopy pullback of

$$E \xrightarrow{\text{diagonal } \circ \text{proj.}} X \times X \leftarrow \xrightarrow{\text{retraction}} Y_1 \wedge_X Y_2 ;$$

then collapsing the subspace consisting of all elements in the homotopy pullback which are mapped to the base point under the forgetful projection to

$$(E/X) \wedge (Y_1/X) \wedge (Y_2/X) .$$

To make this unstable SW product into an SW product on the stable category $sR(X)$ of finitely dominated retractive spaces over $X$, we proceed much as in §2.
Definition 7.1. We let
\[(Y_1, k) \circ (Y_2, \ell) = \text{colim}_n \Omega^{2n+d}(\Sigma^{n-k}Y_1 \wedge \Sigma^{n-\ell}Y_2).\]

More generally we let
\[(Y_1, k) \circ_j (Y_2, \ell) = \text{colim}_n \Omega^{2n+d} \Sigma^j(\Sigma^{n-k}Y_1 \wedge \Sigma^{n-\ell}Y_2),\]

so that \((Y_1, k) \circ (Y_2, \ell) = (Y_1, k) \circ_0 (Y_2, \ell)\), and denote the \(\Omega\)-spectrum with \(j\)-th term \((Y_1, k) \circ_j (Y_2, \ell)\) by \((Y_1, k) \circ \ast (Y_2, \ell)\).

This comes with a structural symmetry \((Y_1, k) \circ \ast (Y_2, \ell) \cong (Y_2, \ell) \circ \ast (Y_1, k)\) determined by the obvious symmetry of \(\ast\). For \(Y_1 = Y_2 = Y\) and \(k = \ell\) we obtain an \(\Omega\)-spectrum \((Y, k) \circ \ast (Y, k)\) with an action of \(\mathbb{Z}/2\).

Note the \(d\) in definition 7.1 which is the formal fiber dimension of \(\xi\). This causes a slight disagreement with the conventions of [34, 1.A.9], but it is convenient here.

Definition 7.1 is essentially insensitive to a stabilization of \(\xi\) by fiberwise suspension. More precisely, a spherical fibration \(\xi\) on \(X\) as above determines an \(\Omega\) product \(\circ\), on \(sR(X)\), and the fiberwise suspension \(\Sigma\xi\) determines another, which we (temporarily) denote by \(\circ'\). There is a natural homotopy equivalence
\[(Y_1, k) \circ' (Y_2, \ell) \rightarrow (Y_1, k) \circ \ast (Y_2, \ell)\]

which respects the canonical involutions. Note also that the \(\xi\)-twisted version of \((Y_1, k) \circ (Y_2, \ell)\) is naturally homeomorphic to the standard one (definition 2.1) if \(\xi\) is a trivial sphere bundle \(S^0 \times X \rightarrow X\).

Definition 2.2 can be re-used with the \(\xi\)-twisted interpretation of the \(SW\) product. The \(\xi\)-twisted versions of corollary 2.5 and definition 2.6 take a slightly more complicated form. For a finitely dominated retractive space \(Y\) over \(X\), let \(Y^\xi\) be the fiberwise smash product of \(Y\) and \(E = E(\xi)\) over \(X\). As before, suppose that the formal fiber dimension of \(\xi\) is \(d\).

Corollary 7.2. There is a natural homotopy fiber sequence of spectra
\[\text{(Y, k) \circ (Y, k)}_{h\mathbb{Z}/2} \longmapsto (\text{(Y, k) \circ (Y, k)})_{\mathbb{Z}/2} \overset{j}{\rightarrow} \Sigma^{\infty - k - d}(Y^\xi / X).\]

Definition 7.3. An \(n\)-dimensional visible hyperquadratic structure on \((Y, k)\) is an element in \(\Omega^n \Omega^\infty \Sigma^{\infty - k - d}(Y^\xi / X)\). An \(n\)-dimensional quadratic structure on \((Y, k)\) is an element of \(\Omega^n \Omega^\infty((\text{(Y, k) \circ (Y, k)})_{h\mathbb{Z}/2})\). Alternatively, an \(n\)-dimensional quadratic structure on \((Y, k)\) can be defined as an element of \(\Omega^n \Omega^\infty\) of the fiber homotopy of the natural map \(J: (\text{(Y, k) \circ (Y, k)})_{\mathbb{Z}/2} \rightarrow \Sigma^{\infty - k - d}(Y^\xi / X)\).

We write \(L_\bullet(X, \xi), V^L_\bullet(X, \xi)\) and \(L_\bullet(X, \xi)\) for the \(L\)-theory spectra determined by the \(\xi\)-twisted SW product on the stable category of finitely dominated retractive spaces over \(X\). Theorem 2.7 remains correct (with essentially the same proof) in this more general setting and can informally be regarded as a statement for spaces \(X\) over \(BG\), the classifying space for stable spherical fibrations. A weak homotopy equivalence, in that setting, is a map of spaces over \(BG\) which is a weak homotopy equivalence of spaces. A Cartesian square, in that setting, is a commutative square of spaces over \(BG\) which is cocartesian as a square of spaces.

We come to an outline of a calculation of \(V L^L_\bullet(X, \xi)\) (which will not be used elsewhere in the paper). Let \(Th(X, \xi)\) be the Thom spectrum of \(X\) and \(\xi\). For
convenience or otherwise, we index that in such a way that the \((d + k)\)–th space is \(S^k \wedge (E/X)\). There is a natural map

\[
\tilde{\sigma}: \text{Th}(X, \xi) \longrightarrow \tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi),
\]

the visible hyperquadratic signature map.

In more detail, \(\pi_n\text{Th}(X, \xi)\) can be identified with the group of bordism classes of normal spaces of formal dimension \(n\) over \((X, \xi)\). A normal space of formal dimension \(n\) over \((X, \xi)\) consists of a finitely dominated space \(Y\), a map \(f: Y \to X\) and a stable (pointed) map \(\eta: S^{n+d} \to E(f^*\xi)/Y\). (The image under the Thom isomorphism of the homology class carried by \([\eta]\) is a class in \(H_n(Y; \mathbb{Z}^w)\), where \(\mathbb{Z}^w\) is the twisted integer coefficient system determined by \(w\).) It is called the fundamental class of the normal space, but it is not subject to any Poincaré duality condition. If it does satisfy the Poincaré duality condition as formulated by Wall [27], then that makes \(Y\) into a Poincaré duality space with Spivak normal bundle \(f^*\xi\). Therefore normal spaces generalize Poincaré duality spaces, and in fact the visible hyperquadratic signature map is a variant of a better known and easier–to–understand map \(\sigma\) from the Poincaré duality bordism spectrum \(\text{Bm}_{PD}(X, \xi)\) of \((X, \xi)\) to the visible symmetric \(L\)–theory spectrum \(\tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi)\). The two maps fit into a commutative square

\[
\begin{array}{ccc}
\text{Bm}_{PD}(X, \xi) & \longrightarrow & \tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi) \\
\downarrow \text{forgetful} & & \downarrow \text{inclusion} \\
\text{Th}(X, \xi) & \longrightarrow & \tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi).
\end{array}
\]

In both the normal bordism theory and the visible hyperquadratic theory, there are external products. They have the form

\[
\text{Th}(X, \xi) \wedge \text{Th}(X', \xi') \longrightarrow \text{Th}(X \times X', \xi \times \xi'),
\]

\[
\tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi) \wedge \tilde{\mathcal{V}}\tilde{L}^\bullet(X', \xi') \longrightarrow \tilde{V}\tilde{L}^\bullet(X \times X', \xi \times \xi')
\]

where \(\xi \times \xi'\) is the external fiberwise smash product of \(\xi\) and \(\xi'\). The composition

\[
\text{Th}(X, \xi) \wedge \tilde{\mathcal{V}}\tilde{L}^\bullet(\ast) \xrightarrow{\sigma \wedge \text{id}} \tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi) \wedge \tilde{\mathcal{V}}\tilde{L}^\bullet(\ast) \xrightarrow{\text{ext. prod.}} \tilde{V}\tilde{L}^\bullet(X, \xi)
\]

is a natural transformation between excisive functors on the category of spaces over \(BG\). It is an equivalence when \(X\) is a point (mapping to the base point of \(BG\)). Hence it is always an equivalence and we have

**Theorem 7.4.** \(\tilde{V}\tilde{L}^\bullet(X, \xi) \simeq \text{Th}(X, \xi) \wedge (S \vee \mathbb{R}P^\infty_{-1})\).

Moreover, the fact that \(\tilde{\sigma}\) commutes with external products immediately leads to a “calculation” of \(\tilde{\sigma}\):

**Proposition 7.5.** The hyperquadratic signature map

\[
\tilde{\sigma}: \text{Th}(X, \xi) \longrightarrow \tilde{\mathcal{V}}\tilde{L}^\bullet(X, \xi) \simeq \text{Th}(X, \xi) \wedge (S \vee \mathbb{R}P^\infty_{-1})
\]

is homotopic to the inclusion of the wedge summand \(\text{Th}(X, \xi) \wedge S \simeq \text{Th}(X, \xi)\).

Next we need a \(\xi\)–twisted version of section 4. We begin with a control space \((\bar{Q}, Q)\) as in 4.3, and a spherical fibration \(\xi: E \to Q\) of formal fiber dimension \(d\), with a distinguished section.

Let \(Y\) and \(Z\) be objects of \(\mathcal{R}^{\text{Bl}}(\bar{Q}, Q)\). To define their \(SW\) product \(Y \odot Z\), we
introduce first an unstable form $Y \times Z$ of it. We define it as the geometric realization of a based simplicial set. An $n$–simplex of this simplicial set is a pair $(f, \gamma)$ where

(i) $f$ is a continuous map from the standard $n$–simplex $\Delta^n$ to the topological inverse limit of the spaces $(Y/Q) \wedge (E^P/Q) \wedge (Z^P/Q)$, where $P$ runs through the large closed subsets of $Q$;

(ii) $\gamma$ is a continuous assignment $c \mapsto \gamma_c$ of paths in $Q$, defined for $c \in \Delta^n$ with $f(c)$ not equal to the base point $\star$.

The paths $\gamma_c$ are to be parametrized by $[-1, +1]$ and must satisfy

$\gamma_c(-1) = r_Y f_Y(c), \quad \gamma_c(+1) = r_Z f_Z(c), \quad \gamma_c(0) = \xi f_E(c)$.

There is a control condition:

For $q \in \bar{Q} \setminus Q$ and any neighborhood $V$ of $q$ in $\bar{Q}$, there exists a smaller neighborhood $W$ of $q$ in $\bar{Q}$ such that, for any $c \in \Delta^n$ with $f(c) \neq \star$, the path $\gamma_c$ either avoids $W$ or runs entirely in $V$.

**Definition 7.6.** For $Y$ and $Z$ in $R_{\text{ld}}(\bar{Q}, Q)$ and integers $k, \ell \in \mathbb{Z}$, let

$$(Y,k) \odot (Z,\ell) = \lim_{\rightarrow} \Omega^{2n+d}(\Sigma^{n-k}Y \wedge \Sigma^{n-\ell}Z).$$

More generally let $(Y,k) \odot (Z,\ell)$ be the spectrum with $j$–th space

$$(Y,k) \odot_j (Z,\ell) = \lim_{\rightarrow} \Omega^{2n+d}\Sigma^j(\Sigma^{n-k}Y \wedge \Sigma^{n-\ell}Z).$$

This is the appropriate definition for the category $sR_{\text{ld}}(\bar{Q}, Q)$. There is also a germ version, for the category $sR_{\text{Gld}}(\bar{Q}, Q)$, which generalizes definition 4.4. With that generalized definition, theorem 4.5 generalizes as follows:

**Theorem 7.7.** The spectrum valued functor $X \mapsto E(X)$ is homotopy invariant and excisive. Here $X$ can be a space over $BG$ or a space with a spherical fibration $\xi$, and $E(X)$ means $L^\bullet(\mathbb{J}X_\infty)$, defined using the $\xi$–twisted SW product.

We leave the detailed formulation to the reader. The proof is essentially identical with the proof of theorem 4.5.

Furthermore definition 4.10 can be re–used in the twisted setting and leads to a generalization of theorem 4.14:

**Theorem 7.8.** The spectrum valued functor $X \mapsto E(X)$ is homotopy invariant and excisive. Here $X$ can be a space over $BG$ or a space with a spherical fibration $\xi$, and $E(X)$ means $VL^\bullet(\mathbb{J}X_\infty)$, defined using the $\xi$–twisted SW product.

The results of sections 5 and 6 also have generalizations to the twisted setting. We do not formulate them here explicitly. They will however be used in sections 12 and 13.

8. **Homotopy invariant characteristics and signatures**

This section is formally analogous to parts of [9, §6]. We begin with a space $X$, a spherical fibration $\xi$ on $X$ (with a distinguished section which is a fiberwise cofibration), and an integer $n \geq 0$. From these data we produce a spectrum $VL^\bullet(X, \xi, n)$. 
In the case where \( X \) is a finitely dominated Poincaré duality space of formal dimension \( n \) and \( \xi \) is the Spivak normal fibration of \( X \), we also construct a characteristic element

\[
\sigma(X) \in F(X, \xi, n) := \Omega^{\infty+n} VLA^\bullet(X, \xi, n).
\]

This refines the Mishchenko–Ranicki (visible) symmetric signature of \( X \) which, in the nonlinear setting, is an element of \( \Omega^{\infty+n} VL^\bullet(X, \xi) \) or of \( \Omega^{\infty+n} L^\bullet(X, \xi) \). The construction has certain naturality properties. As in [9, §1], these properties imply that every family of finitely dominated formally \( n \)-dimensional Poincaré duality spaces \( X_b \), depending on a parameter \( b \in B \), determines a characteristic section \( \sigma(p) \) of a fibration on \( B \) whose fibers are, essentially, the spaces \( F(X_b, \xi_b, n) \).

The \( \xi \)-twisted stable \( SW \)-product on \( sR = sR(X) \) was introduced in section 7. This determines a \( \xi \)-twisted duality involution on \( K(sR) = A(X) \) as explained in [34]. (See remark 14.1 below for a correction.) Using that, we have the map

\[
\Xi: L^\bullet(sR) \to K(sR))^{thZ/2}
\]

of [34]. We also write \( \Xi \) for its restriction to \( VL^\bullet(sR) \). Let \( S^0_l = \mathbb{R}^n \cup \infty \) with the involution \( z \mapsto -z \) for \( z \in \mathbb{R}^n \). The inclusion of \( K(sR) \cong S^0_l \wedge K(sR) \) in \( S^0_l \wedge K(sR) \) induces homotopy equivalence \( t^n \) of Tate spectra,

\[
t^n: K(sR)^{thZ/2} \to (S^0_l \wedge K(sR))^{thZ/2}.
\]

**Definition 8.1.** Writing \( sR = sR(X) \), let \( VL^\bullet(X, \xi, n) = VLA^\bullet(sR, n) \) be the homotopy pullback of

\[
VL^\bullet(sR) \xrightarrow{t^n\Xi} (S^0_l \wedge K(sR))^{thZ/2} \xleftarrow{\text{inclusion}} (S^0_l \wedge K(sR))^{thZ/2}.
\]

For \( \ell = 0, 1, 2, \ldots \) let \( sR^{(\ell)} \) be \( sR \) with the stable \( SW \)-product obtained from the standard \( \xi \)-twisted stable \( SW \)-product in \( sR \) by \( \ell \)-fold looping. That is to say, \( Y_1 \circ_{m} Y_2 \) in \( sR^{(\ell)} \) means the same as \( \Omega^{\ell}(Y_1 \circ_{m} Y_2) \) in \( sR \).

**Lemma 8.2.** There is a chain of natural homotopy equivalences relating the spectra \( VLA^\bullet(sR^{(n)}, 0) \) and \( \Omega^n VLA^\bullet(sR, n) \).

**Proof.** Let \( C \) be any Waldhausen category with a stable \( SW \)-product (subject to the axioms of [34, §2], which include the existence of a cylinder functor). Then \( S_2C \) (the second term of the Waldhausen \( S_\bullet \) construction) inherits a stable \( SW \) product [34, §6]. Essentially the objects of \( S_2C \) are the cofibrations \( C \to D \) in \( C \) and there are functors

\[
\iota: C \to S_2C \ ; \ \iota(D) = (\ast \to D), \quad \partial: S_2C \to C \ ; \ \partial(C \to D) = D.
\]

The second of these respects the stable \( SW \) products. The first one does, too, if we think of \( C \) as \( C^{(1)} \), looping the stable \( SW \) product on \( C \) once. By a routine application of Waldhausen’s approximation and fibration theorems [25], or alternatively by applying his additivity theorem, one deduces a homotopy pullback square of spectra (here with involution)

\[
\begin{array}{ccc}
K(C^{(1)}) & \xrightarrow{\iota_*} & K(S_2C) \\
\downarrow & & \downarrow \partial_* \\
\ast & \longrightarrow & K(C).
\end{array}
\]
This remains a homotopy pullback square after termwise application of \((-)^{h\mathbb{Z}/2}\) or \((-)^{h\mathbb{Z}/2}\). By inspection of homotopy groups, the analogous square

\[
\begin{array}{ccc}
L^\bullet(C^{(1)}) & \longrightarrow & L^\bullet(S_2C) \\
\downarrow & & \downarrow \\
* & \longrightarrow & L^\bullet(C).
\end{array}
\]

is also a homotopy pullback square. Putting these observations together, and adapting the notation introduced in definition 8.1 slightly, we have a homotopy pullback square of \(LK^\bullet\)-spectra

\[
\begin{array}{ccc}
LK^\bullet(C^{(1)}, 0) & \longrightarrow & LK^\bullet(S_2C, 0) \\
\downarrow & & \downarrow \\
* & \longrightarrow & LK^\bullet(C, 0).
\end{array}
\]

This gives an identification

\[
LK^\bullet(C^{(1)}, 0) \simeq \Omega \text{ cone}[\partial: LK^\bullet(S_2C, 0) \rightarrow LK^\bullet(C, 0)].
\]

We now simplify the right-hand side of this identification. By inspection of the homotopy groups, the \(L\)-theory \(L^\bullet(S_2C, 0)\) is contractible. It follows from the additivity theorem that \(K(S_2C, 0)\) is an “induced” spectrum with \(\mathbb{Z}/2\)-action \([34, 9.4]\). More precisely, there is a preferred (non-equivariant) splitting \(u\) of the forgetful map \(\partial: K(S_2C, 0) \rightarrow K(C, 0)\), and this extends to a homotopy equivalence

\[
(u, Tu): K(C) \vee K(C) \longrightarrow K(S_2C, 0)
\]

where \(T\) denotes the (action of the) generator of \(\mathbb{Z}/2\). Hence

\[
\partial: LK^\bullet(S_2C, 0) \rightarrow LK^\bullet(C, 0)
\]

becomes the map between the homotopy pullbacks of the rows in the following commutative diagram:

\[
\begin{array}{ccc}
* & \longrightarrow & (K(C) \vee K(C))^{h\mathbb{Z}/2} \\
\downarrow & & \downarrow \\
L^\bullet(C, 0) & \longrightarrow & (K(C), 0)^{h\mathbb{Z}/2}.
\end{array}
\]

Taking mapping cones of the vertical maps in this diagram, we obtain

\[
\begin{array}{ccc}
L^\bullet(C, 0) & \longrightarrow & (S^1 \wedge K(C, 0))^{h\mathbb{Z}/2} \\
\downarrow & & \downarrow \\
\end{array}
\]

and the homotopy pullback is precisely \(LK^\bullet(C, 1)\). Combining this with the above, we have

\[
LK^\bullet(C^{(1)}, 0) \simeq \Omega LK^\bullet(C, 1) = \Omega LK^\bullet(C^{(0)}, 1)
\]

and consequently, by iteration,

\[
LK^\bullet(C^{(n)}, 0) \simeq LK^\bullet(C^{(n-1)}, 1) \simeq LK^\bullet(C^{(n-2)}, 2) \simeq \cdots \simeq LK^\bullet(C^{(0)}, n).
\]

In the case where \(C = s\mathcal{R} = s\mathcal{R}(X)\), we can add a prefix \(V\) for “visible” where applicable, and the argument goes through.

Now let \(X\) be a finitely dominated Poincaré duality space of formal dimension \(n\), with Spivak normal bundle \(\xi: E \rightarrow X\) and a choice of a stable map \(\eta: S^{n+d} \rightarrow E/X\) such that \([\eta]\) maps to a fundamental class for \(X\) under the Thom isomorphism.
Together, therefore be regarded as the Spivak normal fibration of $s$.

Let $F$ as $\Omega_\infty$ lemma 8.2.) To construct $\sigma$ viewed as a retractive space over $\phi$. This defines an element $v_L$, in the standard simplicial set model for $\Omega_\infty^\infty \mathbf{V}L^* \mathbf{K}(X,\xi,n)$. They also determine a (homotopy) fixed point for the duality involution on $\Omega_\infty^\infty \mathbf{K}(s\mathcal{R}(n)^h)$. The images of $v_L$ and $v_K$ respectively under $\Xi$ and the inclusion $\Omega_\infty^\infty (\mathbf{K}(s\mathcal{R}(n)^h)) \rightarrow \Omega_\infty^\infty (\mathbf{K}(s\mathcal{R}(n)^{h\mathbb{Z}/2}))$

defines an element $\varphi$ in $(S^0 \times X) \circ_0 (S^0 \times X)$, for the $SW$ product in $s\mathcal{R}(n)$. This is fixed under the symmetry involution and nondegenerate, and so $S^0 \times X$ and $\varphi$ together determine a vertex, say $v_L$, in the standard simplicial set model for $\Omega_\infty^\infty \mathbf{V}L^* (s\mathcal{R}(n)^h)$. This was defined, following [34, §9], as the geometric realization of a $\Delta$-set $[m] \mapsto \text{vsp}_0(s\mathcal{R}(n)^h(m))$.

Here $s\mathcal{R}(n)^h(m)$ is a category of certain functors from the poset of faces of $\Delta^m$ to $s\mathcal{R}(n)^h$. The notation $\text{vsp}_0(s\mathcal{R}(n)^h(m))$ means: the set of visible symmetric Poincaré objects of formal dimension zero in $s\mathcal{R}(n)^h(m)$, that is, objects of $s\mathcal{R}(n)^h(m)$ with an appropriate nondegenerate visible symmetric structure. Let $\text{vsp}_0'(s\mathcal{R}(n)^h(m))$ be the classifying space of the category of visible symmetric Poincaré objects of formal dimension zero in $s\mathcal{R}(n)^h(m)$. A morphism between visible symmetric Poincaré objects of formal dimension zero in $s\mathcal{R}(n)^h(m)$ is a morphism in $s\mathcal{R}(n)^h(m)$ respecting the visible symmetric structures; such a morphism is automatically a weak equivalence in $s\mathcal{R}(n)^h(m)$. Then $[m] \mapsto \text{vsp}_0'(s\mathcal{R}(n)^h(m))$

is a $\Delta$-space. Its geometric realization is an enlarged version of the above construction of the 0-th infinite loop space in the $\Omega$-spectrum $\mathbf{V}L^* (s\mathcal{R}(n)^h)$. It is homotopy
equivalent to the original, by a standard argument which exploits the fact that several ways of realizing a bisimplicial set give the same result. The map \( \Xi \) extends easily to this enlarged version. Consequently we end up with an enlarged version of \( F(X, \xi, n) \). Using this, it is easy to promote \( \sigma(X) \in F(X, \xi, n) \) of definition 8.3 to a characteristic.

Namely, let \( X_i \) for \( i = 0, 1, \ldots, k \) be Poincaré duality spaces of formal dimension \( n \), with Spivak normal fibrations \( \xi_i : E_i \to X \) with fibers \( \simeq S^d \) where

\[
d_k \leq d_{k-1} \leq \cdots \leq d_0
\]

and preferred zero sections which are fiberwise cofibrations. Let \( \eta_i : S^{n+d_i} \to E_i/X_i \) be stable maps representing fundamental classes for the \( X_i \). Let homotopy equivalences \( u_i : X_i \to X_{i-1} \) be given for \( i = 1, \ldots, k \), covered by maps

\[
v_i : \Sigma^{d_i-1-d_i} E_i \longrightarrow E_{i-1}
\]

respecting the zero sections, and such that \( v_i \eta_i = \eta_{i-1} \). We can describe these data by a diagram

\[
\begin{array}{c}
(X_0, \xi_0, \eta_0) \leftarrow (X_1, \xi_1, \eta_1) \leftarrow \cdots \leftarrow (X_{k-1}, \xi_{k-1}, \eta_{k-1}) \leftarrow (X_k, \xi_k, \eta_k).
\end{array}
\]

With the new definition of \( F \), the diagram determines an \( n \)-simplex in \( F(X_0, \xi_0, n) \), which we could call the characteristic of the diagram. This assignment extends definition 8.3 and it has the naturality properties which make it into a characteristic (on a certain category \( \mathcal{P}_n \)) as defined in [9, 1.1]. The objects of \( \mathcal{P}_n \) are triples \( (X, \xi, \eta) \) as above. It is very fortunate that we can allow continuous variation of \( \eta \); that is to say, the characteristic \( \sigma \) depends continuously on the “reductions” \( \eta \).

We spell out what [9, 1.6] means here. Let \( p : Y \to B \) be a fibration whose fibers \( Y_b \) are finitely dominated Poincaré duality spaces of formal dimension \( n \). We assume for simplicity that \( B \) is a simplicial complex. We also need a fiberwise Spivak normal fibration. Suppose that this comes in the shape of a fibration \( \xi : E \to Y \) with preferred section, with fibers \( \simeq S^d \), and a map

\[
\eta : B_k \wedge S^{n+d} \longrightarrow E/\sim
\]

over \( B \), where \( E/\sim \) is the pushout of \( E \leftarrow Y \to B \). Every simplex \( K \subset B \) determines a Poincaré duality space \( Y_K = p^{-1}(K) \) with Spivak normal fibration \( \xi_K : E_K \to Y_K \), where \( E_K \) is the portion of \( E \) above \( K \), and a family of reductions \( \eta_b : S^{n+d} \to E_K/Y_K \) where \( b \in K \). These data determine a map

\[
\sigma(Y_K) : K \to F(Y_K, \xi_K, n)
\]

using the continuity property of \( \sigma \). As \( K \) varies, we have maps

\[
\operatorname{hocolim}_K K \longrightarrow \operatorname{hocolim}_K F(Y_K, \xi_K, n) \longrightarrow \operatorname{hocolim}_K \ast.
\]

The space on the right is the barycentric subdivision of \( B \); the map on the right is a quasifibration which we can also describe as \( F_B(Y, \xi, n) \to B \); the composite map from left to right is a homotopy equivalence. Hence the map on the left can be viewed as a “homotopy section” \( \sigma(p) \) of \( F_B(Y, \xi, n) \to B \), and this is what we wanted.

Next we need a generalization or adaptation of definitions 8.1 and 8.3 to the controlled setting. Let \( \bar{X} \) be a control space with compact \( X \), let \( n \) be an integer \( \geq 0 \) and let \( \xi \) be a spherical fibration on \( X \), with fibers \( \simeq S^d \) and with a distinguished section which is a fiberwise cofibration. We have the \( \xi \)-twisted stable
SW-product on $s\mathcal{R} = s\mathcal{R}^\text{id}(\bar{X}, X)$ from section 7. This determines a $\xi$-twisted duality involution on $K(s\mathcal{R})$.

**Definition 8.4.** With $s\mathcal{R} = s\mathcal{R}^\text{id}(\bar{X}, X)$, let $\text{VLA}^\bullet((\bar{X}, X), \xi, n) = \text{VLK}^\bullet(s\mathcal{R}, n)$ be the homotopy pullback of

$$
\text{VL}^\bullet(s\mathcal{R}) \xrightharpoondown_{\text{inclusion}} (S^n \wedge K(s\mathcal{R}))^{th\mathbb{Z}/2} \leftarrow (S^n \wedge K(s\mathcal{R}))^{h\mathbb{Z}/2}.
$$

For $\ell = 0, 1, 2, \ldots$ let $s\mathcal{R}^{(\ell)}$ be $s\mathcal{R}$ with the stable SW-product obtained from the standard $\xi$-twisted stable SW-product in $s\mathcal{R} = s\mathcal{R}^\text{id}(\bar{X}, X)$ by $\ell$-fold delooping. Lemma 8.2 remains correct for this $s\mathcal{R}$. Therefore if we define $F((\bar{X}, X), \xi, n)$ as the homotopy pullback of

$$
\Omega^\infty\text{VLA}^\bullet(s\mathcal{R}^{(n)}) \xrightharpoonup{\text{inclusion}} \Omega^\infty(K(s\mathcal{R}^{(n)})^{th\mathbb{Z}/2}) \leftarrow \Omega^\infty(K(s\mathcal{R}^{(n)})^{h\mathbb{Z}/2})
$$

we may still write informally $F((\bar{X}, X), \xi, n) = \Omega^\infty\text{VLA}^\bullet((\bar{X}, X), \xi, n))$. Now, in order to get a “signature” $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$, we have to throw in a finite domination assumption and a Poincaré duality assumption. Suppose therefore that $S^0 \times X$, as a retractive space over $X$, is finitely dominated in the controlled sense. Suppose that in addition to the data $(\bar{X}, X)$ and $\xi$, we are given a stable map

$$
\eta: S^{n+d} \rightarrow E/\!/X.
$$

There is a diagonal inclusion of $E/\!/X$ in $(S^0 \times X) \circ_0 (S^0 \times X)$ for the SW product in $s\mathcal{R}^{(n)}$ which we are considering. The composition of $\eta$ and the diagonal is then a $\delta$-dimensional visible symmetric structure on $S^0 \times X$ as an object of $s\mathcal{R}^{(n)}$.

**Definition 8.5.** If this is nondegenerate, we call $(\bar{X}, X)$ together with $\xi$ and $\eta$ a controlled Poincaré duality space. In that case let $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$ be the element determined (as in definition 8.3) by $S^0 \times X$ with the above nondegenerate visible symmetric structure.

The naturality properties of $\sigma(\bar{X}, X)$ are similar to those of $\sigma(X)$ in definition 8.3. The morphisms we are most interested in have the form

$$
((\bar{X}_0, X_0), (\xi_0, \eta_0)) \leftarrow ((\bar{X}_1, X_1), (\xi_1, \eta_1))
$$

with an underlying map $f: X_1 \rightarrow X_0$ which takes $X_1$ to $X_0$ and restricts to a homeomorphism from $\bar{X}_1 \setminus X_1$ to $\bar{X}_0 \setminus X_0$. Existence of an inverse up to homotopy $X_0 \rightarrow X_1$ is required; the inverse and the maps in the homotopy are subject to the same conditions as $f$.

**Remark 8.6.** Definition 8.5 is good enough for our purposes, but it is not exactly a generalization of definition 8.3 because of the compactness condition on $\bar{X}$. To obtain a true generalization of definition 8.3, we would need the following setup: a control space $(\bar{X}, X)$ with compact $\bar{X}$, a space $Y$, a map $c: Y \rightarrow X$ and a certain Waldhausen category (with Spanier-Whitehead duality) $s\mathcal{R}^\text{id}(Y; (\bar{X}, X))$ of retractive spectra over $Y$. The class of objects of $s\mathcal{R}^\text{id}(Y; (\bar{X}, X))$ depends on $c: Y \rightarrow X$ but not on the compactification $\bar{X}$ of $X$. The notion of weak equivalence in $s\mathcal{R}^\text{id}(Y; (\bar{X}, X))$ depends on both $c: Y \rightarrow X$ and the compactification $\bar{X}$ of $X$. The space $Y$ together with $c: Y \rightarrow X$ would qualify as a locally finitely dominated space over $(\bar{X}, X)$ if $Y \times S^0$ qualifies as an object of $s\mathcal{R}^\text{id}(Y; (\bar{X}, X))$. If that is the case, then the phrase $Y$ satisfies Poincaré duality with control in $(\bar{X}, X)$ can be given some meaning. (This setup specializes to the situation of definition 8.5 by taking $Y = X$ and $c = \text{id}_X$.)
The goal here is very easy to formulate:

(i) For a closed $n$-manifold $X$ with normal bundle $\xi$, we need to specify a preferred lift

$$\sigma^\%(X) \in F^\%(X, \xi, n)$$

of $\sigma(X) \in F(X, \xi, n)$, across a suitable assembly map.

(ii) For any compact control space $(X, X)$ in which $X$ is an $n$-manifold with normal bundle $\xi$ (without boundary but not necessarily compact), we need to specify a preferred lift

$$\sigma^\%(\bar{X}, X) \in F^\%(X, \xi, n)$$

of $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$, across a suitable assembly map.

To clarify, $\sigma(X) \in F(X, \xi, n)$ and $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$ are the characteristic elements of definition 8.3 and definition 8.5. The space $F^\%(X, \xi, n)$ is designed in such a way that $\pi_* F^\%(X)$ is the homology of $X$ with locally finite coefficients $\text{VLA}^*(x, \xi, n)$, depending on $x \in X$.

The details are slightly more complicated, but we follow [9, §7] closely. In the case (i) we define $F^\%(X, \xi, n)$ and the assembly map to $F(X, \xi, n)$ by means of a homotopy fiber sequence

$$F^\%(X, \xi, n) \longrightarrow F(X, \xi, n) \longrightarrow F(JX, \xi, n)$$

where $JX$ is the control space $(X \times [0, 1], X \times [0, 1])$. The second map in the sequence is induced by the inclusion of $X \simeq X \times \{0\}$ in $X \times [0, 1]$; the spherical fibration $\xi$ is extended to $X \times [0, 1]$ in the obvious way. We have the definition of $F(JX, \xi, n)$ from the end of the previous section. In the case (ii) we introduce

$$J(\bar{X}, X) = \left( \frac{\bar{X} \times [0, 1]}{\sim}, X \times [0, 1] \right)$$

where $\sim$ identifies points in $(\bar{X} \setminus X) \times [0, 1]$ with the same coordinate in $\bar{X} \setminus X$.

Then again $F^\%(\bar{X}, X, \xi, n)$ and the assembly map to $F((\bar{X}, X), \xi, n)$ are defined by means of a homotopy fiber sequence

$$F^\%(\bar{X}, X, \xi, n) \longrightarrow F((\bar{X}, X), \xi, n) \longrightarrow F(J(\bar{X}, X), \xi, n).$$

The second map in the sequence is induced by the inclusion of $(\bar{X} \times \{0\}, X \times \{0\})$ in $J(\bar{X}, X)$.

In case (i), this leaves us with the task of trivializing the image of $\sigma(X)$ under $F(X, \xi, n) \to F(JX, \xi, n)$. We also need to show that $F^\%(X, \xi, n)$ as defined above is excisive etc., so that the forgetful map from $F^\%(X, \xi, n)$ to $F(X, \xi, n)$ can be called an assembly map. In the more general case (ii), our task is to trivialize the image of $\sigma(\bar{X}, X)$ under $F((\bar{X}, X), \xi, n) \to F(J(\bar{X}, X), \xi, n)$, and to establish excision properties etc. for $F^\%(\bar{X}, X, \xi, n)$.

**Trivializing $\sigma(X)$ in $F(JX, \xi, n)$**. We use $F(J*, \zeta, 0)$ where $*$ means “a point” and $\zeta$ is the trivial spherical fibration (with fiber $S^0$). We have $\sigma(*) \in F(J*, \zeta, 0)$. Multiplication with $\sigma(X)$ can be regarded as a based map

$$F(J*, \zeta, 0) \longrightarrow F(JX, \xi, n)$$
which takes $\sigma(*)$ to $\sigma(X)$. Hence it is enough to show that $F(\mathbb{J}*, \zeta, 0)$ is contractible. This can be proved by a standard Eilenberg swindle argument. Think of $\mathbb{J}*$ as $([0, \infty), [0, \infty[)$. Translation by $+1$ is an endofunctor $f$ of $sR(\mathbb{J}*)$. Let

$$g = \bigvee_{i=0}^{\infty} f^i$$

which is again an endofunctor of $sR(\mathbb{J}*)$. For the induced self-maps $f_*$ and $g_*$ of $F(\mathbb{J}*, \zeta, 0)$, we clearly have $f_* \simeq \text{id}$, hence $g_*f_* \simeq g_*$. But since $g \simeq \text{id} \lor gf$, it is also true that $g_* \sim \text{id} + g_*f_*$. Hence the identity of $F(\mathbb{J}*, \zeta, 0)$ is nullhomotopic. □

Remark. This line of reasoning simplifies [9, 7.8]. Observe that it is precisely the manifold property of $X$ which ensures that there is a map multiplication with $\sigma(X)$ from $F(\mathbb{J}*, \zeta, 0)$ to $F(\mathbb{J}X, \xi, n)$. The construction does therefore not generalize to Poincaré duality spaces $X$.

Remark. What the argument really gives us is a lift of $\sigma(X) \in F(X, \xi, n)$ to an element $\sigma(\overline{X})$ in the homotopy pullback of the following diagram of pointed spaces:

$$\begin{array}{ccc}
F(X, \xi, n) & \xrightarrow{\text{incl.}} & F(\mathbb{J}X, \xi, n) \\
\times \sigma(X) & \downarrow & \downarrow \\
F(\mathbb{J}*, \zeta, 0) & \rightarrow & F(\mathbb{J}(\overline{X}, X), \xi, n).
\end{array}$$

Since the right–hand term is contractible, this homotopy pullback is an acceptable substitute for the homotopy fiber of $F(X, \xi, n) \rightarrow F(\mathbb{J}X, \xi, n)$.

Trivializing $\sigma(\overline{X}, X)$ in $F(\mathbb{J}(\overline{X}, X), \xi, n)$. Again we use $F(\mathbb{J}*, \zeta, 0)$ and the element $\sigma(*) \in F(\mathbb{J}*, \zeta, 0)$. Multiplication with $\sigma(\overline{X}, X)$ is a based map $F(\mathbb{J}*, \zeta, 0) \rightarrow F(\mathbb{J}(\overline{X}, X), \xi, n)$ which takes $\sigma(*)$ to $\sigma(X)$. Since $F(\mathbb{J}*, \zeta, 0)$ is contractible, this achieves the trivialization. □

In order to establish excision properties for $F(\mathbb{J}X, \xi, n)$, we begin by clarifying how the categories

$$sR(X), \ sR^\text{id}(\mathbb{J}X), \ sRG^\text{id}(\mathbb{J}X)$$

fit together. (Recall that the decoration “$\text{ld}$” stands for locally finitely dominated.) This will be done from the point of view of Waldhausen's fibration and approximation theorems [25, §1.6]. We assume for now that $X$ is a compact space, not necessarily a manifold. In addition to the standard subcategory $w(sR^\text{id}(\mathbb{J}X))$ of weak equivalences in $sR^\text{id}(\mathbb{J}X)$, we introduce a coarser notion of weak equivalence $\kappa$, that is, a larger subcategory $\kappa(sR^\text{id}(\mathbb{J}X))$. A morphism in $sR^\text{id}(\mathbb{J}X)$ is a $\kappa$-equivalence if its mapping cone is equivalent to zero in $sRG^\text{id}(\mathbb{J}X)$. Adopting Waldhausen’s notation, we write

$$sR^\text{id}(\mathbb{J}X)^\kappa$$

for the full subcategory of $sR^\text{id}(\mathbb{J}X)$ consisting of all objects $\kappa$-equivalent to the zero object. This is a Waldhausen category with the usual $w$-equivalences as the weak equivalences. We also write

$$sR^\text{id}(\mathbb{J}X)_\kappa$$

for $sR^\text{id}(\mathbb{J}X)$ equipped with the coarser notion of weak equivalence, that is, $\kappa$-equivalence.
Lemma 9.1. The forgetful functor $s\mathcal{R}^{id}(\mathcal{J})_\kappa \to s\mathcal{R}^{id}(\mathcal{J})$ satisfies the hypotheses App1 and App2 of Waldhausen’s approximation theorem.

Proof. Property App1 means that morphisms which are taken to weak equivalences by the functor in question are already weak equivalences. This is trivially true in our case. To establish App2 we fix a morphism $f : Y_1 \to Y_2$ in $s\mathcal{R}^{id}(\mathcal{J})$, viewing $Y_1$ as an object of $s\mathcal{R}^{id}(\mathcal{J})_\kappa$. We need to show that $f$ has a factorization

$$Y_1 \xrightarrow{g} \bar{Y}_2 \xrightarrow{h} Y_2$$

where $g$ is a morphism in $s\mathcal{R}^{id}(\mathcal{J})_\kappa$ and $h$ is a weak equivalence in $s\mathcal{R}^{id}(\mathcal{J})$. This will be called an App2 factorization. (Waldhausen’s App2 also requires that $g$ be a cofibration. But if $g$ is not a cofibration, it can easily be converted into one by means of a mapping cylinder construction, so there is no need to pay any attention to that.) Note that

- if $f$ admits an App2 factorization, and $j : Y_2 \to Y_3$ is any weak equivalence in $s\mathcal{R}^{id}(\mathcal{J})$, then $jf : Y_1 \to Y_3$ also admits an App2 factorization;
- if $e : Y_0 \to Y_1$ is a cofibration and a weak equivalence in $s\mathcal{R}^{id}(\mathcal{J})$, then $f$ admits an App2 factorization if and only if $fe$ admits an App2 factorization.

Using these observations and the definition of “locally finitely dominated”, it is not very hard to reduce to a situation where $f$ can be factorized

$$Y_1 \xrightarrow{f_1} Z \xrightarrow{f_2} Y_2$$

in $s\mathcal{R}^{id}(\mathcal{J})$, with $Z$ of type “lf”, so that $Z$ has a locally finite controlled CW-structure relative to $X \times [0,1]$. We may also assume that $f_2$ is a morphism in $s\mathcal{R}^{id}(\mathcal{J})$; if not, replace $Z$ by a sufficiently small relative CW-subspace of $Z$ such that the subspace inclusion is still an isomorphism in $s\mathcal{R}^{id}(\mathcal{J})$. As regards $f_1$, we know that it is a “germ”, defined over $U \cap X \times [0,1]$ for some neighborhood $U$ of $X \times \{1\}$ in $X \times [0,1]$. We can find a smaller neighborhood $V$ of $X \times \{1\}$ in $X \times [0,1]$ and a relative CW-subspace $Z_0$ of $Z$ containing all cells of $Z$ whose image under the retraction meets the complement of $U$, and no cells of $Z$ whose image under the retraction meets $V$. Now let $\bar{Y}_2$ be the double mapping cylinder (homotopy pushout) of the diagram

$$Z/Z_0 \xleftarrow{f_1} Z \xrightarrow{f_2} Y_2$$

in $s\mathcal{R}^{id}(\mathcal{J})$. Let $g : Y_1 \to \bar{Y}_2$ be the composition of $f_1$ with $Z \to Z/Z_0$ and the inclusion $Z/Z_0 \to \bar{Y}_2$. Note that $g$ is everywhere defined, i.e., is a morphism in $s\mathcal{R}^{id}(\mathcal{J})$. On the other hand, $\bar{Y}_2$ is canonically isomorphic in $s\mathcal{R}^{id}(\mathcal{J})$ to the mapping cylinder of $f_2$, so that there is a preferred cylinder projection $h$ from $\bar{Y}_2$ to $Y_2$ in $s\mathcal{R}^{id}(\mathcal{J})$. Then we have $f = hg$ as required. \qed

Lemma 9.2. The maps from $K(s\mathcal{R}(X))$ to $K(s\mathcal{R}^{id}(\mathcal{J})^\kappa)$ and from $\mathbf{VL}^\bullet(s\mathcal{R}(X))$ to $\mathbf{VL}^\bullet(s\mathcal{R}^{id}(\mathcal{J})^\kappa)$ induced by the inclusion $s\mathcal{R}(X) \to s\mathcal{R}^{id}(\mathcal{J})^\kappa$ are homotopy equivalences.

Proof. For the $K$-theory case, apply Waldhausen’s approximation theorem to the inclusion $s\mathcal{R}(X \times [0,1]) \to s\mathcal{R}^{id}(\mathcal{J})^\kappa$. The inclusion of $s\mathcal{R}(X)$ in $s\mathcal{R}(X \times [0,1])$ clearly also induces a homotopy equivalence in $K$-theory. For the $\mathbf{VL}^\bullet$-theory case, argue similarly, using an $\mathbf{VL}^\bullet$-theory version of the approximation theorem. (This has essentially the same hypotheses App1 and App2. The conclusion, that the
functor in question induces a homotopy equivalence of $\mathbf{V}L^\bullet$-spectra, comes from a direct comparison of homotopy groups.)

**Corollary 9.3.** The inclusions $sR(X) \to sR^{ld}(JX)$ and the “passage to germ” functor $sR^{ld}(JX) \to sR^{ld}(JX)$ lead to homotopy fiber sequences of spectra,

$$A(X) \longrightarrow A(JX) \longrightarrow A(JX_\infty) \, ,$$

$$\mathbf{V}L^\bullet(X, \xi, n) \longrightarrow \mathbf{V}L^\bullet(JX, \xi, n) \longrightarrow \mathbf{V}L^\bullet(JX_\infty, \xi, n).$$

**Proof.** By Waldhausen’s fibration theorem there is a homotopy fiber sequence

$$\mathbf{K}(sR^{ld}(JX)^{\kappa}) \longrightarrow \mathbf{K}(sR^{ld}(JX)) \longrightarrow \mathbf{K}(sR^{ld}(JX)_\kappa).$$

With the definition $A(JX) = \mathbf{K}(sR^{ld}(JX))$ and the identifications of lemma 9.1 and lemma 9.2, this turns into a homotopy fiber sequence

$$A(X) \longrightarrow A(JX) \longrightarrow A(JX_\infty).$$

The same reasoning applies in the $\mathbf{V}L^\bullet$-theory case. Of course one needs to know that Waldhausen’s fibration theorem has an analogue in $L$-theory. There is such an analogue, as follows.

Let $\mathcal{C}$ be any Waldhausen category with weak equivalences $w\mathcal{C}$ and with an $SW$ product $\odot$ satisfying the axioms of [34, §2]. Suppose that another subcategory $\kappa \mathcal{C}$ of $\mathcal{C}$ with $w\mathcal{C} \subset \kappa \mathcal{C} \subset \mathcal{C}$ is specified. Suppose that $\mathcal{C}$ with weak equivalences $\kappa \mathcal{C}$ and the same $SW$ product $\odot$ also satisfy the axioms of [34, §2]. As before, let $\mathcal{C}_\kappa$ stand for $\mathcal{C}$ with the coarse notion $\kappa$ of weak equivalences, and $\mathcal{C}^{\kappa}$ for the full Waldhausen subcategory of $\mathcal{C}$ consisting of the objects which are $\kappa$-equivalent to zero. Then there is a homotopy cartesian square of spectra

$$\mathbf{L}^\bullet(\mathcal{C}^{\kappa}) \longrightarrow \mathbf{L}^\bullet(\mathcal{C})$$

$$\downarrow \quad \downarrow$$

$$\mathbf{L}^\bullet((\mathcal{C}^{\kappa})_\kappa) \longrightarrow \mathbf{L}^\bullet(\mathcal{C}_\kappa)$$

with contractible lower left-hand term. We shorten this as usual to a homotopy fiber sequence

$$\mathbf{L}^\bullet(\mathcal{C}^{\kappa}) \longrightarrow \mathbf{L}^\bullet(\mathcal{C}) \longrightarrow \mathbf{L}^\bullet(\mathcal{C}_\kappa).$$

This is the basic fibration theorem for symmetric $L$-theory. There is a version for quadratic $L$-theory and also one for visible symmetric $L$-theory, when that is defined.

The proof of the symmetric $L$-theory fibration theorem is as follows, in outline. Suppose that $(C, D, \varphi)$ is a symmetric Poincaré pair in $\mathcal{C}$. In more detail, we assume that $D$ and $C$ are related by a cofibration $D \to C$ and that $\varphi$ is a homotopy fixed point for the action of $\mathbb{Z}/2$ on $P^n(C \odot C/D \odot D)$, satisfying the appropriate nondegeneracy condition. Then $n$ is the formal dimension of the Poincaré pair.

The symmetric structure $\varphi$ on the pair $(C, D)$ descends to a symmetric structure $\varphi/\partial \varphi$ on $C/D$, which may be degenerate. It is a basic fact of Ranicki’s algebraic theory of surgery that the passage from the Poincaré pair $(C, D, \varphi)$ to the (single) symmetric object $(C/D, \varphi/\partial \varphi)$ is reversible. We have already used it in the proof of theorem 4.5; see [17], [28] and [23] for more details on the inverse construction. These details on the inverse construction imply that $C/D$ with the symmetric structure $\varphi/\partial \varphi$ is nondegenerate in $\mathcal{C}_\kappa$ if and only if $D$ belongs to $\mathcal{C}^{\kappa}$. Hence the bordism theory of Poincaré pairs $(C, D, \varphi)$ with $C$ in $\mathcal{C}$ and $D$ in $\mathcal{C}^{\kappa}$ is “equivalent”
to the bordism theory of Poincaré objects in $C_\kappa$. This amounts to a homotopy equivalence of spectra, from the homotopy cofiber of
\[ L^\bullet(C_\kappa) \to L^\bullet(C) \]
to $L^\bullet(C_\kappa)$. That in turn can be reformulated as a homotopy fiber sequence of spectra $L^\bullet(C_\kappa) \to L^\bullet(C) \to L^\bullet(C_\kappa)$.

**Corollary 9.4.** The inclusions $sR(X) \to sR^\text{ld}(JX)$ and the "passage to germ" functor $sR^\text{ld}(JX) \to sR^\text{ld}(JX)$ lead to a homotopy fiber sequence of spectra $VLA^\bullet(X,\xi,n) \to VLA^\bullet(JX,\xi,n) \to VLA^\bullet(JX,\xi,n)$.

**Proof.** What we are really saying is that $VLK^\bullet$ turns the square of Waldhausen categories with $SW$-duality $sR^\text{ld}(JX)$ into a homotopy cartesian square of spectra, with contractible lower left-hand term. This follows directly from the analogous statements for $K$-theory and $VL^\bullet$-theory, which we have from the previous corollary, and the definition of $VLK^\bullet$ in terms of $K$ and $VL^\bullet$. $\square$

**Excision properties of $F^\%((\bar{X},X),\xi,n)$.** This is similar to the case of $F^\%(X,\xi,n)$. We assume that $\bar{X}$ is compact; we need not assume that $X$ is a manifold. There is a commutative square of Waldhausen categories
\[ sR^\text{ld}(\bar{X},X) \to sR^\text{ld}(\bar{X},X) \]
\[ \to sR^\text{ld}(\bar{X},X) \]
which generalizes ($\ast$). Waldhausen’s fibration theorem can be applied to this, after some minor redefinitions. Hence $F^\%(\bar{X},X,\xi,n)$ is identified (via a chain of natural homotopy equivalences) with
\[ \Omega^{\infty+1}VLK^\bullet(sR^\text{ld}(\bar{X},X)) \]
Finally we note that $sR^\text{ld}(\bar{X},X)$ is exactly the same as $sR^\text{ld}(\bar{X},X)$. Therefore
\[ F^\%(\bar{X},X,\xi,n) \simeq \Omega^{\infty+1}VLK^\bullet(sR^\text{ld}(\bar{X},X)) \]
is essentially a functor of the locally compact space $X$ alone, and as such it is excisive in the locally finite sense of theorem 4.5 and theorem 4.14. Indeed this follows from theorem 4.14 and the analogous theorem for $K$-theory, [30, §6-9]. □

Naturality properties of $\sigma^\% (X)$. The naturality properties of $\sigma^\%$ are analogous to those of $\sigma$ in the previous section. That is, a diagram of closed manifolds, homeomorphisms and stable normal bundle isomorphisms

$$(X_0, \xi_0) \leftarrow (X_1, \xi_1) \leftarrow \cdots \leftarrow (X_{k-1}, \xi_{k-1}) \leftarrow (X_k, \xi_k)$$

determines a $k$-simplex in $F^\% (X_0, \xi_0, n)$. This assignment extends $\sigma^\%$ and commutes with the usual face and degeneracy operators acting on such diagrams. (We omit the details, except for pointing out that each $\xi_i$ can be viewed as a spherical fibration on $X_i$ via fiberwise one-point compactification.)

Naturality properties of $\sigma^\% (\bar{X}, X)$. Let $(\bar{X}_i, X_i)$ for $i = 0, 1, \ldots, k$ be compact control spaces where each $X_i$ is an $n$-manifold. Suppose that they are arranged in a diagram of homeomorphisms of control spaces and stable normal bundle isomorphisms

$$(\bar{X}_0, X_0, \xi_0) \leftarrow ((\bar{X}_1, X_1), \xi_1) \leftarrow \cdots \leftarrow ((\bar{X}_{k-1}, X_{k-1}), \xi_{k-1}) \leftarrow ((\bar{X}_k, X_k), \xi_k).$$

The diagram then determines a $k$-simplex in $F^\% ((\bar{X}_0, X_0), \xi_0)$. This assignment commutes with the usual face and degeneracy operators acting on such diagrams.

10. Discrete homeomorphism groups

Definition 10.1. A stratified space is a space $X$ together with a locally finite partition into locally closed subsets $X_i$, called the strata, such that the closure of each stratum $X_i$ in $X$ is a union of strata. An automorphism of a stratified space $X$ is a homeomorphism $X \rightarrow X$ mapping each stratum $X_i$ to itself.

Definition 10.2. A stratified space is a TOP stratified space if it is paracompact Hausdorff and each stratum $X_i$ is a manifold (of some dimension $n_i$, with empty boundary).

For us, the most important example of a TOP stratified space is the join $X = M \ast S^{i-1}$, where $M$ is a closed manifold. We partition this into two strata. One of these is the embedded copy of $S^{i-1}$. The other stratum, i.e. the complement of $S^{i-1}$ in $M \ast S^{i-1}$, can be identified with $M \times \mathbb{R}^i$. Automorphisms of $M \ast S^{i-1}$, with this stratification, must map the sphere $S^{i-1}$ to itself. (This is not a vacuous condition since, for example, $M$ could also be a sphere in which case $M \ast S^{i-1}$ is homeomorphic to a sphere.)

Another important type of stratified space with two strata is furnished by manifolds with boundary. The boundary can be regarded as one stratum, the complement of the boundary as the other stratum. Combining these two examples, one has a canonical stratification of $M \ast S^{i-1}$ into three strata when $M$ is a manifold with nonempty boundary.

Definition 10.3. The open cone $cL$ on a stratified space $L$ is defined as the quotient of $L \times [0, 1]$ by $L \times \{0\}$. It is canonically stratified with strata $cL \setminus \ast$ and $\ast$, where $L_i$ denotes a stratum of $L$ and $\ast$ is the base of the cone.

The next definition is due to Siebenmann [22]:
Definition 10.4. A stratified space $X$ is locally conelike if, for each stratum $X_i$ and each $x \in X_i$, there exist an open neighborhood $U$ of $x$ in $X_i$, a compact stratified space $L$ and a stratification–preserving homeomorphism (relative to $U$) of $cL \times U$ with an open neighborhood of $x$ in $X$.

Notation 10.5. A locally conelike TOP stratified space will be called a CS space. (This is slightly different from Siebenmann’s definition of a CS space, in which there can be only one stratum of dimension $n$ for each $n \geq 0$, but the partition into strata is not required to be locally finite.) Generalizing some of the conventions of [14], we denote by $\mathcal{H}(X)$ and $\text{Hom}(X)$ the topological group of automorphisms of a CS space $X$ (with the compact–open topology), and the underlying discrete group, respectively. More generally, for a closed subset $A$ of $X$ let $\mathcal{H}(X, \text{rel } A)$ and $\text{Hom}(X, \text{rel } A)$ be the topological group and the underlying discrete group of automorphisms of $X$ which agree with the identity in some neighborhood of $A$. Following [14], we topologize $\mathcal{H}(X, \text{rel } A)$ as a direct limit

$$\text{colim}_U \left\{ h \in \mathcal{H}(X) \left| h(x) = x \text{ for } x \in U \right. \right\}$$

where $\left\{ h \in \mathcal{H}(X) \left| h(x) = x \text{ for } x \in U \right. \right\}$ has the subspace topology inherited from $\mathcal{H}(X)$ and $U$ runs over the set of all open neighborhoods of $A$ in $X$. Let $B\mathcal{H}(X, \text{rel } A)$ be the homotopy fiber of the inclusion $B\mathcal{H}(X, \text{rel } A) \rightarrow B\mathcal{H}(X, \text{rel } A)$.

Lemma 10.6. For any compact $K \subset \mathcal{H}(X, \text{rel } A)$, there exists a neighborhood $U$ of $A$ in $X$ such that $f(U) = \text{id}$ for all $f \in K$. □

Definition 10.7. A clean subspace of a stratified space $X$ is a closed subspace $Y$ of $X$ whose frontier $\text{Fr}(Y)$ in $X$ admits a stratification and a bicollar neighborhood $V \cong \text{Fr}(Y) \times \mathbb{R}$ in $X$ (where the homeomorphism $V \cong \text{Fr}(Y) \times \mathbb{R}$ respects the stratifications.)

The only example of a clean subspace of a stratified space which we really need is as follows. Let $X$ be a locally conelike stratified space. Let $z$ be a point in a stratum $X_i$, let $U$ be a neighborhood of $x$ in $X_i$ and let $e : cL \times U \rightarrow X$ be an open embedding as in definition 10.4. Let $U' \subset U$ be an open ball containing $z$ whose closure in $U$ is a (compact) disk and let $e' L \subset cL$ be an open subcone $L \times [0, r]$ for some $r$ with $0 < r < \infty$. We call the image of $e' L \times U'$ under the embedding $e : cL \times U \rightarrow X$ a quasi–ball (about $z$) in $X$.

Lemma 10.8. The complement of a quasi–ball in a locally conelike stratified space $X$ is a clean subset of $X$.

Proof. The quasi–ball is contained in an open subset of $X$ which is identified with $cL \times U$. (We may as well assume $X = cL \times U$.) Without loss of generality, $U$ can be identified with a euclidean open ball of radius 1. Now the spaces $cL$ and $U$ come with evident actions of the topological monoid $[0, 1]$, a submonoid of $(\mathbb{R}, \cdot)$, and so we get a diagonal action of $[0, 1]$ on $cL \times U$. Without loss of generality the quasi–ball is $\frac{1}{2}(cL \times U)$. A bicollar for its frontier is then defined by the embedding

$$(x, t) \mapsto \frac{e^t}{e^t + 1} \cdot x \in cL \times U$$

for $t \in \mathbb{R}$ and $x$ in the frontier. □
Lemma 10.9. Any point in a Siebenmann stratified space has an open neighborhood which is a quasi–ball.

\[ \square \]

Theorem 10.10. Let \( A \) be closed in a CS space \( X \). Assume that the complement of \( A \) is a quasi–ball. Then \( B\text{Hom}(X, \text{rel } A) \) is acyclic, \( B\text{Hom}(X, \text{rel } A) \) is contractible and hence \( B\text{Hom}(X, \text{rel } A) \) is acyclic.

Proof. The contractibility of \( B\text{Hom}(X, \text{rel } A) \) is a consequence of an Alexander trick showing that in fact \( \text{Hom}(X, \text{rel } A) \) is contractible. Our proof that \( B\text{Hom}(X, \text{rel } A) \) is acyclic is a straightforward generalization of Mather’s proof in the unstratified case \cite{12}. As in the proof of lemma 10.8 we can assume that \( X = cL \times U \) and that the quasi–ball is \( E = \frac{1}{2} \cdot (cL \times U) \). We are looking at automorphisms \( h : cL \times U \to cL \times U \) which are the identity outside \( r_h \cdot (cL \times U) \) for some positive \( r_h < 1/2 \) depending on \( h \). Copying Mather’s argument in the unstratified case, we begin by choosing a sequence of disjointly embedded codimension zero disks \( D_i \subset \frac{1}{2}U \), for \( i = 1, 2, 3, \ldots \). Let \( U_i \) be the relative interior of \( D_i \) in \( U \). Then \( E_i = (2^{-i} \cdot cL) \times U_i \) is a quasi–ball in \( E \) for each \( i \), and \( E_1, E_2, E_3, \ldots \) have disjoint compact closures in \( E \). These choices can easily be made in such a way that there exists \( h_0 \in \text{Hom}(X, \text{rel } X \setminus E) = \text{Hom}(X, \text{rel } A) \) which maps \( E_i \) onto \( E_{i+1} \) for \( i = 1, 2, 3, \ldots \).

Let \( G_i = \text{Hom}(X, \text{rel } X \setminus E_i) \). The inclusion

\[ BG_i \to B\text{Hom}(X, \text{rel } X \setminus E) \]

induces a surjection in integer homology. The reasoning is that any finitely generated subgroup of \( \text{Hom}(X, \text{rel } X \setminus E) \) is conjugate in \( \text{Hom}(X, \text{rel } X \setminus E) \) to a subgroup of \( G_i \). (Namely, for any \( g_1, g_2, \ldots, g_k \) in \( \text{Hom}(X, \text{rel } X \setminus E) \) there is \( r < 1/2 \) such that \( g_1, \ldots, g_k \) agree with the identity outside \( r \cdot (cL \times U) \). Then there exists \( g \in \text{Hom}(X, \text{rel } X \setminus E) \) mapping the closure of \( r \cdot (cL \times U) \) to \( E_i \), so that conjugation with \( g \) takes \( g_1, \ldots, g_k \) to the subgroup \( \text{Hom}(X, \text{rel } X \setminus E_i) \).)

The next step is to introduce the subgroup \( G \) of \( \text{Hom}(X, \text{rel } X \setminus E) \) consisting of all automorphisms \( h \) which have \( h(x) = x \) for all \( x \notin \bigcup E_i \). Then the restriction homomorphisms \( h \mapsto h|E_i \) lead to isomorphisms

\[ G \cong \prod_{i=1}^{\infty} G_i \cong \prod_{i=1}^{\infty} G_1. \]

(The first isomorphism uses the fact that a bijective continuous map between compact Hausdorff spaces is a homeomorphism. The second isomorphism uses conjugation with appropriate powers of \( h_0 \).) Let \( \sigma : G \to \text{Hom}(X, \text{rel } X \setminus E) \) be the inclusion. Define homomorphisms

\[ u, v, w : G_1 \longrightarrow G \cong \prod_{i=1}^{\infty} G_1 \]

by \( u(g) = (g, 1, 1, \ldots) \), \( v(g) = (1, g, g, \ldots) \) and \( w(g) = (g, g, g, \ldots) \). Using conjugation with the element \( h_0 \), we see that \( \sigma v \) and \( \sigma w \) induce the same homomorphism in integral homology,

\[ \sigma_* u_* = \sigma_* w_* : H_*(BG_1) \longrightarrow H_*(B\text{Hom}(X, \text{rel } X \setminus E)). \]

Now assume inductively that our vanishing statement has been established in degrees \( s \) for \( 0 < s < k \). In particular \( H_*(B\text{Hom}(X, \text{rel } X \setminus E)) = 0 \) and \( H_*(BG_1) = 0 \) for \( 0 < s < k \). Writing \( w \) as a composition

\[ G_1 \xrightarrow{\text{diagonal}} G_1 \times G_1 \xrightarrow{id \times v'} G_1 \times G'. \]
where \( G' = \prod_{i=2}^\infty \), and using the Künneth formula for the homology of \( BG_1 \times BG_1 \), we get for any \( z \in H_k(BG_1) \) that
\[
w_z(z) = (z \times 1) + (1 \times v_z(z)) \in H_k(B(G_1 \times G')) = H_k(BG).
\]
Hence \( \sigma_* w_z(z) = \sigma_* u_z(z) + \sigma_* v_z(z) \) in \( H_k(B\text{Hom}(X, \text{rel } X \smallsetminus E)) \). Since \( \sigma_* u_z = \sigma_* w_z \), this means \( \sigma_* u_z(z) = 0 \) and since \( \sigma_* u_z \) is surjective this implies
\[
H_k(B\text{Hom}(X, \text{rel } X \smallsetminus E)) = 0. \quad \square
\]

**Notation 10.11.** Let \( Y \) be a clean subspace of a \( CS \) space \( X \) and let \( A \subset X \) be closed. We assume that \( \text{Fr}(Y) \smallsetminus A \) has compact closure in \( X \). Generalizing some of the notation in [14] again, we write
\[
\mathcal{Hom}(X, Y, \text{rel } A)
\]
for the topological submonoid of \( \mathcal{Hom}(X, \text{rel } A) \) consisting of the automorphisms \( X \to X \) in \( \mathcal{Hom}(X, \text{rel } A) \) which embed \( Y \) in itself. The underlying discrete monoid is \( \text{Hom}(X, Y, \text{rel } A) \). We write \( B\mathcal{Hom}(X, Y, \text{rel } A) \) for the homotopy fiber of the inclusion
\[
B\text{Hom}(X, Y, \text{rel } A) \to B\mathcal{Hom}(X, Y, \text{rel } A).
\]
The identity component of \( \mathcal{Hom}(X, Y, \text{rel } A) \) is denoted by \( \mathcal{Hom}_0(X, Y, \text{rel } A) \) and the underlying discrete monoid by \( \text{Hom}_0(X, Y, \text{rel } A) \). The set–theoretic quotients of \( \text{Hom}(X, Y, \text{rel } A) \) and \( \text{Hom}_0(X, Y, \text{rel } A) \) by the equivalence relation
\[
h \sim h' \iff h = h' \text{ near } Y
\]
are written as \( \text{Emb}^X(Y, \text{rel } A) \) and \( \text{Emb}^0_X(Y, \text{rel } A) \), respectively. There is no preferred topology for these. However, a topological cousin for \( \text{Emb}^X(Y, \text{rel } A) \) can be defined as follows. First we define \( \mathcal{Emb}(Y, \text{rel } Y \cap A) \) as the space of all embeddings \( Y \to Y \) which are the identity on some neighborhood of \( Y \cap A \), with the direct limit topology as before. Then we define
\[
\mathcal{Emb}^0_Y(Y, \text{rel } A) \subset \mathcal{Emb}(Y, \text{rel } Y \cap A)
\]
simply as the path component of the identity in \( \mathcal{Emb}(Y, \text{rel } Y \cap A) \). (We have kept the superscript “X” here for better bookkeeping, although it has clearly no content. Our definition of \( \mathcal{Emb}^0_Y(Y, \text{rel } A) \) is actually not in perfect agreement with that given by McDuff [14] in the unstratified case, but the homotopy types agree, as illustrated by lemma 10.12 below.) Finally
\[
\mathcal{BEmb}^0_Y(Y, \text{rel } A)
\]
is the homotopy fiber of the canonical map \( \mathcal{BEmb}^X(Y, \text{rel } A) \to \mathcal{BEmb}^0_Y(Y, \text{rel } A) \).

**Lemma 10.12.** Let \( X \) be a \( CS \) space, \( Y \) clean in \( X \) and \( A \) closed in \( X \). Suppose that the closure of \( \text{Fr}(Y) \smallsetminus A \) in \( X \) is compact. Then the inclusion–restriction sequence
\[
\mathcal{Hom}(X, \text{rel } Y \cap A) \to \mathcal{Hom}(X, Y, \text{rel } A) \to \mathcal{Emb}^0_Y(Y, \text{rel } A)
\]
is a (weak) homotopy fiber sequence.
Over $P$, relative to some neighborhood of $P \times A$. By an extension germ for $f$ we mean an embedding $P \times U \to P \times X$ over $P$, relative to some neighborhood of $P \times A$, where $U$ is an open neighborhood of $Y$ in $X$. (More precisely, an extension germ is an equivalence class of such extended embeddings.) For maps $f$ as above with extension germs, and homotopies of such, Siebenmann’s isotopy extension theorems [22] imply a homotopy lifting property. That is to say, if we have

- a map $f: P \to \mathcal{E}mb_0^X(Y, \text{rel } A)$, where $P$ is compact $CW$,
- a lift $\bar{f}: P \to \mathcal{H}om(X, Y, \text{rel } A)$ of $f$,
- a homotopy $\Phi: P \times [0, 1] \to \mathcal{E}mb_0^X(Y, \text{rel } A)$ with $\Phi_0 = f$, and
- an extension germ for $\Phi$ compatible with the extension germ for $f$ determined by $\bar{f}$,

then we can find $\Phi: P \times [0, 1] \to \mathcal{H}om(X, Y, \text{rel } A)$ which lifts $\Phi$, has $\Phi_0 = \bar{f}$ and realizes the prescribed extension germ for $\Phi$. For more details, read the remark just below.

Extension germs are easy to come by in the following sense. Suppose that $(P, Q)$ is a compact $CW$ pair and that

$f: P \to \mathcal{E}mb_0^X(Y, \text{rel } A)$

is a map, equipped with an extension germ for $f|Q$. Then there exists a map

g: P \to \mathcal{E}mb_0^X(Y, \text{rel } A)

with $g|Q = f|Q$, homotopic rel $Q$ to $f$, and an extension germ for $g$ which is compatible with the prescribed extension germ for $g|Q = f|Q$. In the case where $Q$ is empty, $g$ can be obtained from $f$ by a formula of type $g_p = f_p \circ e$ for $p \in P$, where $e: Y \to Y$ is a suitable embedding (suitably isotopic to $\text{id}_Y$) which extends to an embedding $U \to Y$ for some neighborhood $U$ of $Y$ in $X$. The existence of such an $e$ is guaranteed by the existence of a bicollar for $\text{Fr}(Y)$ in $X$. In the case where $Q$ is nonempty, a first step for finding $g$ would be to replace $f$ by $f \circ v$ where $v: P \to P$ is a map which extends the identity on $Q$, is homotopic rel $Q$ to the identity, and retracts a neighborhood $V$ of $Q$ to $Q$. This gives a reduction to the case where $f: P \to \mathcal{E}mb_0^X(Y, \text{rel } A)$ is specified with an extension germ for $f|V$, for a neighborhood $V$ of $Q$. Then $g$ can be defined by means of a formula of type $g_p = f_p \circ e_p$, where the $e_p: Y \to Y$ are suitable embeddings, depending this time on $p$. We omit the details.

With these tools available, it is easy to identify the relative homotopy groups of the inclusion $\mathcal{H}om(X, \text{rel } Y \cap A)) \to \mathcal{H}om(X, Y, \text{rel } A)$ with the absolute homotopy groups

$$\pi_s(\mathcal{E}mb_0^X(Y, \text{rel } A)),$$
Remark. Siebenmann states his isotopy extension theorem [22, 6.5] in a slightly unusual way which, in our situation, reads as follows. Let there be given $B$ which is a retract of some cube $I^n$, a point $b \in B$, a map $\Phi$ from $B$ to $\mathcal{E}mb_0^X(Y,\text{rel } A)$ with an extension germ, and a lift $\Phi_b: \{b\} \to \mathcal{H}om(X,Y,\text{rel } A)$ of $\Phi_b = \Phi|\{b\}$ which is compatible with the specified extension germ for $\Phi_b$. Then there exists $\Phi: B \to \mathcal{H}om(X,Y,\text{rel } A)$ which lifts $\Phi$ (and the specified extension germ) and extends the given $\Phi_b$.

To convert this into a homotopy lifting property as above, let $B = P \times [0,1]/P \times \{0\}$. Then $B$ is a compact contractible ENR, and consequently is homeomorphic to a retract of some cube $I^n$. Let $b$ be the natural base point of $B$. With this choice of $B$ and $b$, the Siebenmann formulation solves homotopy lifting problems consisting of a homotopy $\Phi: P \times [0,1] \to \mathcal{E}mb_0^X(Y,\text{rel } A)$, a prescribed initial lift $\Phi_0: P \times \{0\} \to \mathcal{H}om(X,Y,\text{rel } A)$ and suitable extension germ data, provided that $\Phi_0$ is constant. (Then $\Phi$ restricted to $P \times \{0\}$ must be constant, too.) However, the condition $\Phi_0 = \text{const.}$ is not a serious condition! If $\Phi_0$ is not constant, replace $\Phi$ by $\Phi^\#$ where

$$\Phi^\#(x,t) := (\Phi_0(x,0))^{-1} \circ \Phi(x,t),$$

using the fact that $\mathcal{H}om(X,\text{rel } A)$ is a group (although $\mathcal{H}om(X,Y,\text{rel } A)$ might not be). Then find a lifted homotopy $\Phi^\#$ using Siebenmann’s result and working with $\mathcal{H}om(X,\text{rel } A)$ instead of $\mathcal{H}om(X,Y,\text{rel } A)$. Then define $\Phi$ by

$$\Phi(x,t) := \Phi_0(x,0) \circ \Phi^\#(x,t).$$

The following propositions 10.13 and 10.14 as well as corollary 10.15 generalize propositions 2.1, 2.2 and corollary 2.4 in [14], respectively. For the proofs, see [14] and the remark following corollary 10.15 below.

**Proposition 10.13.** Let $X$ be a CS space, $Y$ a clean subset of $X$ and $A$ a closed subset of $X$ such that $\text{Fr}(Y) \setminus A$ has compact closure. Then the inclusion

\[ \overline{\mathcal{H}om}_0(X,Y,\text{rel } A) \longrightarrow \overline{\mathcal{H}om}(X,\text{rel } A) \]

is a weak homotopy equivalence.

**Proposition 10.14.** If $X$, $Y$ and $A$ are as in proposition 10.13, then the sequence

\[ \begin{array}{c}
\overline{\mathcal{H}om}_0(X,\text{rel } Y \cup A) \\
\longrightarrow \\
\overline{\mathcal{H}om}_0(X,Y,\text{rel } A) \\
\longrightarrow \\
\mathcal{E}mb_0^X(Y,\text{rel } A)
\end{array} \]

is an integer homology fibration sequence. In other words, the inclusion of the space $\overline{\mathcal{H}om}_0(X,\text{rel } Y \cup A)$ into the homotopy fiber of the restriction map $\bar{\rho}$ induces an isomorphism of (untwisted) integer homology groups.

**Corollary 10.15.** Let $X$, $Y$ and $A$ be as in proposition 10.13. If $\mathcal{H}om(X,\text{rel } Z)$ is acyclic for $Z = A$, $Y$ and $Y \cup A$, then it is acyclic for $Z = Y \cap A$ also.

The proof of McDuff’s proposition 2.1 in [14], which is the unstratified case of proposition 10.13 above, occupies §4 of [14]. Its backbone is a lemma about topological monoids, lemma 4.1 in [14]. The deduction of McDuff’s proposition 2.1 from that lemma occupies only half a page (right after the statement of the lemma) and carries over to the stratified case with only trivial changes.

Similarly, the proof of McDuff’s proposition 2.2 in [14], which is the unstratified case of proposition 10.14 above, occupies §3 of [14]. It relies on a string of lemmas about topological groups and monoids. The deduction of McDuff’s proposition 2.2 from these lemmas occupies only half a page (at the end of §3 in [14]) and carries over to the stratified case with only trivial changes.

Theorem 10.16. The space $\mathcal{B}(\text{Hom}(X, \text{rel} A))$ is acyclic if $A$ is closed in $X$ and $X \setminus A$ has compact closure in $X$.

Proof. Let $Z$ be a stratum of $X$ which is not contained in $A$ and which is minimal among the strata of $X$ with that property (i.e., all strata of $X$ which belong to the closure of $Z$ in $X$, except $Z$ itself, are contained in $A$). By induction, we may assume:

(i) $\mathcal{B}(\text{Hom}(X, \text{rel} N))$ is acyclic for each closed $N \subset X$ which is a neighborhood of $A \cup Z$.

In addition, we will assume to begin with that

(ii) there exists $z \in Z \setminus A$ and a quasi–ball neighborhood $V$ about $z$ which contains all of $Z \setminus A$.

(This assumption will be removed at a later stage.) Because of (ii), the open subset $Z \setminus A$ of $Z$ is identified with an open subset of a standard euclidean space, and can be triangulated. We fix a triangulation.

For every closed $A'$ which is a neighborhood of $A$ in $X$, we can choose a triangulation of $Z \setminus A$ such that $Z \setminus A'$ is covered by finitely many open stars $\text{st}(z_1), \ldots, \text{st}(z_r)$ where $z_1, \ldots, z_r$ are vertices of the triangulation. Making $V$ sufficiently slim, we can arrange that the portion $V_i$ of $V$ lying over $\text{st}(z_i)$ has empty intersection with $A$. Let $A_i = X \setminus V_i$. For $S \subset \{1, \ldots, r\}$ let $A_S$ be the union of the $A_i$ with $i \in S$. Then

(iii) each $A_S$ is a clean subspace of $X$ and its complement is a quasi–ball, or the empty set. Hence $\mathcal{B}(\text{Hom}(X, \text{rel} A_S))$ is acyclic by theorem 10.10.

Now $\mathcal{B}(\text{Hom}(X, \text{rel} A_1 \cap A_2 \cap \cdots A_i))$ is acyclic for $i = 1, \ldots, r$. This can easily be shown by induction on $i$, using corollary 10.15 and the last part of (iii) just above. Finally choose a closed neighborhood $N$ of $A \cup Z$ such that

$$N \cap A_1 \cap A_2 \cap \cdots \cap A_r \subset A'.$$

By assumption (i), we also know that $\mathcal{B}(\text{Hom}(X, \text{rel} N))$ is acyclic. Hence, by corollary 10.15 again,

$$\mathcal{B}(\text{Hom}(X, \text{rel} N \cap A_1 \cap A_2 \cap \cdots A_r))$$

is acyclic. We have now shown that, for any closed neighborhood $A'$ of $A$, there exists another closed neighborhood $A''$ of $A$ with $A'' \subset A'$ such that $\mathcal{B}(\text{Hom}(X, \text{rel} A''))$ is acyclic. By lemma 10.6, this implies that $\mathcal{B}(\text{Hom}(X, \text{rel} A))$ is acyclic.
Now we repeat the argument, but without hypothesis (ii). As before, we fix a closed neighborhood $A'$ of $A$ in $X$ and we choose finitely many quasi–balls $V_1, \ldots, V_r$ about points in $Z \setminus A$ such that the union of the $V_i$ contains $Z \setminus A'$ and is contained in $X \setminus A$. We define $A_i$ and $A_S$ as before, for $S \subseteq \{1, \ldots, r\}$. We have less information about the $A_S$ this time, but at least we know that condition (ii) is satisfied with $A_S$ in place of $A$. Therefore

(iv) $\mathbb{B}Hom(X, \text{rel } A_S)$ is acyclic for each $S \subseteq \{1, \ldots, r\}$, by the first part of this proof, which relied on (iii). We can now finish the argument as in the first part of the proof, using (iv) instead of (iii). \hfill $\square$

11. Algebraic approximations to structure spaces: Set-up

**Notation 11.1.** We use the symbol $\%$ in the subscript position to describe homotopy fibers of assembly maps. For example: $L^\%_\bullet (X)$ is the homotopy fiber of the assembly map $L^\bullet (X) \rightarrow L^\bullet (X)$, assuming $X$ has the homotopy type of a CW-space. Similarly: $L_\bullet (X)$ is the homotopy fiber of the assembly map in quadratic $L$-theory and $\mathbb{L}A_\bullet (X, \xi, n)$ is the homotopy fiber of the assembly map in quadratic $\mathbb{L}A$-theory.

From now on we will often suppress the $\bar{X}$ in a control space $(\bar{X}, X)$ and make up for that with a semicolon followed by a “; $c$” to indicate a “controlled” context. For example, instead of writing $F^\%((\bar{X}, X), \xi, n)$ and $F((\bar{X}, X), \xi, n)$ as in section 9, we may write $F^\%(X, \xi, n; c)$, $F(X, \xi, n; c)$.

In that spirit, $F^\% (X, \xi, n; c)$ is the homotopy fiber of the assembly map alias forgetful map $F^\%(X, \xi, n; c) \rightarrow F(X, \xi, n; c)$ which we have from section 9. This assumes that $X$ is the nonsingular part of a control space $(\bar{X}, X)$ with compact $\bar{X}$.

**Construction 11.2.** Let $M$ be a compact $m$-manifold with normal bundle $\nu$. We introduce a map $\varphi$ from the honest structure space $S(M)$ to $F^\% (M, \nu, m) \simeq \Omega^{\infty + m} \mathbb{V}LA_\bullet (M, \nu, m) \simeq \Omega^{\infty + m} \mathbb{L}A_\bullet (M, \nu, m)$.

**Remark.** The setting is away from $\partial M$. Points of $S(M)$ correspond to homotopy equivalences of pairs $f: (N, \partial N) \rightarrow (M, \partial M)$ where the induced map $\partial N \rightarrow \partial M$ is a homeomorphism.

**Construction 11.3.** Notation being as in the previous construction, there is a “local degree” homomorphism from $\pi_m L^\%_\bullet (M, \nu)$ to the group $L_0(\mathbb{Z})^{\pi_0 M}$. It is onto.

This is the composition of Poincaré duality, $\pi_m L^\%_\bullet (M, \nu) \cong H^0(M; \mathbb{L}a_\bullet (\ast))$, with an evaluation map from $H^0(M; \mathbb{L}a_\bullet (\ast))$ to $L_0(\mathbb{Z})^{\pi_0 M}$. We admit that $L_0(\mathbb{Z})$ could also be described as $L_0(\ast)$.

**Theorem 11.4.** Assume $\dim(M) \geq 5$. The diagram

$$
S(M) \xrightarrow{\varphi} \Omega^{\infty + m} \mathbb{L}A_\bullet (M, \nu, m) \xrightarrow{\text{local degree}} L_0(\mathbb{Z})^{\pi_0 M}
$$

is a homotopy fiber sequence in the concordance stable range.

In this theorem, $L_0(\mathbb{Z}) \cong 8\mathbb{Z}$ is viewed as a discrete space. The local degree is defined on $\pi_m L^\%_\bullet (M, \nu, m)$ via the forgetful map to $\pi_m L^\%_\bullet (M, \nu)$. Details on the meaning of “concordance stable range” are given in the definition which follows.
Definition 11.5. For a compact manifold $N$ let $k_N$ be the minimum of all positive integers $k$ such that the stabilization map of concordance spaces
\[ \Omega \mathcal{H}(N \times D^i) \longrightarrow \Omega \mathcal{H}(N \times D^{i+1}) \]
is $k$-connected for $i = 0, 1, 2, \ldots$. The precise meaning of theorem 11.4 is that

- the induced map from $\pi_0 \mathcal{S}(M)$ to
\[ \ker[\pi_0 \mathbf{LA}_{\nu}(M, \nu, m) \to L_0(\mathbb{Z})^{\pi_0 M}] \]
is bijective;

- the restricted map, from a component of $\mathcal{S}(M)$ represented by a homotopy equivalence $(N, \partial N) \to (M, \partial M)$ to the corresponding component of $\Omega^{\infty+m} \mathbf{LA}_{\nu}(M, \nu, m)$, is $(k_N+1)$-connected.

Remark. It follows from [10] and smoothing theory that
\[ k_N \geq \min \left\{ \frac{m-7}{2}, \frac{m-4}{3} \right\} \]
if $N$ admits a smooth structure, $m = \dim(N)$. Similar estimates for non-smooth manifolds were part of the topology folklore in the 70’s, but completely convincing proofs of these have apparently not been found. Unfortunately, even if $M$ admits a smooth structure, there may be some components of $\mathcal{S}(M)$, represented by $(N, \partial N) \to (M, \partial M)$ say, whose source manifold $N$ does not admit a smooth structure.

Our only goal in the rest of the paper is to prove theorem 11.4. We conclude this section with an overview of the proof; more can be found in the remaining sections. For brevity, information on “which spherical fibration” will be suppressed where it seems superfluous; for example, we write $F_{\nu}(M, \nu, m)$ instead of $F_{\nu}(M, \nu, m)$.

Lemma 11.6. The map in construction 11.2 identifies $\pi_0 \mathcal{S}(M)$ with the kernel of the local degree homomorphism from $\pi_0 \mathbf{LA}_{\nu}(M, \nu) \to L_0(\mathbb{Z})^{\pi_0 M}$.

The verification is “manual” (but is not left to the reader). From then on, the proof of theorem 11.4 that we have relies on a downward induction argument. In order to make this work, we need to extend construction 11.2 (more precisely, the looping of that) as follows. Let $M$ be given as before. For every $i \geq 0$ we introduce the controlled (honest) structure space $\mathcal{S}(M \times \mathbb{R}^i; c)$, using the compactification $M \star S^{i-1}$ of $M \times \mathbb{R}^i$ to define the control criteria. More specifically let
\[ S^i(M \times \mathbb{R}^i; c) \subset \mathcal{S}(M \times \mathbb{R}^i; c) \]
be the union of the connected components of $\mathcal{S}(M \times \mathbb{R}^i; c)$ which are in the image of the map $\pi_0 \mathcal{S}(M) \to \pi_0 \mathcal{S}(M \times \mathbb{R}^i; c)$ induced by $\times \mathbb{R}^i$.

Construction 11.7. For $i \geq 0$ we construct a map $\varphi$ from $S^i(M \times \mathbb{R}^i; c)$ to
\[ F_{\nu}(M \times \mathbb{R}^i, m + i; c) \simeq \Omega^{\infty+m+i} \mathbf{LA}_{\nu}(M \times \mathbb{R}^i, m + i; c) . \]
This map agrees with construction 11.2 when $i = 0$. The following commutes for every $i \geq 0$:
\[
\begin{array}{ccc}
S^i(M \times \mathbb{R}^i; c) & \xrightarrow{\varphi} & F_{\nu}(M \times \mathbb{R}^i, m + i; c) \\
\downarrow \text{product with } \mathbb{R} & & \downarrow \text{product with } \mathbb{R} \\
S^i(M \times \mathbb{R}^{i+1}; c) & \xrightarrow{\varphi} & F_{\nu}(M \times \mathbb{R}^{i+1}, m + i + 1; c).
\end{array}
\]
Lemma 11.8. There is a homotopy fibration sequence
\[ \mathcal{H}(M \times \mathbb{R}^i; c) \longrightarrow \mathcal{S}(M \times \mathbb{R}^i; c) \xrightarrow{\times \mathbb{R}} \mathcal{S}(M \times \mathbb{R}^{i+1}; c). \]

Lemma 11.9. There is a homotopy fibration sequence
\[ \Omega^\infty A_{\mathbb{R}}(M \times \mathbb{R}^i; c) \longrightarrow F_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c) \xrightarrow{\times \mathbb{R}} F_{\mathbb{R}}(M \times \mathbb{R}^{i+1}, m + i + 1; c). \]

Write \( \chi \) for the map between vertical homotopy fibers in the diagram of construction 11.7. Then with the identifications of the previous two lemmas, we can write
\[ \Omega \chi: \Omega \mathcal{H}(M \times \mathbb{R}^i; c) \rightarrow \Omega^\infty A_{\mathbb{R}}(M \times \mathbb{R}^i; c). \]

Proposition 11.10. This map \( \Omega \chi \) is \( (k_M + i) \)-connected (see definition 11.5).

One more ingredient that we need for our downward induction procedure is the controlled version of the Casson-Sullivan-Wall-Quinn-Ranicki main theorem of surgery theory (which we regard as well-known). This involves controlled block structure spaces. We state it here using the decoration \( h \) for “controlled homotopy equivalences”. More details on the meaning of the \( h \) will be given in section 13.

Theorem 11.11. For each \( i \geq 0 \), there are homotopy fibration sequences
\[ \tilde{S}^i(M \times \mathbb{R}^i; c) \longrightarrow \Omega^{\infty + i} L_{\mathbb{R}}^i(M \times \mathbb{R}^i; c) \longrightarrow L_0(\mathbb{Z})\pi_0 M. \]

Lemma 11.12. In the (homotopy co-)limit \( i = \infty \), the (first) map in theorem 11.11 and the map of construction 11.7 are identical (up to certain homotopy equivalences, and on the base point component of the source).

Proof of theorem 11.4. Consider a component of \( \mathcal{S}(M) \) represented by some homotopy equivalence \( g: N \rightarrow M \), say, restricting to a homeomorphism of the boundaries. Then \( g \) determines a homotopy equivalence \( g_*: \mathcal{S}(N) \rightarrow \mathcal{S}(M) \) and also a homotopy equivalence \( g_*: F_{\mathbb{R}}(N, m) \rightarrow F_{\mathbb{R}}(M, m) \). Once the details of construction 11.2 emerge (in the next section), it will be clear that the square
\[ \begin{array}{ccc} \mathcal{S}(N) & \longrightarrow & F_{\mathbb{R}}(N, m) \\ \downarrow g_* & & \downarrow g_* \\ \mathcal{S}(M) & \longrightarrow & F_{\mathbb{R}}(M, m) \end{array} \]
commutes up to a translation, i.e., up to addition of a constant in the infinite loop space \( F_{\mathbb{R}}(M, m) \). Combining that observation with lemma 11.6, we see that it is enough to verify the claim of theorem 11.4 for the base point components of \( \mathcal{S}(M) \) and \( F_{\mathbb{R}}(M, m) \) only. In other words it is enough to prove the looped statement.

Let \( z \) be a nonzero element in some homotopy group \( \pi_k \) of the map
\[ \Omega \varphi: \Omega \mathcal{S}(M) \rightarrow \Omega F_{\mathbb{R}}(M, m). \]

We have to show that \( k > k_N \). By theorem 11.11 and lemma 11.12, there exists an \( i \geq 0 \) such that the image of \( z \) in \( \pi_k \) of the map
\[ \Omega \mathcal{S}(M \times \mathbb{R}^{i+1}; c) \longrightarrow \Omega F_{\mathbb{R}}(M \times \mathbb{R}^{i+1}, m + i + 1; c) \]
(of construction 11.7) is zero. With the minimal choice of such an \( i \), we obtain that \( \pi_k \) of the map of vertical homotopy fibers in the once-looped diagram of construction 11.7 is nonzero. By proposition 11.10, it follows that \( k > k_N + i \). Since \( i \geq 0 \), this implies \( k > k_N \). \( \square \)
12. Algebraic approximations to structure spaces: Details

We begin with construction 11.2, imitating both [9] and Waldhausen’s approach to h-cobordisms. Suppose to begin that \( \partial M \) is empty.

**Definition 12.1.** Let \( \mathcal{D} \) be the following category. The objects are finitely dominated Poincaré duality spaces \( X \) of formal dimension \( m \), together with a spherical fibration \( \xi: E \to X \) with fiber \( \simeq S^d \), a preferred section for that, and a stable map \( \eta: S^{m+d} \to E/X \) which carries a fundamental class. Mostly for convenience, we require \( X \) to be compact Hausdorff and homotopy equivalent to \( M \). A morphism from \((X_1, \xi_1, \eta_1)\) to \((X_0, \xi_0, \eta_0)\) is a pair \((u, v)\) where \( u \) is a homotopy equivalence from \( X_1 \) to \( X_0 \) and

\[
v: \Sigma_{X_1}^{d_1-d_0} E_1 \to E_0
\]

is a homotopy equivalence which covers \( u \), respects the zero sections and satisfies \( v\eta_1 = \eta_0 \). (It is understood that \( d_1 \geq d_0 \), where \( d_1 \) and \( d_0 \) are the formal fiber dimensions of \( \xi_1: E_1 \to X_1 \) and \( \xi_0: E_0 \to X_0 \).)

We allow continuous variation of \( \eta \) in objects \((X, \xi, \eta)\). This makes the set of objects of \( \mathcal{D} \) into a space. (Some conditions on underlying sets should be added to our definition of objects to ensure that the class of objects of \( \mathcal{D} \) is indeed a set.) We also allow continuous variation of \( v \) in morphisms \((u, v)\) as above. This makes the set of morphisms of \( \mathcal{D} \) into a space, in such a way that “source” and “target” are continuous maps from the morphism space to the object space. Hence \( \mathcal{D} \) is a topological category. (By allowing continuous variation of the \( \eta \)'s and the \( v \)'s, we achieve that \( BD \) is homotopy equivalent to \( BG(M) \), where \( G(M) \) is the topological monoid of homotopy automorphisms of \( M \). This follows from the uniqueness of Spivak normal fibrations; see section 14.)

The category \( \mathcal{D} \) has a subcategory \( \mathcal{C} \) which is defined like this. An object \((X, \xi, \eta)\) of \( \mathcal{D} \), as above, belongs to \( \mathcal{C} \) precisely if \( X \) is a closed \( m \)-manifold, \( \xi \) is a sphere bundle, and \( \eta: S^{m+d} \to E/X \) restricts to a homeomorphism from \( \eta^{-1}(E \setminus X) \) to \( E \setminus X \). (The condition on \( \eta \) means that \( \eta \) is the “Thom collapse” associated with an embedding of \( N \) in some euclidean space.) A morphism \((u, v)\) in \( \mathcal{D} \), as above, belongs to \( \mathcal{C} \) if its source and target are in \( \mathcal{C} \), and both \( u \) and \( v \) are homeomorphisms. Continuous variation of \( \eta \) in objects \((X, \xi, \eta)\) and of \( v \) in morphisms \((u, v)\) are allowed as before. The result is that \( BC \) is homotopy equivalent to a disjoint union of classifying spaces \( B\text{Hom}(N) = B\text{TOP}^\delta(N) \), where \( N \) runs through a maximal set of pairwise non-homeomorphic closed \( m \)-manifolds which are homotopy equivalent to \( M \). (Recall from section 10 that \( \text{Hom}(N) \) is the discrete homeomorphism group of \( N \).)

The manifold \( M \) itself, equipped with a euclidean normal bundle \( \nu \) etc., can be viewed as an object of \( \mathcal{C} \). We note that

\[
\mathcal{S}_{\text{cat}}(M) := \text{hofiber}_{\mathcal{D}[M] \to \mathcal{D}[BD]}
\]

is a good combinatorial model for \( \mathcal{S}(M) \). More precisely, \( \mathcal{S}_{\text{cat}}(M) \) comes with a forgetful map to \( \mathcal{S}(M) \) which is a homology equivalence. We like to think of that map as an inclusion.

The homotopy invariant signature \( \sigma \) which we have constructed in section 8 determines or “is” a point in \( \text{holim} F[\mathcal{D}] \). To be more precise, \( F[\mathcal{D}] \) means the functor taking \((X, \xi, \eta)\) in \( \mathcal{D} \) to \( F(X, \xi, m) \) as defined in section 8; then we have \( \sigma(X) \in F(X, \xi, m) \) with the naturality properties discussed at the end of that same
section. Similarly, the “excisive” signature $\sigma^\%$ which we have constructed in section 9 is a point in $\text{holim} F^\% C$. Here $F^\% C$ means the functor taking $(N, \xi, \eta)$ to $F^\%(N, \xi, m)$ as defined in section 9.

We now use an adjunction principle relating homotopy limits to homotopy colimits. It says, applied to our situation, that $\sigma^\% \in \text{holim} F^\% D$ determines a “homotopy section” of the projection $p : \text{hocolim} F^\% D \to BD$. This comes in the shape of a map

$$\bar{\sigma} : BD^\big \longrightarrow \text{hocolim} F^\% D$$

such that $p \circ \bar{\sigma} : BD^\big \to BD$ is a homotopy equivalence. (Ignoring subtleties related to the topology on $D$, we can say that a point in $\text{holim} F^\% D$ is by definition a natural transformation from a certain functor on $D$ with contractible values to $F^\% D$. The space $BD^\big$ can be defined as the homotopy colimit of that functor with contractible values. Then $\bar{\sigma}$ can be defined as the map of homotopy colimits induced by the natural transformation $\sigma$.) We also get

$$\bar{\sigma}^\% : B^\big \longrightarrow \text{hocolim} F^\% C$$

by applying the same adjunction principle to $\sigma^\%$. The fact that $\sigma^\%$ “lifts” $\sigma$ means that the diagram

$$\begin{array}{ccc}
BC^\big & \xrightarrow{\text{inclusion} \circ \sigma^\%} & \text{hocolim} F^\% D \\
\downarrow & & \downarrow \\
BD^\big & \xrightarrow{\sigma} & \text{hocolim} F^\% D
\end{array}$$

is commutative up to homotopy; more precisely we get a specific homotopy between the two resulting maps from $BC^\big$ to $\text{hocolim} F^\% D$. To finish, we pass to vertical homotopy fibers in that commutative square. In the left-hand column, we take the homotopy fiber over the base point (determined by $M$), so that the homotopy fiber is $\mathcal{S}^\text{cat}(M)$. In the right-hand column, we take the homotopy fiber over the point in $\text{hocolim} F^\% D$ determined by $M$ and $\sigma(M)$. That homotopy fiber then becomes $F^\%(M, \nu, m)$. (Here we are using the fact that the two homotopy colimits are quasi-fibered over $BD$, since $F^\%$ and $F$ take all morphisms to homotopy equivalences.) Hence the resulting map between vertical homotopy fibers has the form

$$\mathcal{S}^\text{cat}(M) \longrightarrow F^\%(M, \nu, m).$$

By obstruction theory, and since each component of $F^\%(M, \nu, m)$ is a simple space, this map extends in an essentially unique way to a map

$$\mathcal{S}(M) \longrightarrow F^\%(M, \nu, m).$$

Extending the construction to the case where $M$ has a nonempty boundary is an easy matter. We use doubles: Write $M^\sharp = M \amalg_{\partial M} M$. The inclusion of $M$ in $M^\sharp$ (as the “first” amalgamated summand, say) induces a map $\mathcal{S}(M) \to \mathcal{S}(M^\sharp)$. On the algebraic side, the fold map $M^\sharp \to M$ induces

$$F^\%(M^\sharp, \nu^\sharp, m) \longrightarrow F^\%(M, \nu, m)$$

where $\nu^\sharp$ is the pullback of $\nu$ to $M^\sharp$, or equivalently, the normal bundle of $M^\sharp$. Hence we can construct $\varphi_M$ by composing

$$\mathcal{S}(M) \to \mathcal{S}(M^\sharp) \longrightarrow F^\%(M^\sharp, \nu^\sharp, m) \to F^\%(M, \nu, m).$$
It is easy to verify that this construction is consistent with the previous one in the case where \( \partial M = \emptyset \).

This completes construction 11.2. We have \( \mathbf{VLA}^\bullet_{\mathbb{R}}(M, m) \simeq \mathcal{L}^\bullet_{\mathbb{R}}(M, m) \) by theorem 2.7.

Construction 11.7 uses very similar ideas. A new difficulty here is that we do not have a good “discrete homology approximation theorem” (as in section 10) for the topological group of controlled automorphisms of \( M \times \mathbb{R}^1 \), in other words, the topological group of homeomorphisms \( M \ast S^{i-1} \to M \ast S^{i-1} \) which restrict to the identity on \( S^{i-1} \). We do have such a theorem for the topological group of homeomorphisms \( M \ast S^{i-1} \to M \ast S^{i-1} \) taking \( S^{i-1} \) to \( S^{i-1} \); so we have to make the best of that. Suppose to begin that \( M \) is closed.

**Definition 12.2.** Generalizing definition 12.1, we introduce a category \( \mathcal{D} \) whose objects are certain control spaces \( (\tilde{X}, X) \) with compact Hausdorff \( \tilde{X} \), together with a spherical fibration \( \xi : E \to X \) with fiber \( \simeq S^d \), a preferred section for that, and a stable map \( \eta : S^{m+d} \to E/\pi X \). Compare definition 4.1. These data are required to satisfy the conditions of definition 8.5 for a controlled Poincaré duality space. In addition, however, we require that \( (\tilde{X}, X) \) be homotopy equivalent as a control space to \( (M \ast S^{i-1}, M \times \mathbb{R}^1) \); and moreover we mean a homotopy equivalence such that the maps involved restrict to homeomorphisms of the singular sets, and the homotopies involved are isotopies on the singular sets. A morphism from \( (\tilde{X}_1, X_1, \xi_1, \eta_1) \) to \( (\tilde{X}_0, X_0, \xi_0, \eta_0) \) is a pair \((u, v)\) where \( u \) is a map of control spaces from \( (\tilde{X}_1, X_1) \) to \( (\tilde{X}_0, X_0) \) and \( v : \Sigma^{d_1 - d_0} E_1 \to E_0 \) covers \( u|X_1 \), respects the zero sections and satisfies \( v\eta_1 = \eta_0 \). Continuous variation of \( \eta \) in objects \( (\tilde{X}, X, \xi, \eta) \) and continuous variation of \( v \) in morphisms \( v \) is allowed. The result is that

\[
BD \simeq B(G(M \times \mathbb{R}^1; c) \times \text{Hom}(S^{i-1}))
\]

where \( G(M \times \mathbb{R}^1; c) \) is the topological monoid of controlled homotopy automorphisms of \( M \times \mathbb{R}^1 \) and \( \text{Hom}(S^{i-1}) \) is the discrete group of homeomorphisms from \( S^{i-1} \) to \( S^{i-1} \), as usual. Note that \( \text{Hom}(S^{i-1}) \) acts on \( G(M \times \mathbb{R}^1; c) \) by monoid automorphisms, and we have used \( \times \) to indicate a semidirect product.

The category \( \mathcal{D} \) has a subcategory \( \mathcal{C} \) which is defined like this. An object \( (\tilde{X}, X, \xi, \eta) \) of \( \mathcal{D} \) belongs to \( \mathcal{C} \) precisely if \( X \) is an \( m \)-manifold, \( \xi \) is a sphere bundle and \( \eta : S^{m+d} \to E/\pi X \) restricts to a homeomorphism from \( \eta^{-1}(E \setminus X) \) to \( E \setminus X \). A morphism \( (u, v) \) in \( \mathcal{D} \), as above, belongs to \( \mathcal{C} \) if its source and target are in \( \mathcal{C} \), and both \( u \) and \( v \) are homeomorphisms. Continuous variation of \( \eta \) in objects \( (\tilde{X}, X, \xi, \eta) \) and continuous variation of \( v \) in morphisms \( v \) is allowed. The result is that

\[
BC \simeq \prod_\beta B\text{Hom}(\tilde{X}_\beta, X_\beta).
\]

Here each \( (\tilde{X}_\beta, X_\beta) \) is a control space, with compact \( \tilde{X}_\beta \), whose nonsingular part \( X_\beta \) is an \( m + i \)-manifold without boundary. The control space \( (\tilde{X}_\beta, X_\beta) \) is homotopy equivalent to \( (M \ast S^{i-1}, M \times \mathbb{R}^1) \), and as before the kind of homotopy equivalence we mean is one where the maps and homotopies involved restrict to homeomorphisms/isotopies of the singular sets. We select a maximal set of pairwise non-homeomorphic control spaces \( (\tilde{X}_\beta, X_\beta) \) of this kind. For each of them,
\( \text{Hom}(X, X) \) is the discrete group of homeomorphisms \((X, X) \to (X, X)\). If \((X, X)\) is homeomorphic to \((N \times S^{i-1}, N \times \mathbb{R}^1)\) for some closed manifold \(N \simeq M\), then we have valuable information about the homology of \(B\text{Hom}(X, X)\) from section 10.

Generalizing from \(i = 0\) to \(i \geq 0\), we obtain functors \(F|\mathcal{D}\) and \(F^\%|\mathcal{D}\) and elements \(\sigma \in \text{holim} F|\mathcal{D}\) as well as \(\sigma^\% \in \text{holim} F^\%|\mathcal{C}\). They lead to a homotopy commutative diagram

\[
\begin{array}{ccc}
BC^\text{big} & \xrightarrow{\text{inclusion} \circ \sigma^\%} & \text{holim} F^\%|\mathcal{D} \\
\downarrow & & \downarrow \\
BD^\text{big} & \xrightarrow{\sigma} & \text{holim} F|\mathcal{D}
\end{array}
\]

as before. At this point it is not a good idea to take vertical homotopy fibers, unless \(i = 0\). Instead we enlarge the diagram by adding another row:

\[
\begin{array}{ccc}
BC^\text{big} & \xrightarrow{\text{inclusion} \circ \sigma^\%} & \text{holim} F^\%|\mathcal{D} \\
\downarrow & & \downarrow \\
BD^\text{big} & \xrightarrow{\sigma} & \text{holim} F|\mathcal{D} \\
\uparrow a & & \uparrow \\
B\text{Hom}(S^{i-1}) & \xrightarrow{=} & B\text{Hom}(S^{i-1})
\end{array}
\]

The map \(a\) in the lower half of the diagram is obtained by composing the forgetful map \(B\text{Hom}(S^{i-1}) \to B\text{Hom}(S^{i-1})\) with a standard “diagonal” homotopy equivalence \(B\text{Hom}(S^{i-1}) \to B\text{Hom}(S^{i-1})^\text{big}\) and the map \(B\text{Hom}(S^{i-1})^\text{big} \to BD^\text{big}\) induced by the inclusion functor \(\text{Hom}(S^{i-1}) \to \mathcal{D}\).

We are now interested in the homotopy pullbacks of the columns of (**) in particular we let \(S_\text{cat}(M \times \mathbb{R}^1; c) := \text{holim} \text{ of left-hand column in (**)}\).

**Lemma 12.3.** There is a map from \(S_\text{cat}(M \times \mathbb{R}^1; c)\) to \(\mathcal{S}(M \times \mathbb{R}^1; c)\) which induces a bijection on \(\pi_0\) and a homology isomorphism on the components corresponding to “flat” structures (given by controlled homotopy equivalences \(N \times \mathbb{R}^1 \to M \times \mathbb{R}^1\), where \(N\) is closed).

**Proof.** By section 10 applied to the (unstratified) sphere \(S^{i-1}\), there is a homotopy fiber sequence

\[B\text{Hom}(S^{i-1}) \to BD^\text{big} \to B\left(G(M \times \mathbb{R}^1; c) \times \mathcal{H}(\text{om}(S^{i-1}))\right).\]

Hence the base point component of \(S_\text{cat}(M \times \mathbb{R}^1; c)\) is homotopy equivalent to the base point component of

\[Z = \text{hofiber}[B\text{Hom}(M \times S^{i-1}, M \times \mathbb{R}^1) \to B\left(G(M \times \mathbb{R}^1; c) \times \mathcal{H}(\text{om}(S^{i-1}))\right)].\]

By section 10 again, this time applied to \(M \times S^{i-1}\) with strata \(M \times \mathbb{R}^1\) and \(S^{i-1}\), the space \(Z\) admits a map to

\[Z' = \text{hofiber}[B\text{Hom}(M \times S^{i-1}, M \times \mathbb{R}^1) \to B\left(G(M \times \mathbb{R}^1; c) \times \mathcal{H}(\text{om}(S^{i-1}))\right)]\]

which induces an isomorphism in homology. The description of \(Z'\) simplifies to

\[\text{hofiber}[B\text{Hom}(M \times \mathbb{R}^1; c) \to B\left(G(M \times \mathbb{R}^1; c)\right)].\]
which is a union of components of $\mathcal{S}(M \times \mathbb{R}^i; c)$. We can argue similarly for the other components of $\mathcal{S}_{\text{cat}}(M \times \mathbb{R}^i; c)$ corresponding to flat structures. 

\textbf{Lemma 12.4.} The homotopy pullback (homotopy limit) of the right-hand column in (**) is homotopy equivalent to

$$F_{\mathbb{R}}(M \times \mathbb{R}^i, \nu, m + i; c) \times \mathcal{B}\text{Hom}(S^{i-1}).$$

\textbf{Proof.} We recall that hocolim $F_{\mathbb{R}}|\mathcal{C}$ and hocolim $F|\mathcal{C}$ are (total spaces of) quasi-fibrations over $BD$. It is enough to show that the associated fibrations are trivial over the acyclic subspace $B\text{Hom}(S^{i-1})$ of $BD$. By obstruction theory, it is also enough to show that the automorphisms $h_*$ of the spaces

$$F_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c), \quad F(M \times \mathbb{R}^i, m + i; c)$$

induced by a homeomorphism $h : S^{i-1} \to S^{i-1}$ are based homotopic to the respective identity maps, provided $h$ itself is homotopic to the identity of $S^{i-1}$. This follows from homotopy invariance properties of the functors $F_{\mathbb{R}}$ and $F$ (on control spaces) which we have established. \hfill $\square$

Because of lemma 12.4, the map between homotopy pullbacks of columns in diagram (**) takes the form

$$\mathcal{S}_{\text{cat}}(M \times \mathbb{R}^i; c) \longrightarrow F_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c) \times \mathcal{B}\text{Hom}(S^{i-1}).$$

We drop the second component and obtain

$$\mathcal{S}_{\text{cat}}(M \times \mathbb{R}^i; c) \longrightarrow F_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c).$$

By lemma 12.3 and obstruction theory, this restricts/extends in an essentially unique way to a map

$$\mathcal{S}^i(M \times \mathbb{R}^i; c) \longrightarrow F_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c). \quad (***)$$

This is what we call construction 11.7. The homotopy equivalence

$$F_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c) \simeq \Omega^{\infty + m + i} \mathbf{LA}_{\mathbb{R}}(M \times \mathbb{R}^i, m + i; c)$$

follows from theorem 5.1.

As in the case $i = 0$, the construction can be extended to the case of a compact $M$ with nonempty boundary by doubling.

\section{13. Algebraic models for structure spaces: Proofs}

\textbf{Proof of lemma 11.8.} There is the following commutative square of controlled structure spaces:

$$\begin{array}{ccc}
\mathcal{S}(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c) & \longrightarrow & \mathcal{S}(M \times \mathbb{R}^i \times [0, \infty[; c) \\
\downarrow & & \downarrow \\
\mathcal{S}(M \times \mathbb{R}^{i+1} \times ]-\infty, 1[; c) & \longrightarrow & \mathcal{S}(M \times \mathbb{R}^{i+1}; c). \\
\end{array} \quad (\bullet)$$

\textbf{Details on the spaces involved:} All manifolds here are codimension zero submanifolds of $M \times \mathbb{R}^{i+1}$. We make them into control spaces by taking closures inside $M * S^i$. This amounts to adding the equator $S^{i-1}$ of $S^i$ to $M \times \mathbb{R}^i \times [0, 1]$, the closed upper hemisphere of $S^i$ to $M \times \mathbb{R}^i \times [0, \infty]$ and the closed lower hemisphere of $S^i$ to $M \times \mathbb{R}^i \times ]-\infty, 1[$. Beware that structure space notation of the form $\mathcal{S}(C, D)$ usually assumes that $C$ is a manifold, $D$ is a codimension zero closed submanifold of
∂C and the structures considered are “trivial” over the closure of the complement of D in ∂C. Hence there is a homotopy fibration sequence $S(C) \rightarrow S(C, D) \rightarrow S(D)$.

Details on the maps: The maps in the square are given by obvious extension of structures in all cases. For the left-hand vertical map for example, we extend by gluing with the trivial structure (identity) on $M \times R^i \times [-\infty, 0]$. For the top horizontal map, we extend a structure $f$ by gluing with $\partial f \times \text{id}$, where $\partial f$ is the boundary structure (on $M \times R^i \times 1$) determined by $f$ and $\text{id}$ means the identity map on $[1, \infty[$.

In the square (●), the upper left-hand term is clearly homotopy equivalent to the controlled $h$-cobordism space $H(M \times R^i; c)$. The lower left-hand term is homotopy equivalent to $S(M \times R^i; c)$, via restriction of structures to the boundary. With this identification, the lower horizontal arrow is just $\times R$ up to homotopy. The lower right-hand term is contractible. The square is homotopy cartesian. (Modulo a replacement of controlled structure spaces by the homotopy equivalent bounded structure spaces, this goes back to Anderson and Hsiang [3]. See also [31] for some added details.) Therefore the square amounts to a homotopy fibration sequence

$$
\begin{array}{ccc}
H(M \times R^i; c) & \longrightarrow & S(M \times R^i; c) \\
\times R & \longrightarrow & S(M \times R^{i+1}; c).
\end{array}
$$

This completes the proof. \qed

The proof of lemma 11.9 uses an analogue of (●) on the algebraic side. As a preparation for that, we need to come to terms with pairs of spaces and even pairs of control spaces as inputs for $A$-theory and $L$-theory.

**Notation 13.1.** Let $g: X_1 \rightarrow X_2$ be any map of “ordinary” spaces. There are a number of ways in which one can make sense of notation such as

$$A(X_1 \rightarrow^g X_2).$$

What we mean by it is the $K$-theory spectrum of a certain Waldhausen category $R(X_1 \rightarrow X_2)$ whose objects are pairs of finitely dominated retractive spaces

$$Y_2 \simeq X_2, \quad Y_1 \simeq X_1,$$

together with a cofibration $g_* Y_1 \rightarrow Y_2$ (in the Waldhausen category of finitely dominated retractive spaces over $X_2$). There is an obvious inclusion of $A(X_2)$ in $A(X_1 \rightarrow X_2)$. There is also an obvious forgetful map $A(X_1 \rightarrow X_2) \rightarrow A(X_1)$. It is an easy consequence of the additivity theorem that

$$A(X_2) \longrightarrow A(X_1 \rightarrow X_2) \longrightarrow A(X_1)$$

is a homotopy fibration sequence, and that it splits up to homotopy. Hence

$$A(X_1 \rightarrow X_2) \simeq A(X_1) \vee A(X_2),$$

showing that $A(X_1 \rightarrow X_2)$ as we have defined it should never be confused with the mapping cone of $g_* : A(X_1) \rightarrow A(X_2)$.

We can also introduce an involution on $A(X_1 \rightarrow X_2)$, using an SW-product on the stable version $sR(X_1 \rightarrow X_2)$ of $R(X_1 \rightarrow X_2)$. Namely, let $(Y_2, Y_1)$ and $(Y'_2, Y'_1)$ be objects of $R(X_1 \rightarrow X_2)$. We have the standard SW products in $R(X_1)$ and $R(X_2)$ and we have a map of spectra

$$Y_1 \circ \cdot Y'_1 \longrightarrow Y_2 \circ \cdot Y'_2.$$
induced by $g$. We take its homotopy cofiber (mapping cone), convert it to an $\Omega$-spectrum, and call the result $$(Y_2, Y_1) \odot (Y'_2, Y'_1).$$

This defines a perfectly good $SW'$-product in $sR(X_1 \to X_2)$, and so determines an involution on $A(X_1 \to X_2)$. This works equally well in the twisted setting: we would then assume that $X_2$ comes equipped with a spherical fibration $\xi$, and we would note that $X_1$ comes equipped with $g^*\xi$. Twisted or not, the homotopy fiber sequence

$$A(X_2) \to A(X_1 \to X_2) \to A(X_1)$$

then becomes a homotopy fibration sequence of spectra with an action of $\mathbb{Z}/2$. As such it typically does not split. In the important special case where $X_1 = X_2 = X$ and $g$ is the identity, the canonical splitting map $v: A(X) \to A(X \to X)$ is non-equivariant to such a shocking extent that

$$(vT, Tv): A(X) \vee A(X) \to A(X \to X)$$

is a homotopy equivalence (where $T$ denotes the action of the generator of $\mathbb{Z}/2$, both in the source and in the target of $v$). This observation goes back to [32].

There are analogues of this in $L$-theory. Given $g: X_1 \to X_2$ as before, we use $sR(X_1 \to X_2)$ to define to define $L_*(X_1 \to X_2)$. Then we have (by inspection) homotopy fiber sequences

$$\Omega L_*(X_2) \to \Omega L_*(X_1 \to X_2) \to L_*(X_1),$$

$$\Omega VL_*(X_2) \to \Omega VL_*(X_1 \to X_2) \to VL_*(X_1).$$

It is then rather easy to identify $L_*(X_1 \to X_2)$ with the mapping cone of the map $g_* : L_*(X_1) \to L_*(X_2)$, and to identify $VL_*(X_1 \to X_2)$ with the mapping cone of $g_* : VL_*(X_1) \to VL_*(X_2)$.

To make matters more complicated, it could happen that $g: X_1 \to X_2$ is the non-singular part of a morphism of control spaces

$$\bar{g} : (\bar{X}_1, X_1) \to (\bar{X}_2, X_2).$$

Then the definitions of $A(X_1 \to X_2)$, $L_*(X_1 \to X_2)$ and $VL_*(X_1 \to X_2)$ have controlled analogues which we denote informally by

$$A(X_1 \to X_2; c), \quad L_*(X_1 \to X_2; c), \quad VL_*(X_1 \to X_2; c).$$

There are still homotopy fiber sequences

$$A(X_2; c) \to A(X_1 \to X_2; c) \to A(X_1; c),$$

$$\Omega L_*(X_2; c) \to \Omega L_*(X_1 \to X_2; c) \to L_*(X_1; c),$$

$$\Omega VL_*(X_2; c) \to \Omega VL_*(X_1 \to X_2; c) \to VL_*(X_1; c).$$

The first of these splits, but typically does not split equivariantly. Again, if $\bar{g}$ is an identity, $(\bar{X}_1, X_1) = (\bar{X}_2, X_2) = (X, X)$, then

$$(vT, Tv) : A(X; c) \vee A(X; c) \to A(X \to X; c)$$

is a homotopy equivalence.

Finally, we allow ourselves to write $A(X_2, X_1; c)$ etc. instead of $A(X_1 \to X_2; c)$ if $\bar{g} : (X_1, X_1) \to (X_2, X_2)$ is the inclusion of a control subspace. Also, we may allow triads of control spaces, consisting of $(X_2, X_2)$ and control subspaces $(\bar{X}_1, X_1)$ as well as $(X_0, X_0)$ where $X_0 \cap X_1 = \emptyset$. Then we write $A(X_2, X_1, X_0; c)$ and the like.
Using these conventions, we can set up an algebraic analogue of the commutative square (●). The idea is, essentially, to substitute $F_{\%}$ for $S$. We need an abbreviation:

$$M^{(i)}_J = M \times \mathbb{R}^i \times J$$

for any closed interval $J \subset \mathbb{R}$. If $J$ is a singleton $\{b\}$, we also write $M^{(i)}_b$ for this. The analogue of diagram (●) is then

$$\text{hofib}_q \left( F_{\%}(M^{(i)}_{[0,1]}, M^{(i)}_0, M^{(i)}_1, \ell; c) \right) \longrightarrow \text{hofib}_q \left( F_{\%}(M^{(i)}_{[0,\infty]}, M^{(i)}_0, \ell; c) \right)$$

$$\downarrow \quad \downarrow$$

$$F_{\%}(M^{(i)}_{[-\infty,1]}, M^{(i)}_0, \ell; c) \longrightarrow F_{\%}(M^{(i+1)}, \ell; c).$$

The maps in square (●●) are defined in close analogy with those in square (●). In particular the composite map from upper left to lower right in the diagram is induced by an exact functor of Waldhausen categories which takes a triad $(Y, Y_1, Y_0)$ of retractive spaces (where $Y$ is retractive over $M \times \mathbb{R}^i \times [0,1]$ and $Y_1$ is retractive over $M \times \mathbb{R}^i \times 1$ and $Y_0$ is retractive over $M \times \mathbb{R}^i \times 0$) to the pushout of

$$Y \leftarrow Y_0 \sqcup Y_1 \rightarrow Y_0 \times ]-\infty,0] \sqcup Y_1 \times [1,\infty[$$

which is a retractive space over $M \times \mathbb{R}^i \times \mathbb{R} = M \times \mathbb{R}^{i+1}$. More precisely, this exact functor induces a map

$$F_{\%}(M^{(i)}_{[0,1]}, M^{(i)}_0, M^{(i)}_1, \ell; c) \longrightarrow F_{\%}(M^{(i+1)}, \ell; c)$$

and we pre-compose with the projection from the homotopy fiber

$$\text{hofib}_q \left( F_{\%}(M^{(i)}_{[0,1]}, M^{(i)}_0, M^{(i)}_1, \ell; c) \right) \downarrow$$

$$F_{\%}(M^{(i)}_0, M^{(i)}_1, \ell; c)$$

The homotopy fiber is to be taken over the point

$$q \in F_{\%}(M^{(i)}_0, \ell; c)$$

which is the characteristic element or signature determined by $M \times \mathbb{R}^i$.

For the square (●●) we must now establish the analogues of many of the observations we have made about square (●).

**Lemma 13.2.**

1. The upper left-hand term in (●●) is homotopy equivalent to

$$\Omega^\infty A_{\%}(M \times \mathbb{R}^i \times [0,1]; c).$$

2. The upper right-hand term is contractible.

3. The lower left-hand term is homotopy equivalent, by a forgetful map, to $F_{\%}(M \times \mathbb{R}^i \times 1, \ell - 1; c)$. With that identification the lower horizontal map turns into the map “product with $\mathbb{R}$.”
Proof. We begin with the proof of statement (1). As the forgetful map

\[ \text{Forgetful}(M^{(i)}(0,1), M_0^{(i)}, M_1^{(i)}, \ell; c) \rightarrow \text{Forgetful}(M_0^{(i)}, \ell; c) \]

is a map of infinite loop spaces, all its homotopy fibers are canonically homotopy equivalent and we may focus on the homotopy fiber over the base point of \( \text{Forgetful}(M_0^{(i)}, \ell; c) \). As a consequence of the additivity theorem, this is homotopy equivalent to

\[ \text{Forgetful}(M^{(i)}(0,1), M_0^{(i)}, \ell; c) \].

The inclusion of \( M \times \mathbb{R}^i \times 1 \) in \( M \times \mathbb{R}^i \times [0, 1] \) is a controlled homotopy equivalence. Hence, as explained in 13.1, the spectrum \( \mathbb{L}_*(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c) \) is contractible. By the same reasoning, the same holds with \( \mathbb{L}_* \) replaced by \( \mathbb{L}_*^{\%} \), and then also with \( \mathbb{L}_* \) replaced by \( \mathbb{L}_*^{\%} \). It was also explained that \( A(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c) \) is an “induced” spectrum with \( \mathbb{Z}/2 \)-action; by the same reasoning, that also holds with \( A \) replaced by \( A^{\%} \) or \( A^{\%} \). Therefore

\[ A^{\%}(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c) \]

is also contractible. Hence all that remains of the upper left-hand term in (\( \bullet \bullet \)) is \( \Omega^{\infty+\ell} \) of the spectrum

\[ (S^\ell \wedge A^{\%}(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c))^{h\mathbb{Z}/2} \].

Because of the “induced” property, the forgetful map from that spectrum to

\[ S^\ell \wedge A^{\%}(M \times \mathbb{R}^i \times [0, 1]; c) \]

is a homotopy equivalence. Applying \( \Omega^{\infty+\ell} \) we get \( \Omega^{\infty}A^{\%}(M \times \mathbb{R}^i \times [0, 1]; c) \), as claimed in (1).

The proof of statement (2) comes next. Reasoning as in step (1), we see that the upper right-hand term in (\( \bullet \bullet \)) is homotopy equivalent to

\[ \text{Forgetful}(M \times \mathbb{R}^i \times [0, \infty[, \ell; c) \]

which is the homotopy fiber of an assembly map

\[ \text{Forgetful}(M \times \mathbb{R}^i \times [0, \infty[, \ell; c) \rightarrow \text{Forgetful}(M \times \mathbb{R}^i \times [0, \infty[, \ell; c) \]

We are going to show separately that both source and target of this assembly map are contractible. In fact, the \( \text{Forgetful} \) case is obvious from the excision properties of \( \text{Forgetful} \). To understand \( \text{Forgetful}(M \times \mathbb{R}^i \times [0, \infty[, \ell; c) \), replace the control conditions indicated by “\( c \)” by “boundedness” conditions indicated by “\( b \)”. By the argument of [2], this does not alter the homotopy type. But the space \( \text{Forgetful}(M \times \mathbb{R}^i \times [0, \infty[, \ell; b) \) is contractible since the corresponding Waldhausen category is flasque.

Statement (3) is a consequence of statement (2). Namely, by construction there is
a homotopy fibration sequence of infinite loop spaces

\[ F_\% (M \times \mathbb{R}^i \times ] - \infty, 1] ; c) \]
\[ \downarrow \]
\[ F_\% (M \times \mathbb{R}^i \times ] - \infty, 1], M \times \mathbb{R}^i \times 1, \ell ; c) \]
\[ \downarrow \]
\[ F_\% (M \times \mathbb{R}^i \times 1, \ell ; c) \]
which is clearly split. But \( F_\% (M \times \mathbb{R}^i \times ] - \infty, 1] ; c) \) is contractible. \( \square \)

**Proof of lemma 11.9.** It remains only to prove that the square (••) as a whole is homotopy cartesian. We may replace the reference point \( q \) in \( \text{hofib}_q \) by the base point zero. This shows that square (••) is equivalent to

\[ F_\% (M_{[0,1]}^{(i)}, M_1^{(i)}), \ell ; c) \longrightarrow F_\% (M_{[0,\infty]}^{(i)}, \ell ; c) \]
\[ \downarrow \]
\[ F_\% (M_{-\infty,1]}^{(i)}, M_1^{(i)}), \ell ; c) \longrightarrow F_\% (M_{[i+1]}^{(i)}, \ell ; c). \]

To unravel diagram (a) we set up a whole string of auxiliary commutative squares (explanations provided further down):

\[ A(M \times \mathbb{R}^i \times ] 0, 1] ; c) \longrightarrow A(M \times \mathbb{R}^i \times ] 0, \infty] ; c) \]
\[ \downarrow \]
\[ A(M \times \mathbb{R}^i \times ] - \infty, 1] ; c) \longrightarrow A^h(M \times \mathbb{R}^{i+1} ; c) \]
\[ \downarrow \]
\[ A(M \times \mathbb{R}^i \times ] 0, 1] ; c) \longrightarrow A(M \times \mathbb{R}^i \times ] 0, \infty] ; c) \]
\[ \downarrow \]
\[ A(M \times \mathbb{R}^i \times ] - \infty, 1] ; c) \longrightarrow A^h(M \times \mathbb{R}^{i+1} ; c) \]
\[ A_\% (M \times \mathbb{R}^i \times ] 0, 1], M \times \mathbb{R}^i \times 1 ; c) \longrightarrow A_\% (M \times \mathbb{R}^i \times ] 0, \infty] ; c) \]
\[ \downarrow \]
\[ A_\% (M \times \mathbb{R}^i \times ] - \infty, 1], M \times \mathbb{R}^i \times 1 ; c) \longrightarrow A^h_\% (M \times \mathbb{R}^{i+1} ; c) \]
\[ E_\% (M \times \mathbb{R}^i \times ] 0, 1], M \times \mathbb{R}^i \times 1 ; c) \longrightarrow E_\% (M \times \mathbb{R}^i \times ] 0, \infty] ; c) \]
\[ \downarrow \]
\[ E_\% (M \times \mathbb{R}^i \times ] - \infty, 1], M \times \mathbb{R}^i \times 1 ; c) \longrightarrow E_\% (M \times \mathbb{R}^{i+1} ; c). \]

The details are as follows. The maps in square (b) are obvious inclusion-induced maps. The square is homotopy cartesian by (the proof of) theorem 6.1. The \( h \) superscript in the lower right-hand term should be read as “locally finite and finite dimensional up to controlled homotopy”. The maps in square (c) are induced by exact functors between the corresponding Waldhausen categories, the same functors that we used to set up square (a). Square (c) is again homotopy cartesian. One way
to show this is to use the inclusion (b) → (c). The associated square of homotopy cofibers has a homotopy equivalence in the top row and contractible spectra in the bottom row, which clearly makes it homotopy cartesian. It follows immediately that (d) is also homotopy cartesian. Square (e) is obtained from (d) by applying first $S^i \wedge$ to each term, then $\Omega^\infty \mathbb{R}_i^+$, then the operation homotopy fixed points for the action of $\mathbb{Z}/2$. Each of these three operations preserves homotopy cartesian squares, so that (e) is again homotopy cartesian. (Note also, for use later on, that the first two operations obliterate the difference between $A_{\mathbb{Z}/2}^h$ and $A_{\mathbb{Z}/2}$. This makes the definition of (e) simpler.) Now, considering the definition of square (a), we have

(a) → (e).

It turns out that the resulting cube is homotopy cartesian (and consequently, that (a) is homotopy cartesian). To see this, first look at the face of that cube determined by the top rows of (a) and (e). This is homotopy cartesian since one of its edges is a map between two contractible spaces and the edge opposite to that is a homotopy equivalence, as we saw in the proof of statement (1) of lemma 13.2. Next, look at the face of the cube determined by the bottom rows of (a) and (e). This is homotopy cartesian by statement (2) of lemma 13.2 and lemma 6.7. □

Remark 13.3. Let $\mathbf{ullet \flat}$ be the square obtained from square $\mathbf{\bullet}$ by deleting in each term the non-flat components. (We call an element of, for example, $\pi_0 S(M \times [0,1], M \times \mathbb{R}_i^\times 1 ; c)$ flat if it is in the image of the map

$$\pi_0 S(M \times [0,1], M \times 1 ; c) \rightarrow \pi_0 S(M \times \mathbb{R}_i^\times [0,1], M \times \mathbb{R}_i^\times 1 ; c)$$

given by product with the identity $\mathbb{R}_i^\times \rightarrow \mathbb{R}_i^\times$.) It is now straightforward to produce a map from square $\mathbf{\bullet \flat}$ to square $\mathbf{\bullet \bullet}$ by imitating the construction of the map $\mathbf{\bullet \bullet \bullet}$ at the end of section 12.

Proof of proposition 11.10. So far we do know that the map of vertical homotopy fibers in the square of construction 11.7, restricted to the base point component, has the form

$$\mathcal{H}_0(M \times \mathbb{R}_i ; c) \xrightarrow{\chi} \Omega^\infty A_{\mathbb{Z}/2}(M \times \mathbb{R}_i ; c) .$$

But we also have a fairly good description of $\chi$ which comes from combining remark 13.3 and part (1) of lemma 13.2. It is simply “the” map which to a controlled $h$-cobordism on $M \times \mathbb{R}_i^\times$ associates its characteristic (relative to $M \times \mathbb{R}_i^\times$) in $\Omega^\infty A_{\mathbb{Z}/2}(M \times \mathbb{R}_i^\times ; c)$. In the case $i = 0$, this is familiar from [9] ; and of course, in a slightly different form it is really due to Waldhausen. At this point, although we are still assuming that $M$ is closed, it is worth noting that the construction of $\chi$ (as a map defined on $\mathcal{H}_0(M \times \mathbb{R}_i ; c)$) goes through for any compact manifold with boundary. To estimate the connectivity of $\Omega \chi$, we observe that $\chi$ has a stability property, as follows. On the geometric side there is a well-known homotopy cartesian square of $h$-cobordism spaces

$$\begin{array}{ccc}
\mathcal{H}(M \times [-1,+1] \times \mathbb{R}_i^{-1} ; c) & \longrightarrow & \mathcal{H}(M \times [-1, +\infty] \times \mathbb{R}_i^{-1} ; c) \\
\downarrow & & \downarrow \\
\mathcal{H}(M \times [-\infty,+1] \times \mathbb{R}_i^{-1} ; c) & \longrightarrow & \mathcal{H}(M \times \mathbb{R}_i^\times ; c)
\end{array}
$$

(1)
with contractible off-diagonal terms. Hence
\[ \mathcal{H}(M \times \mathbb{R}^{i-1} \times [-1, +1]; \epsilon) \simeq \Omega \mathcal{H}(M \times \mathbb{R}^{i-1} \times [-\infty, +\infty]; \epsilon) \]
for \( i > 0 \). On the more algebraic side, there is a similar homotopy cartesian square
\[ \Omega^\infty A_{\mathbb{R}}(M \times [-1, +1] \times \mathbb{R}^{i-1}; \epsilon) \longrightarrow \Omega^\infty A_{\mathbb{R}}(M \times [-1, +\infty] \times \mathbb{R}^{i-1}; \epsilon) \]
\[ \Omega^\infty A_{\mathbb{R}}(M \times ] - \infty, +1] \times \mathbb{R}^{i-1}; \epsilon) \longrightarrow \Omega^\infty A_{\mathbb{R}}(M \times \mathbb{R}^i; \epsilon) \]
with contractible off-diagonal terms; we had this before in the proof of theorem 6.1. Let \((1)_0\) be the square obtained from \((1)\) by throwing away the non-base point components. It is easy to see that \( \chi \) extends to a map of squares, from square \((1)_0\) to square \((2)\). We can deduce two useful facts from that. Firstly,
\[ \mathcal{H}_{0}(M \times \mathbb{R}^i; \epsilon) \longrightarrow \Omega^\infty A_{\mathbb{R}}(M \times \mathbb{R}^i; \epsilon) \]
duces an isomorphism in \( \pi_1 \). To see this, think of \( \pi_1 \) of the source and target as \( \pi_0 \) of \( \mathcal{H}(M \times [-1, +1] \times \mathbb{R}^{i-1}; \epsilon) \) and \( A_{\mathbb{R}}(M \times [-1, +1] \times \mathbb{R}^{i-1}; \epsilon) \), respectively. Describe the homomorphism in these terms. It is still given by relative characteristics and it is therefore bijective by the “controlled \( h \)-cobordism theorem”. Secondly, we have a commutative square
\[ \Omega^2 \mathcal{H}(M \times \mathbb{R}^i; \epsilon) \longrightarrow \Omega^{i+2} A_{\mathbb{R}}(M \times \mathbb{R}^i; \epsilon) \]
\[ \Omega \mathcal{H}(M \times D^1 \times \mathbb{R}^{i-1}; \epsilon) \longrightarrow \Omega^{i+1} A_{\mathbb{R}}(M \times D^1 \times \mathbb{R}^{i-1}; \epsilon) \]
Induction on \( i \) using these two facts then shows that
\[ \Omega \mathcal{H}(M \times \mathbb{R}^i; \epsilon) \longrightarrow \Omega^{i+1} A_{\mathbb{R}}(M \times \mathbb{R}^i; \epsilon) \]
is \((k_M + i)\)-connected. Indeed
\[ \Omega \mathcal{H}(M \times D^i) \longrightarrow \Omega^{i+1} A_{\mathbb{R}}(M \times D^i) \]
can be identified with the inclusion of \( \Omega \mathcal{H}(M \times D^i) \) in \( \text{colim}_j \Omega \mathcal{H}(M \times D^{i+j}) \) by Waldhausen’s \( h \)-cobordism theory, and is therefore \( k_M \)-connected by the definition of \( k_M \).

**Remark 13.4.** Although theorem 11.11 is meant as a fact and a quotation, a socio-historical essay (if not strictly mathematical) essay around it is in order, with a view to the proof of lemma 11.12. Perhaps the best known, most polished and least complicated method for establishing a homotopy fibration sequence
\[ \tilde{S}_k^*(M \times \mathbb{R}^i; \epsilon) \longrightarrow \Omega^{i+m+1} L_k^d(M \times \mathbb{R}^i; \epsilon) \longrightarrow L_0(\mathbb{Z})^n M \]
is the one developed by Ranicki [19, §18]. We summarize the key points, referring to [19] for definitions and clarifications.

1. The cases \( i > 0 \) can be reduced to the case \( i = 0 \) by a torus trick.
2. A compact polyhedron \( X \) and a homotopy equivalence \( e:M \to X \) transverse to the triangulation of \( X \) are chosen. A degree one normal map \( f:N \to M \) between closed manifolds determines (if \( f \circ e \) is transverse to the triangulation of \( X \)) chain complexes \( C(N) \) and \( C(M) \) “dissected” over \( X \), with dissected nondegenerate symmetric structures of formal dimension
The map \( f_\ast : C(N) \rightarrow C(M) \) respects the symmetric structures and so determines a splitting

\[
C(N) \simeq D \oplus C(M)
\]

where the “kernel” \( D \) is again dissected and comes with a nondegenerate symmetric structure. But \( D \) is globally (after assembly) contractible. Hence the dissected symmetric structure has an automatic refinement to a dissected quadratic structure.

(3) \( L_\ast \mathbb{Z}(M) \) admits a description as the bordism theory of chain complexes dissected over \( X \), with a dissected quadratic Poincaré structure. With that description, the assembly map is induced by the assembly (“universal” assembly) of dissected chain complexes.

(4) The local degree homomorphism from \( \pi_m L_\ast \mathbb{Z}(M) \) to \( L_0(\mathbb{Z}) \pi_0 M \) is zero on elements determined by degree one normal maps \( f : N \rightarrow M \). (The reason is that an oriented map \( f \) between manifolds of the same dimension which has “global” degree one, in the sense that it respects fundamental classes, will also have local degree one, i.e., the generic cardinality of \( f^{-1}(x) \) is 1 for any \( x \) in the target. It is this relationship between global and local degree which has been shown to fail spectacularly \([4]\) in the world of ANR homology manifolds.)

These ideas, generalized to a “parameterized” setting (where the parameter space is \( \Delta^k \) for \( k = 0, 1, 2, \ldots \)) lead to a map

\[
\tilde{S} \ast(M \times \mathbb{R}^i ; c) \rightarrow \Omega^{\infty+n+i} L_\ast \mathbb{Z}(M \times \mathbb{R}^i ; c).
\]

Proving that the map is a homotopy equivalence except for a deviation in \( \pi_0 \) is another matter and there is no need to go into that here. What we need to do is this: unravel each of the items (1)–(3) and relate it to the methods (e.g., the “characteristic element” method and the “control method”) which we have favored in this paper.

(1a). The torus trick consists in using that \( S \ast(M \times \mathbb{R}^i ; c) \) and \( L_\ast \mathbb{Z}(M \times \mathbb{R}^i ; c) \) are homotopy retracts of \( S \ast(M \times (S^1)^i) \) and \( L_\ast \mathbb{Z}(M \times (S^1)^i) \), respectively. The “retracting” maps

\[
S \ast(M \times (S^1)^i) \rightarrow S \ast(M \times \mathbb{R}^i ; c),
\]

\[
L_\ast \mathbb{Z}(M \times (S^1)^i) \rightarrow L_\ast \mathbb{Z}(M \times \mathbb{R}^i ; c)
\]

are obvious transfer maps in both cases.

(2a). Dissection theory works with retractive spectra over \( X \) and visible symmetric structures just as well as with chain complexes and symmetric structures. It remains true that a dissected visible symmetric structure on a dissected retractive spectrum \( Y \) over \( X \) (subject to some finiteness conditions) automatically lifts to a dissected quadratic structure if \( Y \) is globally weakly equivalent to zero.

(3a). The interpretation of the assembly map in terms of “assembly of dissected chain complexes” works just as well with dissected retractive spaces and spectra. Given a degree one normal map \( f : N \rightarrow M \) which is a homotopy equivalence, there are two slightly different ways of using the dissection argument to extract \( L \)-theoretic information.

- One way is to form the “visible symmetric kernel” (mapping cone of the stable Umkehr map \( M \mathbb{II} X \rightarrow N \mathbb{II} X \), with a nondegenerate visible symmetric
structure). It is dissected over $X$ along with its nondegenerate visible symmetric structure. Since $f$ is a homotopy equivalence, the kernel is globally contractible and the dissected nondegenerate visible symmetric structure on it automatically lifts to a dissected nondegenerate quadratic structure. The global contractibility of the kernel implies a preferred “global nullbordism” of the kernel (i.e., a nullbordism of the assembled kernel with the assembled quadratic structure). The dissected kernel and the global nullbordism then determine a point in the homotopy fiber (over the base point “zero”) of the assembly map

$$\Omega^{\infty+m}L_\bullet(X) \to \Omega^{\infty+m}L_\bullet(X).$$

Another way is to note that $M \amalg X$ and $N \amalg X$ can themselves be dissected over $X$, and come with dissected nondegenerate visible symmetric structures. Before assembly, these may not be equivalent; after assembly they are certainly equivalent via $f \amalg \text{id} : N \amalg X \to M \amalg X$. Hence these data determine a point in a homotopy fiber of the assembly map

$$\Omega^{\infty+m}VL_*^\infty(X) \to \Omega^{\infty+m}VL_*(X).$$

This time we take the homotopy fiber over the point in $\Omega^{\infty+m}VL_*(X)$ determined by $M \amalg X$ with its (assembled) nondegenerate visible symmetric structure.

There is an obvious “compatibility” between the two methods. By theorem 2.7, we lose no $L$-theoretic information by relying on the second method. But now another little problem remains. We have two descriptions of the assembly map in $L$-theory, one in terms of “dissections” and another one using control. How are they related? An easy way to make a connection is to use both approaches simultaneously. Very schematically, we have one description of $VL_*^\infty$-theory as

$$VL_*(X, \text{dissected})$$

and another as the homotopy fiber of an inclusion

$$VL_*(X) \to VL_*(X \times [0, 1], \text{controlled}),$$

as in section 9. There is a third description of $VL_*^\infty(X)$ as the homotopy fiber of an inclusion

$$VL_*(X, \text{dissected}) \to VL_*(X \times [0, 1], \text{controlled and dissected}).$$

(The dissections are always over $X$, even where we are dealing with retractive spaces over $X \times [0, 1]$. “Control” refers to the control space $JX$.) Validating this third formula amounts to showing that

$$VL_*(X \times [0, 1], \text{controlled and dissected})$$

is contractible. (This is left to the reader.) This makes the connection which we were after. Moreover it does that in such a way that the two standard methods (by dissection and by control) of lifting “manifold signatures” in $VL$-spaces across the assembly map are seen to agree.

Proof of lemma 11.12. The homotopy colimit of

$$A^R_\bullet(M) \to A^R_\bullet(M \times \mathbb{R}; c) \to A^R_\bullet(M \times \mathbb{R}^2; c) \to \cdots$$
is contractible by lemma 6.5. It follows that
\[
\text{hocolim}_m (A_{\mathbb{Z}}(M \times \mathbb{R}^i; m + i; c))_{h\mathbb{Z}/2}
\]
is also contractible. (The dimension indicator \(m + i\) here specifies the involution or
the SW-product, which is obtained from the standard one by \((m + i)\)-fold looping).
Therefore the inclusion
\[
\text{hocolim}_m (A_{\mathbb{Z}}(M \times \mathbb{R}^i; m + i; c))_{h\mathbb{Z}/2} \to \text{hocolim}_m (A_{\mathbb{Z}}(M \times \mathbb{R}^i; m + i; c))_{h\mathbb{Z}/2}
\]
is a homotopy equivalence. Therefore the forgetful map
\[
\text{hocolim}_m F_{\mathbb{Z}}(M \times \mathbb{R}^i; c) \to \text{hocolim}_m \Omega^{\infty + m + i} L_* \mathbb{R}_{\mathbb{Z}/2}(M \times \mathbb{R}^i; c)
\]
is also a homotopy equivalence. Hence in the limit \(i \to \infty\), it does not matter
whether we use the map \(\varphi\) in construction 11.7 or a simplified version
constructed purely in terms of \(L\)-theoretic signatures (as in sections 8 and 9, but
without any algebraic \(K\)-theory). It is easy to factorize this simplified map through
\[
\tilde{S}_0(M \times \mathbb{R}^i; c).
\]
The reason is that we can extend \((\circ)\) by adding on another simplicial direction. In
other words, there are maps
\[
\tilde{S}_0(M \times \Delta_k \times \mathbb{R}^i; c) \to \Omega^{\infty + m + i} L_* \mathbb{R}_{\mathbb{Z}/2}(M \times \Delta_k \times \mathbb{R}^i; c) \quad (\circ)
\]
for every \(k \geq 0\), generalizing \((\circ)\); the notation \(\Delta_k\) means that we think of \(\Delta_k\) as a
diagram of manifolds (the faces of any codimension), not as a single manifold with
boundary. By taking geometric realizations over \(k\) (and noting, as we have done
before, that on the target side all the face operators are homotopy equivalences),
we obtain a single map
\[
\tilde{S}_0(M \times \mathbb{R}^i; c) \to \Omega^{\infty + m + i} L_* \mathbb{R}_{\mathbb{Z}/2}(M \times \Delta_k \times \mathbb{R}^i; c).
\]
Now we let the new simplicial direction take over by restricting \((\circ\circ)\) to appropriate
0-skeletons for each \(k\), and noting that that does not affect the homotopy type of the
geometric realization over \(k\). Then what we have is just the “standard” map of
\(\Omega\) theory using the decoration \(p\) throughout. This is explained in [31].

**Proof of lemma 11.6.** We normally define \(L\)-theory in terms of \(L\)-theory and \(A\)-theory using the decoration \(p\) throughout. However, it follows easily from lemma 6.7
that it does not make a difference to the meaning of \(\pi_m L A_{\mathbb{Z}/2}(M, m)\) if we use the
decoration \(h\) throughout. Then \(\pi_m L A_{\mathbb{Z}/2}(M, m)\) becomes \(\pi_m\) of the homotopy fiber
of a certain map
\[
L^h \mathbb{R}_{\mathbb{Z}/2}(M) \to S^1 \wedge \bigl(S^m \wedge A^h(M)\bigr)_{h\mathbb{Z}/2}.
\]
We simplify this to: \(\pi_0\) of the homotopy fiber of a certain map
\[
u: \Omega^m L^h \mathbb{R}_{\mathbb{Z}/2}(M) \to B(\mathbb{Z} \otimes_{\mathbb{Z}/2} \text{Wh})
\]
where \(\text{Wh} = \pi_0 A^h(M)\) is the Whitehead group of \(M\). (According to this definition,
the Whitehead group comes with a standard action of \(\mathbb{Z}/2\) by automorphisms. Also,
$\mathbb{Z}/2$ acts on $\mathbb{Z}$ via multiplication with $(-1)^m$. Now it will be convenient to identify $\Omega^m \mathbf{L}_{\mathbb{Z}/2}^h(M)$ with the block structure space $\tilde{S}^h(M)$.

This uses the decoration $h$ version of theorem 11.11. (A word of warning is in order. Two homotopy equivalences $f_1: N_1 \to M$ and $f_2: N_2 \to M$ are in the same component of this block structure space if and only if they are $h$-cobordant over $M$. This does not force $N_1$ and $N_2$ to be homeomorphic.) Using such an identification, i.e., using any of the equivalent ways of making such an identification described in remark 13.4, we may write

$$u: \tilde{S}^h(M) \to B(\mathbb{Z} \otimes_{\mathbb{Z}/2} \text{Wh}).$$

In this form, $u$ is easy to unravel. It takes all 0-simplices in $\tilde{S}^h(M)$ to the base point of $B(\mathbb{Z} \otimes_{\mathbb{Z}/2} \text{Wh})$. It takes a 1-simplex corresponding to an $h$-cobordism (over $M$) of torsion $t$ to the loop in $B(\mathbb{Z} \otimes_{\mathbb{Z}/2} \text{Wh})$ determined by $1 \otimes t$. (This partial definition or description of $u$ can be extended recursively to the higher skeleta because $B(\mathbb{Z} \otimes_{\mathbb{Z}/2} \text{Wh})$ has vanishing higher homotopy.) From that description of $u$, there is a canonical map $e: \pi_0 \mathbf{S}(M) \to \pi_0 \text{hofiber}(u)$ and that is precisely the map $\pi_0 \mathbf{S}(M) \to \pi_m \mathbf{L}_{\mathbb{Z}/2}^h(M, m)$ which we are investigating. It should now be clear that $e$ is bijective. □

14. Appendix: Corrections and Elaborations

Remark 14.1. There is an unfortunate oversight in [34], as follows. In [34, 7.1] we have an enlarged model $xK(C)$ of the $K$–theory space $K(C)$ of a Waldhausen category $C$ with Spanier–Whitehead product,

$$xK(C) = \Omega[xwS_C].$$

Here $xwS_C$ is an enlarged model of Waldhausen’s $wS_C$. It is a simplicial category with a degreewise involution which anticommutes with the simplicial operators. The involution at the category level induces an involution on $[xwS_C]$. It should have been pointed out just before [34, 7.2] that this must be combined with the “reverse loops” operation to give the preferred involution on $\Omega[xwS_C] = xK(C)$. With that convention, the standard inclusion of $[xwC]$ in $\Omega[xwS_C]$ respects the preferred involutions. This is used in [34, §9].

Remark 14.2. We have frequently encountered the following constellation in this paper: a Waldhausen category $\mathcal{D}$ with $SW$-product satisfying all the usual axioms and a Waldhausen subcategory $\mathcal{C} \subset \mathcal{D}$ closed under weak equivalences and “duals”. Then it is often useful to have something like a “quotient” of $\mathcal{D}$ by $\mathcal{C}$, designed in such a way that the algebraic $K$-theory spectrum of the quotient is homotopy equivalent to the mapping cone of $K(C) \to K(D)$, and similarly for the various $L$-theory spectra. From the point of view of algebraic $K$-theory the easiest and best approach is to continue using $\mathcal{D}$, but with a new notion of weak equivalence where all morphisms in $\mathcal{D}$ whose mapping cones are in $\mathcal{C}$ qualify as weak equivalences. From an $L$-theory point of view, this seems (at first) less fortunate because the old $SW$ product in $\mathcal{D}$, with the new notion of weak equivalence, will normally violate one of the basic conditions for an $SW$ product [34, 1.1]. It is probably possible to repair this by introducing a new $SW$ product to go along with the new
the required control condition. Here is a corrected proof. For each integer $i \geq 0$, define $\psi_i: [0, 1] \to [0, 1]$ so that $\psi_i(t) = (1 - \log_2(t))/(1 + i)$ if $t \geq 2^{-i}$ and $\psi_i(t) = 1$ otherwise. There is an endomorphism of the control space $(Z \times [0, 1], Z \times [0, 1])$ given by the formula

$$(x, s, t) \mapsto ((x, s \cdot \psi_i(t)), t)$$

for $(x, s) \in Z = X \times [0, 1]$. Denote the induced endofunctor of $A^p(JZ)_{\infty}$ by $\sigma_i$. Note that $\sigma_0 = \text{id}$, and all the $\sigma_i$ are related by invertible natural transformations (so that $\sigma_i$ is isomorphic to $\sigma_j$ for any $i, j \geq 0$). We re-define $\tau$ by the formula

$$\tau(A) = \bigoplus_{i \geq 0} \sigma_i$$

and this time it is an endofunctor of $A^p(JZ)_{\infty}$. There is an Eilenberg swindle in the shape of a natural isomorphism of functors $\tau \cong \text{id} \oplus \tau$. This uses the natural isomorphisms of functors $\sigma_i \to \sigma_{i+1}$ mentioned earlier. Hence, for the self-map $\tau_*$ of the infinite loop space $K(A^p(JZ)_{\infty})$ we have $\tau_* + \text{id} \simeq \tau_*$.

Remark 14.4. The existence and uniqueness of Spivak normal fibrations for a Poincaré duality space $X$ of formal dimension $n$ has been repeatedly used in this paper. What does it mean? Let $G_k$ be the space of based homotopy automorphisms of $S^{k-1}$. Let $\xi_k$ be the canonical quasi-fibration on $BG_k$ with fibers $\simeq S^{k-1}$. Form the space $U_k(X)$ of pairs $(g, \eta)$ where $g: X \to BF_k$ and $\eta$ is a based map from $S^{n+k}$ to the Thom space of $g^*\xi_k$. (Here, Thom space means the mapping cone of the projection.) The existence and uniqueness claim is that

$$U_{\infty}(X) = \text{colim}_k U_k(X)$$

is contractible. Wall [26] shows that $U_{\infty}(X)$ is connected, and it seems likely that similar (Spanier-Whitehead duality) arguments could be employed to show that $U_{\infty}(X)$ is contractible. A different argument is as follows. Suppose first that $X$ has the homotopy type of a compact CW-complex. For $k \gg 0$ we consider pairs $(N, e)$ where $N$ is a compact smooth codimension zero submanifold of $\mathbb{R}^{n+k}$, the inclusion $\partial N \to N$ induces in isomorphism on $\pi_1$, and $e: N \to X$ is a homotopy equivalence. We can call such a thing a regular neighborhood of $X$ in $\mathbb{R}^{n+k}$. These regular neighborhoods can be regarded as 0-simplices of a suitable simplicial set where the $j$-simplices are certain regular neighborhoods of $X \times \Delta^j$ in $\mathbb{R}^{n+k} \times \Delta^j$. Let $V_k(X)$ be its geometric realization. It is relatively easy to verify that

$$V_{\infty}(X) = \text{colim}_k V_k(X)$$

is contractible; this amounts to an existence and uniqueness statement for regular neighborhoods of $X$ in euclidean space. Hence it is enough to show that there exist
compatible homotopy equivalences

\[ V_k(X) \to U_k(X) \]

for large enough \( k \). Indeed, for any \( (N, e) \) in \( V_k(X) \), the homotopy fibers of \( \epsilon|_{\partial N} : \partial N \to X \) are \( (k-1) \)-spheres [17]. Then \( N/\partial N \) can be regarded as the Thom space of a spherical fibration on \( X \) and the Pontryagin-Thom collapse from \( \mathbb{R}^{n+k} \cup \infty \to N/\partial N \) is a Spivak reduction. This gives maps \( V_k(X) \to U_k(X) \). To understand why these maps \( V_k(X) \to U_k(X) \) should be highly connected, fix some \( (g, \eta) \) in \( U_k(X) \). The mapping cone of \( g^*\xi_k \) is a quotient of the mapping cylinder. The mapping cylinder comes with a map \( \delta \) to \([0, 1] \) measuring the “distance” to the zero section \( X \). If \( \eta \) is transverse to \( \delta^{-1}(1/2) \), then \( \eta^{-1} \) of \( \delta^{-1} \) of \([0, 1/2] \) is a codimension zero smooth compact submanifold \( N \) of \( S^{n+k} \) avoiding the base point of \( S^{n+k} \). This comes with an obvious map \( e : N \to X \), and more importantly, with a map of pairs \( \tilde{e} \) from \( (N, \partial N) \) to the (disk bundle, sphere bundle) pair associated with \( g^*\xi_k \). Now \( e \) may not be a homotopy equivalence. But embedded surgery on \( (N, \partial N) \), with the goal of making \( \tilde{e} \) embedded bordant to a homotopy equivalence of pairs, will repair that. Hence the map \( V_k(X) \to U_k(X) \) is 0-connected. A parameterized version of this argument shows that \( V_k(X) \to U_k(X) \) is \( j \)-connected for any \( j \).

If \( X \) is finitely dominated with nonzero finiteness obstruction, then \( X \times S^1 \) has zero finiteness obstruction, thanks to M. Mather; see also [26]. But \( R_\infty(X) \) is a homotopy retract of \( R_\infty(X \times S^1) \); hence \( R_\infty(X) \) is again contractible.

References


E-mail address: m.weiss@maths.abdn.ac.uk
E-mail address: williams.40@nd.edu