Disjunctive Syllogism:
the universal characterization of multiplicative "or"

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The inference pattern known as disjunctive syllogism (DS) appears as a derived rule in Gentzen's natural deduction calculuses NI and NK. This is a paradoxical feature of Gentzen's calculi in so far as DS is sometimes thought of as appearing intuitively more elementary than the rules $\vee$E, $\neg$E, and EFQ that figure in its derivation. For this reason, many contemporary presentations of natural deduction depart from Gentzen and include DS as a primitive rule. However, such departures violate the spirit of natural deduction, according to which primitive rules are meant to relationally define logical connectives *via* universal properties (§2). This situation raises the question: Can disjunction be relationally defined with DS instead of with Gentzen's $\vee$I and $\vee$E rules? We answer this question in the affirmative and explore the duality between Gentzen's definition and our own (§3). We argue further that the two universal characterizations, rather than provide competing relational definitions of a single disjunction operator, disambiguate natural language's "or" (§4). Finally, this disambiguation is shown to correspond exactly with the additive and multiplicative disjunctions of linear logic (§5). The hope is that this analysis sheds new light on the latter connective, so often deemed mysterious in writing about linear logic.

1. Sextus Empiricus attributed to Chrysippus of Soli the claim that hunting dogs exhibit a particular inference pattern in their observed behavior:

> And according to Chrysippus, who shows special interest in irrational animals, the dog even shares in the far-famed "dialectic." This person, at any rate, declares that the dog makes use of the fifth complex indemonstrable syllogism when, on arriving at a spot where three ways meet, after smelling at the two roads by which the quarry did not pass, he rushes off at once by the third without stopping to smell. For, says the old writer, the dog implicitly reasons thus: "The creature went either by this road, or by that, or by the other: but it did not go by this road or by that: therefore it went by the other."[1]

The modern name of the inference pattern governing the dogs' alleged implicit reasoning is "disjunctive syllogism." It is schematically presented as an inference from the premises

[1] Sextus Empiricus *Outlines of Pyrrhonism* I.69, from *Bury 1933*

"Either A or B" and "not A" to the conclusion "B." The anecdote Sextus Empiricus provided seems to demonstrate not only the basic pattern but also its iterated use: Our "quarry" has taken one of paths A, B, or C. Ruling out path A leaves the question whether it took path B or path C. Further ruling out path B allows us to conclude that it took path C. If we denote the inference figures $\dfrac{\neg A \qquad A \vee B}{B}$ and $\dfrac{\neg B \qquad A \vee B}{A}$ by DS, we can represent the alleged canine reasoning by

$$\dfrac{\neg B \qquad \dfrac{\neg A \qquad A \vee (B \vee C)}{B \vee C}\,\text{DS}}{C}\,\text{DS}$$

Following Bobzien (*1996*) we suggest that the Stoics would have formalized the hunting dog example as we have done here, as an iterated application of the rule DS, first to a set of premises the major one of which has a compound clause, then to a set of premises the major one of which is that same clause. An alternative conceptualization of disjunctive syllogism (DSA) would avail itself of the associativity of the disjunction and apply singly to n premises one of which is a disjunction of n alternatives, the others of which are the negations of all but one of those alternatives: $\dfrac{\neg A_1 \ldots \neg A_{n-1} \qquad A_1 \vee \ldots \vee A_{n-1} \vee B}{B}$. What is important is that DS or DSA, whichever they intended, is labeled "indemonstrable": for Chrysippus and his followers, the disjunctive syllogism is one of the five or so inference patterns that arbitrary arguments can be reduced to. It does not stand to be further reduced to other simpler inference patterns.[2]

2. The claim that disjunctive syllogism is indemonstrable intrigues. On the one hand, as we shall see, subsequent developments in propositional logic strongly indicate that its validity can be reduced to that of other clearly identifiable and more elementary inference patterns. On the other hand, many people have an intuitive sense that disjunctive syllogism is more evidently valid than the inferences it is thereby shown to reduce to. One approach to this tension is to distinguish logical and psychological senses in which an inference pattern can be thought of as basic. An example of a vindication of the Stoics against the apparent commination from formal logic along these lines is *López-Astorga 2015*. In this paper we propose a different style of vindication drawn from formal logic's own further details.

First let us consider the way in which disjunctive syllogism appears as a derived inference, rather than as a basic one, in modern propositional logic. In Gerhard Gentzen's *natural deduction* systematization of logic, introduced in *1934–35*, the propositional fragment of the classical calculus NK consists of rules

[2] *Bobzien 2020* defends one understanding of the Stoic doctrine of the ἀναπόδεικτοί against several alternatives. But all interpretations agree that by calling an inference pattern "indemonstrable," the Stoics meant that it does not stand to be further reduced to more elementary inference patterns.

$$\frac{A \supset B \qquad A}{B}\supset\text{E} \qquad\qquad \frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}\supset\text{I}$$

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}\wedge\text{E} \qquad\qquad \frac{A \qquad B}{A \wedge B}\wedge\text{I}$$

$$\frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\vee\text{E} \qquad\qquad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B}\vee\text{I}$$

$$\frac{\neg A \qquad A}{\bot}\neg\text{E} \qquad\qquad \frac{\begin{matrix}[A]\\ \bot\end{matrix}}{\neg A}\neg\text{I}$$

$$\frac{\bot}{A}\text{EFQ} \qquad\qquad \frac{\neg\neg A}{A}\text{DN}$$

Natural deduction proofs are trees constructed by iterating these rules. Applications of $\vee$-Elim, $\supset$-Intro, and $\neg$-Intro are labeled with natural numbers 1, 2, . . ., to indicate which, if any, assumptions (identified in brackets) are discharged, i.e., those assumptions, if any, bearing the same label as the rule. Discharge of assumptions is inessential to these rules so that, for example, $B \supset A$ is derivable from A by one application of $\supset$-Intro with "vacuous discharge." When all the assumptions in such a proof are discharged by the application of a rule, the sentence at the tree's single root node is said to be proved. Otherwise this sentence is said to be derived from the set of assumptions that remain open at leaf nodes or, more succinctly, the inference scheme whose premises are the proof's open assumptions and whose conclusion is the sentence at the tree's root is said to be derived. Gentzen called the result of deleting the rule DN NI, the natural deduction calculus for intuitionistic propositional logic. Shortly after Gentzen published his thesis, Ingebrigt Johansson (*1937*) studied the result of deleting also the rule EFQ. This system with only the introduction and elimination rules for each of the canonical propositional connectives is called NM, the natural deduction calculus for minimal propositional logic. An elementary observation is that the rule EFQ is redundant in NK although it is not so in NI.

Here is a simple natural deduction derivation of DS:

$$\frac{A \vee B \qquad \dfrac{\dfrac{\overset{1}{[A]} \qquad \neg A}{\bot}\neg\text{E}}{B}\text{EFQ} \qquad \overset{1}{[B]}}{B}\vee\text{E}_1$$

The rule DN is nowhere used in this derivation. On the other hand, Johansson (*1937*) showed that the use of EFQ is essential. Thus $\neg A, A \vee B \vdash_{\text{NI}} B$ but $\neg A, A \vee B \nvdash_{\text{NM}} B$. On its surface, this state of affairs draws into question the "indemonstrable" status of DS. Here its validity is reduced to that of ¬E, ∨E, and EFQ. Logic instructors are familiar with the irony of the reduction. DS strikes typical students as far more basic an inference scheme than ∨E, with its network of subproofs. Especially surprising is the news that EFQ is unavoidable, for this inference seems very strong to most students and even fallacious to many, whereas very few students express any reservations about DS. Here is an occasion to say, "If anyone still harbors misgivings about EFQ, it would appear that they should also give up DS."

One might wonder if situating DS in natural deduction is therefore misleading in some way. Maybe Gentzen's rules are not the right grid to impose on the disjunctive syllogism. Perhaps in another framework, its indemonstrability can be preserved and its association with what are often regarded as intuitively more suspicious principles can be avoided. In this spirit, one even finds textbooks in which classical propositional logic is presented in the style of natural deduction, but in place of Gentzen's I/E rules are found an assortment of other basic inference figures. Stoic "indemonstrables" such as DS and *modus tollens* are common entries in such treatments. In fact, Pelletier (*1999*) surveyed the most prominent 20th century textbook presentations of natural deduction and observed that "the Gentzen ideal of introduction and elimination rules for each connective was followed only by Fitch [*1952*]" (p. 23).

It is true that there is something arbitrary about the choice of basic principles in any formulation of propositional logic. But natural deduction style presentations of logic that depart from Gentzen's I/E ideal overlook something else non-arbitrary about the choice of basic principles. This something else is the main idea behind natural deduction, an idea that lends added force to the claim that DS is demonstrable.

The way Gentzen put it is not particularly lucid, but the idea has been refined and clarified into one of the major conceptual developments of 20th century logic, so we can see retrospectively just what he meant. Gentzen's original statement was that natural deduction's introduction rules "represent the definitions of" the logical constants and that its elimination rules are "consequences of" these definitions (*1934–35*, p. 80).

For an illustrative example, consider Gentzen's introduction rule for ∨, commonly referred to as "addition":

$$A \vdash A \vee B \text{ and } B \vdash A \vee B$$

To say that this rule is not only valid but defining of the ∨ connective is to say that any other sentence that can legitimately fill in the blanks in A ⊢_____ and B ⊢_____ , any sentence, that is, that stands in the same inferential relationships as A ∨ B, must itself be inferable from A ∨ B:

$$\text{For all C, if } A \vdash C \text{ and } B \vdash C, \text{ then } A \vee B \vdash C$$

But notice that this last expression, commonly referred to as "proof-by-cases," is just Gentzen's elimination rule for ∨! Thus one can derive ∨E, not (as Gentzen's own words suggest) from the corresponding $\vee$I rule, but (as he surely meant) from the conception of that rule as defining of $\vee$. When one defines the disjunction of A and B as the thing that can be inferred from both A and B, $\vee$I captures the part about it being so inferable, whereas $\vee$E captures the part about it being "*the thing*" that is.

Gentzen's idea of characterizing $\vee$ with an I/E rule pair is an example of definition with a *universal property*[3]: specifying a way that a particular sentence is related to other sentences such that if any other sentence is related to those same sentences in the same way, then that sentence will bear a further relation to the specified sentence indicating that it is the extreme, ideal, or archetypical sentence standing in this relation. Here "addition" is said to hold of $\vee$ definitionally; "proof-by cases" then follows by universality.[4]

In the same fashion as for $\vee$, the I/E rules Gentzen specified for each of the propositional connectives $\wedge$, $\neg$, $\supset$ provide a definition of the connective in the form of a universal property. Thus natural deduction exemplifies the phenomenon of defining something in terms of its situation in the network of relations it bears to other things of the same type, just as truth-functional semantics exemplifies the phenomenon of defining something in terms of the abstract object it refers to or its internal constitution. You can think of these as the relational and essential styles of definition.

The idea of relationally defining logical connectives *via* universal properties will evoke in many readers' minds connections with the proof-theoretic semantics program's idea of "logical harmony." The idea of logical harmony is that inference rules can properly be thought of as specifying a connective's meaning only when certain conditions are met, conditions such as invertibility (*Schroeder-Heister 2006*), normalization (*Prawitz 1965*), or maximal inferential strength (*Tennant 1978*). The proof-theoretic literature on logical harmony is extensive and arguably tendentious, culminating with Steinberger's (*2013*) observation of the inequivalence of its various proposed precisifications, "harmony–as–conservative extension," "harmony–as–leveling procedure," and "harmony–as–deductive equilibrium." The conception of logical connectives as universal properties is meant to adhere to the core of Gentzen's idea and to support a coherent notion of relational definition without favoring any particular elaboration of the harmony concept.

[3]Formally introduced in *Samuel 1948*. A modern treatment is *Bergman 2015*.

[4]As we stressed above, Gentzen did not present his natural deduction calculi in terms of universal constructions. He could not have. Although latent in many mathematical arenas dating back as far as the ancient Greek study of a pair of integers' greatest common factor, the concept of a universal property was only formally introduced in *Samuel 1948*. The association of universal properties with logical connectives came later still in Lawvere's (*1963*) thesis. Even though the idea is so central today that a textbook on *Basic Category Theory* can begin with the declaration that "[t]he most important concept in this book is that of *universal property*" (*Leinster 2014*, p. 1), its explicit formulation evaded Gentzen. With hindsight, Lawvere's observation reveals that the meet, join, and adjoint objects in the ordered structures that figure in the algebraic semantics of subclassical logics were all along universal characterizations of logical connectives. We submit that similar hindsight provides a full analysis of the sense in which Gentzen intended natural deduction rules to define logical connectives and the sense in which Gentzen meant that a connective's I/E rules relate to one another.

For an illustration of the sort of simple analysis facilitated by the relational definition of logical particles in terms of universal properties and of its aptness as an interpretation of Gentzen's thought, consider the EFQ inference. Gentzen notoriously labeled this inference $\bot$E, suggesting that it—unlike DN—can be accommodated in the relational definition scheme. But the absence of a $\bot$I rule to accompany $\bot$E might seem problematic, if the idea of I/E rules is, as we are suggesting, to pairwise describe a universal property. To see why it is not, consider EFQ rewritten in turnstile notation:

$$\bot \vdash \text{A, for all A}$$

To say that this rule is not only valid but defining of the $\bot$ connective is to say that any other sentence that can legitimately fill in the blank in "_____$\vdash$ A, for all A," any sentence, that is, that stands in the same inferential relationships as $\bot$, must itself allow inference to $\bot$:

$$\text{For all C, if C} \vdash \text{A, for all A, then C} \vdash \bot$$

The reader might have already noticed why Gentzen would have not have included this $\bot$I rule in his natural deduction calculi, even if the point of natural deduction was to provide universal characterizations of logical particles with I/E rule pairs: If C $\vdash$ A, for all A, then in particular, for any given D, C $\vdash$ D and C $\vdash$ $\neg$D, from which $\bot$ follows by Gentzen's own $\neg$E rule. In other words, $\bot$E (EFQ) and its universalization $\bot$I do indeed relationally define the $\bot$ connective, but the $\bot$I rule is redundant in NM, NI, and NK and therefore omitted.

On the other hand, when an inference like DN is observed to be underivable in NM, one can conclude with Gentzen that its alleged validity cannot be accounted for by the relational meaning of the $\neg$ connective.[5] But of course it is classically valid, so apparently the relational and essential styles of definition come apart.

Observing how the I/E rules relationally define the logical connectives may well vindicate the Stoic doctrine of the indemonstrability of *modus ponens*: Not only is this inference figure justified by the meaning of the $\supset$ connective, it is one clause of the very definition of $\supset$ in the relational style, the elimination rule for $\supset$. But the same observation seems to strengthen the case for the demonstrability of DS. On the relational scheme $\vee$I and $\vee$E are defining of $\vee$, whereas DS is not. Moreover DS cannot even be accounted for by the relational meanings of the $\vee$ and $\neg$ connectives. One also needs EFQ. The derivation of DS in natural deduction is no reduction of Stoic reasoning to arbitrarily chosen rules. It makes plain exactly in what ways DS turns on the meanings of the $\vee$ and $\neg$ connectives as well as how it extends them.

[5] Adhering to his characteristic phraseology, Gentzen wrote in *1934–35* that DN "falls outside the framework [of introduction and elimination rules], because it represents a new elimination of the negation whose admissibility does not follow at all from our method of introducing the $\neg$-symbol by $\neg$I" (p. 81). In *1936* he wrote that DN "conflicts in fact quite categorically with the remaining forms of inference" (p. 169).

3. Nevertheless, the intuition that disjunctive syllogism is basic—valid just by virtue of the meanings of "or" and "not"—is resilient. Can $\vee$ not be relationally defined in some other way that captures this intuition?

In fact one can treat DS itself, not just as one among the countless valid inference patterns licensed by the $\vee$ connective, but as defining of $\vee$. To say that

$$\mathrm{A} \vee \mathrm{B}, \neg\mathrm{A} \vdash \mathrm{B} \text{ and } \mathrm{A} \vee \mathrm{B}, \neg\mathrm{B} \vdash \mathrm{A}$$

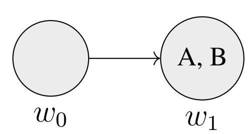
is universal for $\vee$ is to say that any other sentence that can legitimately fill in the blanks in _____, $\neg$A $\vdash$ B and _____, $\neg$B $\vdash$ A must be something that A $\vee$ B is inferable from:

$$\text{For all C, if C}, \neg\mathrm{A} \vdash \mathrm{B} \text{ and C}, \neg\mathrm{B} \vdash \mathrm{A}, \text{ then C} \vdash \mathrm{A} \vee \mathrm{B}.$$

The natural deduction presentation of this last rule is

$$\dfrac{\begin{matrix}[\neg \mathrm{A}] \\ \mathrm{B}\end{matrix} \qquad \begin{matrix}[\neg \mathrm{B}] \\ \mathrm{A}\end{matrix}}{\mathrm{A} \vee \mathrm{B}}\ \mathrm{DS}^-$$

Like DS, DS$^-$ is underivable in NM. Unlike DS, it is underivable even in NI: In the two node Kripke frame



$w_0 \Vdash \neg\mathrm{A} \supset \mathrm{B}$ and $w_0 \Vdash \neg\mathrm{B} \supset \mathrm{A}$ but $w_0 \nVdash \mathrm{A} \vee \mathrm{B}$. It is, of course, derivable in NK (let π_1 denote any NK proof of A $\vee$ $\neg$A and π_2 denote an NK derivation of B from $\neg$A):

$$\dfrac{\begin{matrix}\pi_1 \\ \mathrm{A} \vee \neg\mathrm{A}\end{matrix} \qquad \dfrac{\overset{1}{[\mathrm{A}]}}{\mathrm{A} \vee \mathrm{B}}\ \vee\mathrm{I} \qquad \dfrac{\begin{matrix}\overset{1}{[\neg\mathrm{A}]} \ \ldots \ \overset{1}{[\neg\mathrm{A}]} \\ \pi_2 \\ \mathrm{B}\end{matrix}}{\mathrm{A} \vee \mathrm{B}}\ \vee\mathrm{I}}{\mathrm{A} \vee \mathrm{B}}\ \vee\mathrm{E}_1$$

A curiosity of this demonstration of DS^- is that only one of its two sub-derivations is used.[6] We will return to this point shortly.

Johansson observed that, over the base system NM, EFQ and DS are interderivable. Thus NM+DS is equivalent to NI. However, if instead of starting with NM one replaces Gentzen's rules $\vee$I and $\vee$E with DS and DS^-, the situation is more interesting. Call the system NR. Notice first that EFQ is not derivable in NR. This is because in the canonical NM+DS proof of EFQ,

$$\dfrac{\dfrac{\bot}{\bot \vee A}\,\vee\mathrm{I} \qquad \dfrac{\overset{1}{[\bot]}}{\neg\bot}\,\neg\mathrm{I}_1}{A}\,\mathrm{DS}$$

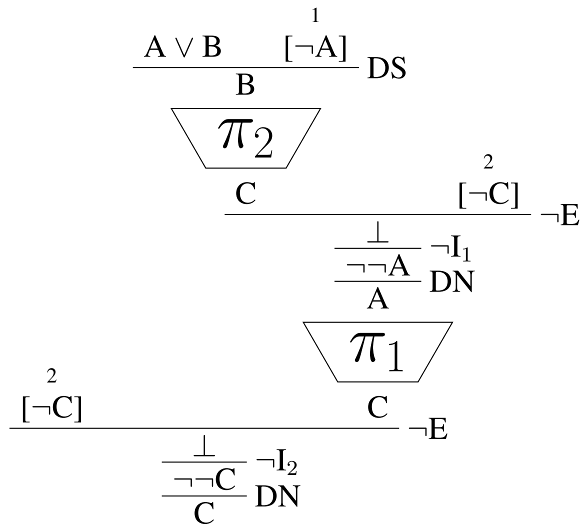
the $\vee$I rule is unavoidable. One might have supposed that $\vee$I is derivable in NR and therefore available to feature in an NR derivation of EFQ, but it is not. Here is a simple natural deduction derivation of the addition rule in NR+EFQ (notice that the application of DS^- uses vacuous discharge of $\neg$B along the right branch):

$$\dfrac{\dfrac{\dfrac{A \qquad \overset{1}{[\neg A]}}{\bot}\,\neg\mathrm{E}}{B}\,\mathrm{EFQ} \qquad A}{A \vee B}\,\mathrm{DS}^-_1$$

As we will see later, the use of EFQ is unavoidable. From the Chrysippian point of view, $\vee$E is more complex still, for it is underivable even in NR+EFQ. The following derivation, witnessing the full classical strength of NR+DN, appears to be the simplest one[7] (here π_1 and π_2 denote derivations of C from A and from B):

[6] Weir (*1986*) and Choi (*2021*) present natural deduction systems of classical propositional logic in which DS is a primitive rule in place of Gentzen's $\vee$E. Weir calls it $\vee$E*, and Choi calls it $\underline{\vee}$E. Neither author seeks to universalize DS in order to arrive at the appropriate corresponding introduction rule. They are focused instead on the problem of proving normalization for classical logic in the full $\langle \neg, \supset, \vee, \wedge, \bot \rangle$ signature. It is not surprising, therefore, that Weir's $\vee$I* consists of two templates $\dfrac{\overset{[\neg B]}{A}}{A \vee B}$ and $\dfrac{\overset{[\neg A]}{B}}{A \vee B}$. This rule suffices for normalization and *in the classical setting* is all one ever needs. It is not, however, the universalization of DS and therefore cannot be said to relationally define $\vee$ together with DS. Remarkably, Choi's $\underline{\vee}$I is exactly our DS^-. Neither author discusses the relationship between their alternative rules for disjunction and Gentzen's, absent the principles of classical logic which obscure the point.

[7] Weir (*1986*, p. 470) presents a very different derivation of $\vee$E based on his system with $\vee$E* and $\vee$I*. It also has only seven inferences, but the uncanny one-line proof of A $\vee$ $\neg$A from the discharged assumption $\neg$A it contains can obviously not be replicated when using DS^- in place of Weir's stronger $\vee$I* rule.



Thus the connective $\vee$ submits to two different relational definitions. If one takes the "addition" property of disjunction as definitional of $\vee$, then by universality the "proof-by-cases" inference pattern comes along with it. DS, however, does not: its demonstration depends on a combination of the connectives' relational meanings and the EFQ inference. If instead one takes DS as definitional of $\vee$, then the additional rule DS^- is secured by universality. Now addition stands as something to be demonstrated with the connectives' relational meanings. Again, EFQ is needed to complete the demonstration.

One conspicuous asymmetry between the pairs of rules $\vee$I/$\vee$E and DS/DS^- is that the former are "pure"; they define $\vee$ without reference to other connectives. Gentzen's ability to treat each logical connective individually with such pure definitions is regarded as one of his emblematic achievements. The absence of this feature from the DS/DS^- pair might therefore seem like a step backwards towards the cumbersome entanglement of connectives characteristic of Frege-style proof systems. As purity is a hallmark of the proof-theoretic semantics program Gentzen inaugurated, defining $\vee$ with the $\vee$I/$\vee$E pair could appear, for this reason alone, preferable.

A simple response to this asymmetry is just to point out that the relational definition scheme depends in no way on purity. To define a logical operator as a universal construction, all that is needed is to identify a property that the operator not only has but has definitionally, so that it is the characteristic or extremal example of an object having that property—extremal in the sense that it stands in a specific relation to every other object with the same property. This relational approach to meaning does not operate at cross purposes to the idea of proof-theoretic semantics. When, as in the example of logical operators, the relations in question are valid inferences, it can be thought of as that idea's precisification: The meaning of a logical operator is given by the defining inferential relations it stands in, "defining" in the sense that any other object that stands in those same inferential relations does so only because part of its meaning is given by that same logical operator as made evident by the fact that the operator can be inferred from the object or vice-versa (according to whether the construction is right-universal or left-universal, as described by Bergman in *2015*, page 71.) Purity does not enter into this idea.

A more subtle, but equally important, response is that the relational scheme exposes the appearance of asymmetry as a mirage. Gentzen's rules for $\neg$, to take a well worked example, do not exhibit purity: These rules define the $\neg$ symbol partially in terms of $\bot$. In the name of purity, this "problem" is eliminated by treating $\neg$A as an abbreviation of A $\supset \bot$, revealing that the natural deduction rules for $\neg$ are themselves just special cases of those for $\supset$. But to what benefit? The Elim rule for $\supset$ (*modus ponens*) itself states that A $\supset$ B together with A supports an inference to the conclusion B. To say that this rule is defining of the conditional is to say that any other object (C) that, together with A, supports an inference to the conclusion B, must be such that it alone supports an inference to the conclusion A $\supset$ B: You cannot pair-up with A to infer B unless you are already in a position to to infer A $\supset$ B. This is the Intro rule for $\supset$.

What are the relations that this universal characterization is built of? Plainly, there is the inference from the arbitrary object C to A $\supset$ B. But this is the only relation between A $\supset$ B and another object, and it is guaranteed to attain only when there is a valid inference from C together with A to B. When is there such an inference? According to the universal characterization of conjunction, there is no way to have both C and A without begin in a position to infer C $\wedge$ A, *the* object that supports an inference to anything that can be inferred from C together with A. In other words, in addition to the inference from the arbitrary object C to A $\supset$ B, there is an inference from C$\wedge$A to B, and the existence of the former depends on the existence of the latter. Finally, there is the independent inference of B from A $\supset$ B together with A, i.e., an inferential relation from $(\mathrm{A} \supset \mathrm{B}) \wedge \mathrm{A}$ to B. No less than the $\bot$ operator figures in the defining rules for $\neg$, the $\wedge$ operator figures in the defining rules for $\supset$. In the language of category theory, *exponential objects* (of which the logical conditional is an example) are defined in terms of *products* (such as (*additive*) conjunctions). Impurity is endemic to the relational definition scheme, even as natural deduction's notation occasionally disguises it.

Upon reflection, the situation is just what one should expect. The whole idea of the relational scheme is to characterize an object in terms of the relations it bears to other objects, as opposed to its internal constitution. On the essentialist scheme, an object is to be understood in isolation from everything else. The relations that attain between it and other objects are then determined by reflecting on its and those other objects' inherent meanings. Here impurity is an obstacle to complete definition. But when relations are constitutive of meaning, there is nothing to understand about an object beyond how it relates to other objects. If the object to be characterized is a conditional, and among the other objects to which it is definitionally related is the conjunction, then so be it. By the same token, the role of $\neg$ in the rules DS/DS^{-} in no way compromises their effectiveness as meaning constitutive and is in fact perhaps more typical of the relational definition scheme. Impurity is no fault but rather exactly what one would expect to find featured in a typical relational definition.

Faced with a logical particle like "or," multiple universal characterizations are available, but each fails to account for some intuitively valid inferential aspect of the word's meaning. Without a compelling reason to prefer one characterization over the other, Gentzen's analysis of propositional logic no longer appears definitive against the Stoic doctrine of the indemonstrability of disjunctive syllogism.

4. It might seem tempting to conclude from the inability of any universal characterization of "or" to account for all the word's intuitively valid inference patterns that logical disjunction resists relational definition. The $\vee$I/$\vee$E pair captures some of its meaning. The DS/DS$^-$ pair captures some more. No single inference rule is defining as the relational style of definition demands.

One might even have the further thought that part of the meaning of disjunction is not relational at all. Any specification of its meaning in the relational style has to be supplemented with principles of reasoning like EFQ or even DN in order to recover some other of its valid inference patterns. But, as Gentzen first noted, EFQ is independent of the relational definition of $\vee$, and DN lies outside the relational scheme of meaning altogether.

Both of these thoughts are natural given the plausible assumption that some one disjunction operator licenses the two basic inference patterns, addition and disjunctive syllogism. A less natural thought, but a more straightforward moral from the analysis above, is that this plausible assumption is false. What the relational style of definition unearths is that

1. the logical operator $\oplus$ that not only licenses the inferences from "A" to "$A \oplus B$" and from "B" to "$A \oplus B$," but in fact does so definitionally, does not further license the inference from "$\neg A$" and "$A \oplus B$" to "B";

2. the logical operator $⅋$ that not only licenses the inferences from "$\neg A$" and "$A ⅋ B$" to "B" and from "$\neg B$" and "$A ⅋ B$" to "A," but does so definitionally, does not further license the inference from "A" to "$A ⅋ B$";

3. "or" is ambiguous between $\oplus$ and $⅋$;

4. only in the presence of EFQ and DN are "$A \oplus B$" and "$A ⅋ B$" interderivable, so that it makes sense to use a single connective $\vee$ for both.

A similar point has been pressed by advocates of "relevance logic." Here are Stephen Read's words from *1981*:

> Now "or" is ambiguous: for "A or B" *can* mean "if not A then B." . . . When so used, the disjunction "A or B" does not follow from A alone—Addition fails. . . . Yet sometimes we do use "or" in such a way that Addition is valid. . . . But if "or" is taken in such a way . . . , then the disjunction is not equivalent to a conditional, it lacks inferential force, and DS is invalid. (p. 68)

In saying that "or," taken in such a way that addition is valid, "lacks inferential force," Read meant that nothing can be concluded from the knowledge of such an "or" statement, because logical inference is "intensional" (whereas this sort of disjunction is "extensional.") We can agree with this idea only in part: Surely DS is invalid when the disjunction in question is of the sort "A ⊕ B"; the inference from "¬A" to "B" is not licensed by "A ⊕ B" alone. But other inferences certainly are licensed by "A ⊕ B," for example the inference to "C" from the fact that it follows from "A" as well as from "B."

By the same token, saying that "A or B" has "inferential force" when it is taken to mean "if not A then B" seems only partially correct: "A ⅋ B" licenses the inference from "¬A" to "B," but just as "addition fails" for this connective so too does inference following the "proof by cases" paradigm.

Read's own disambiguations of the natural language "or" seem perfectly suited to our analysis, though. One example involves a conversation. The first speaker says, "So either you posted the letter or you burned it." The second speaker replies, "You mean you really think that if I didn't post it I burned it?" and the two continue:

—"That's right."

—"Why do you think that?"

—"Because you posted it."

The first speaker inferred "Either A or B" from "A." Their "or" was the additive ⊕. When the second speaker rephrased the claim as "If not A, B," the first speaker's assent committed them to ⅋. The second speaker is correct to conclude: "Just because I posted the letter, you cannot licitly infer that either I posted it or [⅋] I burned it—nothing follows about what happened if I didn't post it" (p. 68).

The second example involves a government benefit: "You qualify for a grant if either you are over 65 or you earn less than £2000 a year." Read wrote, "To satisfy a disjunctive condition" like this one "it suffices to satisfy one disjunct" (Ibid.). He only meant to provide one example of an unambiguously additive disjunction. One could add: If you can verify neither your age nor your salary, the receptionist at the grant office will not be impressed with your proof that if you exceed the income threshold it must be because of your seniority. It not only suffices but also is necessary to satisfy one disjunct, for even if each of condition A and condition B qualify one for the grant, A ⅋ B does not.

None of this has anything to do with the intensionality of inference or the relevance of implication, however compatible those doctrines are with the distinction between ⊕ and ⅋.

What about Chrysippus's hunt? If upon nearing the fork, you saw the quarry already some ways down path A, could you claim "It either took A or B?" Only if you aim to deceive, for the ordinary understanding of those words are just that you have ruled out C—that if it didn't take A it took B. The "or" is ⅋. But having witnessed it take path A puts you in no position to say what follows if it hadn't. It could just as well have taken C as B. On the other hand, if in addition to seeing your prey on A you saw a predator on C, you could sensibly say, "So long as it took either A or B, there's still a chance to complete the hunt." And because it did take A, the hunt continues. This disjunct is the additive $\oplus$; it admits proof by cases.

If alternatively you didn't see any continuation beyond the fork, then upon sniffing A without picking up the scent you can conclude that the quarry either took B or ($= \parr$) C. There are no other alternatives. But if your employer says at this point, "If you've secured our prize down one of B or ($= \oplus$) C, your work is done—I can take it from here," you can't retire early by demonstrating that it most certainly is secure in either one or the other. The boss needs to know which.

5. A different motivation than the ones described above spurred the original disambiguation of $\vee$ into the two disjunction-like operators $\oplus$ and $\parr$: the distillation of logical operations from structural rules of reasoning. The first step in this program was Gentzen's own (*1934–35*) reformulation of classical and intuitionistic propositional logic in the sequent calculi LK and LI. These calculi present logical inference in the form of sequents: expressions of the form "$\Gamma \vdash \Delta$" in which Greek letters stand for (possibly empty) finite multi-sets of formulas.[8] Associated with each propositional connective are a "left" and "right" rule. For example, the rules for $\vee$ are:

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \vee(L)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee(R)$$

These L/R rules for the propositional connectives function alongside another set of rules of structural reasoning. The classical calculus LK has five:

$$\frac{\Gamma \vdash \Theta}{D, \Gamma \vdash \Theta} \text{thinning(L)} \qquad\qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, D} \text{thinning(R)}$$

$$\frac{D, D, \Gamma \vdash \Theta}{D, \Gamma \vdash \Theta} \text{contraction(L)} \qquad\qquad \frac{\Gamma \vdash \Theta, D, D}{\Gamma \vdash \Theta, D} \text{contraction(R)}$$

$$\frac{\Gamma \vdash \Theta, D \qquad D, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda} \text{cut}$$

[8] In Gentzen's original formulation, sequents were instead presented as *sequences* of antecedent and succedent formulas separated by a sequent arrow $\longrightarrow$. In order to permute formulas in the antecedent and succedent positions, additional structural rules of exchange(L) and exchange(R) were included. Using the sequent arrow in place of the turnstile avoids occasional confusion between the presentation of a sequent and the claim that such a sequent is provable. However, for reasons that will become clear, we depart from Gentzen in conformity with standard usage in linear logic research.

A proof in LK is a finite branching tree built from iterations of these logical and structural rules, each leaf node of which is a basic sequent of the form $A \vdash A$, with A atomic. The sequent with which the root node is labeled is called the endsequent.

Gentzen observed that if one modifies LK by requiring the righthand side of the turnstile to have at most one formula, the resulting calculus LI is such that $\vdash_{\text{LI}} A$ if, and only if, $\vdash_{\text{NI}} A$. (Obviously the rule contraction(R) plays no role in LI and can be omitted.)

We noted earlier that Gentzen was bothered by the presence of the rule DN in classical natural deduction. As this rule "represents a new elimination of the negation whose admissibility does not follow at all from our method of introducing the ¬-symbol by ¬I" it "falls outside the framework" of the I/E scheme. Because LK and LI differ only with respect to their structural rules, the seemingly differential treatment given to ¬ by classical logic is shown to be an illusion. Gentzen wrote, "The special position of negation, which makes for an annoying exception in the natural calculus, is [in the sequent calculus] lifted away as if by magic" (*1938*, §1.6).

In this way, what appears to be a use of ¬ unlicensed by the meaning invested in it by the I/E rules of natural deduction is shown in fact to be a residue of purely structural features of reasoning characteristic of classical logic. One can press further in this vein. It is well known that the operational rules of LK can be presented in either "context-sharing" or "context-independent" styles. For example, a context-sharing (cs) presentation of ∨(L) was given above. In the context-independent (ci) style, one instead has:

$$\dfrac{A, \Gamma \vdash \Delta \qquad B, \Theta \vdash \Lambda}{A \vee B, \Gamma, \Theta \vdash \Delta, \Lambda}\ \vee(\text{L})\text{-ci}$$

The equivalence of these presentations is easy to verify:

$$\dfrac{\dfrac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma, \Gamma \vdash \Delta, \Delta}\ \vee(\text{L})\text{-ci}}{A \vee B, \Gamma \vdash \Delta}\ \}\text{multiple contractions}$$

$$\dfrac{\dfrac{A, \Gamma \vdash \Delta}{A, \Gamma, \Theta \vdash \Lambda, \Delta}\ \}\text{multiple thinnings} \qquad \dfrac{B, \Theta \vdash \Lambda}{B, \Gamma, \Theta \vdash \Lambda, \Delta}\ \}\text{multiple thinnings}}{A \vee B, \Gamma, \Theta \vdash \Lambda, \Delta}\ \vee(\text{L})\text{-cs}$$

but we see that their equivalence depends on the presence of the structural rules of thinning and contraction.[9] If one thinks of L/R rules as defining of a logical operator, then a natural

[9]In our presentation of LK above we have followed Gentzen in providing context-sharing presentations of most rules but a context-independent presentation of cut. *Franks 2010* presents a possible rationale for this choice.

next thought is that the context-sharing and context-independent rules define two distinct operators. This idea motivates the development of linear logic (*Girard 1987*), which features two conjunction-like operators:

$$\frac{\mathrm{A}, \mathrm{B}, \Gamma \vdash \Delta}{\mathrm{A} \otimes \mathrm{B}, \Gamma \vdash \Delta}\otimes(\mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, \mathrm{A} \qquad \Theta \vdash \Lambda, \mathrm{B}}{\Gamma, \Theta \vdash \Delta, \Lambda, \mathrm{A} \otimes \mathrm{B}}\otimes(\mathrm{R})$$

$$\frac{\Gamma, \mathrm{A} \vdash \Delta}{\mathrm{A} \,\&\, \mathrm{B}, \Gamma \vdash \Delta} \quad \frac{\mathrm{B}, \Gamma \vdash \Delta}{\mathrm{A} \,\&\, \mathrm{B}, \Gamma \vdash \Delta}\&(\mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, \mathrm{A} \qquad \Gamma \vdash \Delta, \mathrm{B}}{\Gamma \vdash \Delta, \mathrm{A} \,\&\, \mathrm{B}}\&(\mathrm{R})$$

and two disjunction-like operators:

$$\frac{\mathrm{A}, \Gamma \vdash \Delta \qquad \mathrm{B}, \Gamma \vdash \Delta}{\mathrm{A} \oplus \mathrm{B}, \Gamma \vdash \Delta}\oplus(\mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, \mathrm{A}}{\Gamma \vdash \Delta, \mathrm{A} \oplus \mathrm{B}} \quad \frac{\Gamma \vdash \Delta, \mathrm{B}}{\Gamma \vdash \Delta, \mathrm{A} \oplus \mathrm{B}}\oplus(\mathrm{R})$$

$$\frac{\mathrm{A}, \Gamma \vdash \Delta \qquad \mathrm{B}, \Theta \vdash \Lambda}{\mathrm{A} \parr \mathrm{B}, \Gamma, \Theta \vdash \Delta, \Lambda}\parr(\mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, \mathrm{A}, \mathrm{B}}{\Gamma \vdash \Delta, \mathrm{A} \parr \mathrm{B}}\parr(\mathrm{R})$$

The connectives $\&$ and $\oplus$ with context-sharing rules are called "additives"; $\otimes$ and $\parr$ with context-independent rules are called "multiplicatives." But linear logic has no structural rules for thinning and contraction, so the additive rules are not interderivable with their multiplicative counterparts.[10]

The designations "additive" and "multiplicative" in linear logic seem to originate from an analogy to the distributive laws of elementary arithmetic: $\mathrm{A} \otimes (\mathrm{B} \oplus \mathrm{C})$ and $(\mathrm{A} \otimes \mathrm{B}) \oplus (\mathrm{A} \otimes \mathrm{C})$ are interderivable, as are $\mathrm{A} \parr (\mathrm{B} \,\&\, \mathrm{C})$ and $(\mathrm{A} \parr \mathrm{B}) \,\&\, (\mathrm{A} \parr \mathrm{C})$; but as addition does not distribute over multiplication, the results of replacing in these linear identities each connective with its counterpart do not hold. But the label "additive" is apt also in another way: $\oplus$(R) is just the familiar inference pattern known as addition.

In fact, the $\oplus$ of linear logic is the very same operator as the one bearing that label in the previous section, the operator defined by Gentzen's original I/E rules of natural deduction. Not only addition, but also the proof-by-cases inference holds of $\oplus$, as

$$\frac{\mathrm{A} \vdash \mathrm{C} \qquad \mathrm{B} \vdash \mathrm{C}}{\mathrm{A} \oplus \mathrm{B} \vdash \mathrm{C}}\oplus(\mathrm{L})$$

is just an instance of the $\oplus$(L) rule. However, neither law is valid for the multiplicative $\parr$, as one can readily check by applying cut-elimination (*hint*: first observe that in any MALL provable sequent containing only the connectives $\otimes$ and $\parr$, the number of occurrences of each atom must be the same in the antecedent as in the succedent.).

What about linear logic's $\parr$? Again, it corresponds exactly with the operator relationally defined by DS and DS$^-$. Here is the linear derivation of DS:

[10] For a complete presentation of the multiplicative/additive fragment of classical propositional linear logic (MALL)—the fragment of linear logic that interests us here—see *Bellin 1990*.

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \parr B \vdash A, B}\ \parr(L)}{A \parr B, \neg A \vdash B}\ \neg(L)$$

For the linear derivation of DS^-, let π denote any MALL proof of $\Gamma, \neg A \vdash B$ and observe:

$$\dfrac{\dfrac{\begin{array}{c}\pi \\ \Gamma, \neg A \vdash B\end{array} \qquad \dfrac{A \vdash A}{\vdash \neg A, A}\ \neg(R)}{\Gamma \vdash A, B}\ \text{cut}}{\Gamma \vdash A \parr B}\ \parr(R)$$

Again, cut-elimination can be applied to show that neither DS nor DS^- is derivable for $\oplus$ (*hint*: if $A \oplus B, \neg A \vdash B$ had a MALL proof, then after one application of $\oplus(R)$ so would $A \oplus B, \neg A \vdash A \oplus B$).

A remarkable feature of MALL is the fact that, like classical logic, it has an involutive negation (among its derivable rules is DN in the form $\neg\neg A \vdash A$). But whereas the analysis provided by natural deduction suggests that $\oplus$ and $\parr$ are interderivable in the presence of DN, linear analysis reveals that their distinction depends rather on disentangling these logical operations from contraction and thinning, rules which are not explicit in the setting of natural deduction.

One striking parallel to classical natural deduction persists, however. Recall that the relational definition of $\parr$ is:

$$A \parr B, \neg A \vdash B \text{ and } A \parr B, \neg B \vdash A$$

and

$$\text{For all C, if } C, \neg A \vdash B \text{ and } C, \neg B \vdash A, \text{ then } C \vdash A \parr B.$$

In the MALL derivation of DS^-, only one clause of the universality condition from the relational definition of $\parr$ is needed: From only the assumption that there is a MALL proof of $\Gamma, \neg A \vdash B$, one is able to construct a MALL proof of $\Gamma \vdash A \parr B$. At first glance, the situation suggests that linear logic's $\parr$ is not quite the same operator as the one defined by the universal property.

Again, this is too hasty a verdict. The conditions that warrant inference to $A \parr B$ are the availability of inferences both of A from $\neg B$ and of B from $\neg A$. That is how $\parr$ is defined on the relational scheme. It just happens that in MALL, another one of Chrysippus's indemonstrable laws, *modus tollens*, holds. Thus if π is any MALL proof of $\Gamma, \neg A \vdash B$ one has:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \neg A}\,\neg(R) \qquad \begin{matrix}\Pi \\ \Gamma, \neg A \vdash B\end{matrix}}{\Gamma \vdash A, B}\,\text{cut}}{\Gamma, \neg B \vdash A}\,\neg(R)$$

6. The linear logic literature abounds with declarations that the multiplicative disjunction ⅋ defies intuitive understanding. On the *n*Lab website, it is described as "probably also the hardest to understand intuitively of any of the linear logic connectives." In an n-category cafe post, Michael Shulman wrote: "Linear logic, regarded as a logic, has always been rather mysterious to me, especially the multiplicative disjunction ⅋ (and I know I'm not alone in that)." One finds an entry on the Mathematics Stackexchange titled "What is the intuition behind the 'par' operator in linear logic?" followed by a spirited discussion spanning several years.

Perhaps the multiplicative disjunction does not readily submit to a straightforward reading in the game semantics or resource tracking interpretations that typically motivate linear logic. But the connective has a life outside of MALL. It is the disjunction underlying the simple inference known as disjunctive syllogism. If one's impression is that this inference pattern is intuitively obvious, rather than something whose validity awaits demonstration in more elementary terms, then a plausible analysis of the word "or" appearing in it is just that: "A or B" is, by definition, the thing that licenses the inferences from "not A" to "B" and from "not B" to "A." Is such a stipulation well-defined? On the relational scheme it is. DS captures the part about "A or B" licensing those inferences. DS^- captures the part about it being "*the thing*" that does so. In some logical environments, a connective so defined also admits such inferences as addition and proof-by-cases. In others it does not. In neither situation do those inferences follow from the way "or" has been defined.

An alternative definition of the word "or" in terms of addition is both possible and more familiar today. As an analysis of the word's occurrence in the disjunctive syllogism, this alternative definition makes that inference appear problematic. For in only some logical environments will DS appear valid when subjected to this analysis, and in none of them will it follow simply from the way "or" has been defined. Proceeding in this way, DS appears at best demonstrably valid, and that demonstration indicates structural aspects of reasoning on which its validity depends.

What the Stoics must have meant when they said that DS is indemonstrable is that as a definition, the rules of addition and proof by cases do not capture the logical force of the "or" that appears in DS. This could be because the Stoics rejected those rules as defining of "or" out of a preference for DS and DS^-, but it does not have to be. Chrysippus and his followers might have meant just that DS is valid by definition. If so, we now know what definition that is.[11]

[11] To be sure, there are other features of Stoic logic unaccounted for by both the additive and multiplicative "or"; for example, the fourth indemonstrable suggests that their disjunction was "exclusive."

Whether addition is true by definition as well is beside to point, so long as one understands that, if it is, that definition is of a different logical operator we recognize today as $\oplus$. DS is valid because the "or" that appears in it is $\parr$. It is indemonstrable because A $\parr$ B literally means that it is valid.

Works Cited

Bellin, G. 1990. *Mechanizing Proof Theory: Resource-Aware Logics and Proof-Transformations to Extract Explicit Information*, PhD. thesis, Stanford University.

Bergman, G. M. *An Invitation to General Algebra and Universal Constructions*. Second Edition. 2015. New York: Springer.

Bobzien, S. 1996. "Stoic syllogistic," *Oxford Studies in Ancient Philosophy* **14**, 133–92.

Bobzien, S. 2020. "Demonstration and the indemonstrability of the Stoic indemonstrables," *Phronesis*, **65**(3), pp. 355–78.

Bury, R. G. 1933 (trans.) *Sextus Empiricus: Outlines of Pyrrhonism*, Cambridge: Harvard University Press.

Choi, S. 2021. "The normalization theorem for the first-order classical natural deduction with disjunctive syllogism," *Korean Journal of Logic*, **2**(24), 143–168.

Fitch, F. B. 1952. *Symbolic logic. An introduction.* New York: The Ronald Press Company.

Franks, C. 2010. "Cut as consequence," *History and Philosophy of Logic*, **31**(4), 349–79.

Gentzen, G. 1934–35. 'Untersuchungen über das logische Schließen'. Gentzen's doctoral thesis at the University of Göttingen, translated as 'Investigations into logical deduction' in *Szabo 1969*, 68–131.

Gentzen, G. 1938. "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie," *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften*, **34**(4), 19–44. Translated as "New version of the consistency proof for elementary number theory," in *Szabo 1969*, 252–86.

Girard, J.-Y. 1987. "Linear logic," *Theoretical Computer Science*, **50**, 1–102.

Johansson, I. 1937. "Der Minimalkalkul, ein reduzierter intuitionistischer Formalismus," *Compositio Mathematica*, **4**, 119–36.

Lawvere, F. W. 1963. "Functorial semantics of algebraic theories." PhD thesis, Columbia University. Republished in *Reprints in Theory and Applications of Categories*, No. 5 (2004), 1–121.

Leinster, T. 2014. *Basic Category Theory*, Cambridge Studies in Advanced Mathematics, Vol. 143, Cambridge: Cambridge University Press.

López Astorga, M. 2015, "Chrysippus' indemonstrables and mental logic," *Croatian Journal of Philosophy*, **XV**,(43), 1–15.

Pelletier, F. J. 1999. "A brief history of natural deduction," *History and Philosophy of Logic*, **20**(1), 1–31.

Prawitz, D. 1965. *Natural Deduction: a proof-theoretic study*, reprinted by Dover, 2006.

Read, S. 1981. "What is wrong with disjunctive syllogism?" *Analysis*, **41**(2), 66–70.

Samuel, P. 1948. "On universal mappings and free topological groups," *Bulletin of the American Mathematical Society* **54**, 591–598.

Schroeder-Heister, P. 2006. "Validity Concepts in Proof-Theoretic Semantics," *Synthese*, **148**: 525–571.

Shulman, M. 2018. "Linear logic for constructive mathematics," *the n-Category Café*,

`https://golem.ph.utexas.edu/category/2018/05/linear_logic_for_constructive.html`

Steinberger, F. 2013. "On the equivalence conjecture for proof-theoretic harmony," *Notre Dame Journal of Formal Logic*, **54**(1), 79–86.

Szabo, M. E. 1969. *The Collected Papers of Gerhard Gentzen*, London: North Holland.

Tennent, N. 1978. *Natural Logic*, Edinburgh: Edinburgh University Press.

Weir, A. 1986. "Clasical harmony," *Notre Dame Journal of Formal Logic*, **27**(4), 459–482.

"Multiplicative disjunction," *nlab*,

`https://ncatlab.org/nlab/show/multiplicative+disjunction`