Encodable by thin sets

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April 17, 2018

Midwest Computability Theory Seminar
Joint with Ludovic Patey.

Preliminary Report
https://www3.nd.edu/~cholak/papers/chicago18.pdf
\( RT^n_{<\infty,l}\)-encodable

- Let \( c \) be a coloring of all finite sets of size \( n \) (all subsets of \( \omega \)) by finitely many colors, not necessarily computable.
- A set \( T \) is \( l\)-thin iff \( c \) uses at most \( l \) colors to color all the sets of size \( n \) from \( T \) and \( T \) is infinite. So \( |c([T]^n)| \leq l \).
- A set \( S \) is \( RT^n_{<\infty,l}\)-encodable iff there is a coloring \( c \) (as above) such that every \( l\)-thin set \( T \) computes \( S \), i.e. \( S \leq_T T \).
$RT^n_{<\infty,l}$-encodable sets are always hyperarithmetic.

- Assume $c$ witness that $S$ is $RT^n_{<\infty,l}$-encodable.
- Given $X$ there is an infinite thin set $H$ for $c$ such that $H \subseteq X$.
- A set $S$ is computably encodable if for every infinite set $X$, there is an infinite subset $H$ of $X$ such that $H$ computes $S$.
- By theorems of Jockusch and Soare and Solovay, the computable encodable sets are exactly the hyperarithmetic sets.
The $RT^2_{<\infty,1}$-encodable sets includes all hyperarithmetic sets

- The 1-thin sets are exactly the homogenous sets.
- (Solovay) $S$ is hyperarithmetical iff $S$ has a modulus, i.e. a function $g$ such that if $h \geq g$ then $h \geq_T S$.
- The interval $[x, y]$ is $g$-large iff $y > g(x)$.
- $c(x, y) = 1$ iff $[x, y]$ is $g$-large. (A unbalanced coloring.)
- Let $H$ be a homogenous set for $c$. Fix $x \in H$. Then, for almost all $y \in H$, $[x, y]$ is $g$-large. So, for all $y \in H$, $[x, y]$ is $g$-large.
- Hence $p_H \geq g$.

For every hyperarithmetical set $S$ there is a coloring such that every homogenous set computes $S$. 
The $RT^n_{<\infty,l}$-encodable sets, for small $l$

**Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2016)**

For small $l$, the $RT^n_{<\infty,l}$-encodable sets are exactly the hyperarithmetic sets. Here small is defined as $l < 2^{n-1}$.

**Sketch.**

Again code in a modulus into all thin sets of a coloring. For $l \geq 2^{n-1}$, the coding does not work.
The $RT^n_{<\infty,l}$-encodable sets, for large $l$

Theorem (Wang, 2014)

For large $l$, the $RT^n_{<\infty,l}$-encodable sets are exactly the computable sets.

Use the strong cone avoiding of $RT^n_{<\infty,l}$. So given any $c$ and any noncomputable $S$, there is a $l$-thin set $T$ for $c$ that avoids the cone above $S$ (even if $c \geq_T S$).

This is an inductive forcing proof and relies on (strong) cone avoiding of earlier and other principles, like $COH$, $WKL$, $RT^1_{<\infty,1}$, $RT^2_{3,1}$, etc. Use Mathias like conditions where the reservoir avoids the cone above $S$ plus some. Example at end of talk.

The earlier coloring where every homogenous set computes a noncomputable $S$ is a counterexample of strong cone avoidance.
Theorem (Cholak, Patey)

For medium $l$, the $RT^n_{<\infty,l}$-encodable sets include the arithmetic sets. Another coding is needed. This coding is also a counterexample of strong cone avoidance.

Conjecture (Cholak, Patey)

For medium $l$, the $RT^n_{<\infty,l}$-encodable sets are exactly the arithmetic sets.

Given any nonarithmetical set $S$ avoid the cone above $S$. Like above but relies on (strong, nonarithmetical) cone avoiding of earlier and other principles, like $COH$, $WKL$, $RT^1_{<\infty,1}$, $RT^2_{3,1}$, etc. Use Mathias like conditions where $S$ is still nonarithmetical over the reservoir.
Bounds

Conjecture (Cholak, Patey)
There is an elementary function defining when \( l \) is greatest medium number in terms of \( n \).

Theorem

• For \( n = 2 \), small is less than 2 and large is 2 or greater.
• For \( n = 3 \), small is less than 4, medium is 4, and large is 5 or greater.
• For \( n = 4 \), small is less than 8, medium is 8 to 13, and large is 14 or greater.
• For \( n = 5 \), small is less than 16, medium is 16 to . . .
LARGE vs MEDIUM vs small

- Large needs strong cone avoidance.
- The line between medium and large is drawn by where strong cone avoiding fails. This counterexample to strong cone avoidance also shows that the arithmetic sets are encodable.
- For medium $l$ we cannot use strong cone avoidance but nonarithmetical cone avoidance. (Conjecture for all $n > 2$.)
- The original coding determines small.
Counterexample to strong cone avoiding

**Theorem**

*There is a $\Delta^0_2$ coloring $c : [\omega]^3 \to 5$ such that every 4-thin set for $c$ computes $0'$.*
Our first attempt

Let \( g \) be a modulus of \( 0' \).

- Recall \([a, b]\) is \( g\)-large iff \( b \geq g(a) \). Otherwise it is \( g\)-small.
- Let \( i(x, y) = 1 \) if \([x, y]\) is \( g\)-large and 0 otherwise.
- Let \( c(x, y, z) = \langle i(x, y), i(y, z), i(x, z) \rangle \). This is a 5 coloring, some colors are missing.
- Apply \( RT^3_{5,4} \) to \( c \) to get a thin set \( T \).
- If any color but \( \langle 0, 0, 1 \rangle \) is missed, \( T \) or an straightforward reduction of \( T \) has all \( g\)-large intervals and hence computes \( 0' \).
- Need to learn more about missing the color \( \langle 0, 0, 1 \rangle \).
- A set \( H \) is \( g\)-transitive iff for all \( x < y < z \) in \( H \) if \([x, y]\) and \([y, z]\) are \( g\)-small so is \([x, z]\). This is the color \( \langle 0, 0, 1 \rangle \).
GAP

GAP is the statement that, for all $g$, an infinite $g$-transitive set exists. So the existence of a 4-thin set (when colored as above) without the color $\langle 0, 0, 1 \rangle$.

**Theorem**

GAP follows from $\text{RT}_5^3$, and satisfies strong cone avoidance.

**Conjecture**

$\text{RT}_2^2$ does not imply GAP.
Back to $RT^3_{5,4}$ and coding

Left c.e. increasing functions

Refine the above coloring $c$. We need to make it harder to avoid the color $\langle 0, 0, 1 \rangle$. So we have to color more triples with color $\langle 0, 0, 1 \rangle$ and less with color $\langle 0, 0, 0 \rangle$.

The modulus $g$ of $0'$ is a left c.e. increasing function with approximations $g_0, g_1, \ldots$ (the approximations are increasing).

- Define $j(x, y, z)$ is 1 iff $[x, z]$ is $g$-large or $[x, y]$ is $g_z$-large.
- Let $c(x, y, z) = \langle i(x, y), i(y, z), j(x, y, z) \rangle$.
- Apply $RT^3_{5,4}$ to $c$ to get a thin set $T$.
- For all possible missed colors, a straightforward reduction of $T$ has all $g$-large intervals and hence computes $0'$.

**Theorem**

$c$ is $\Delta^0_2$ and every 4-thin set for $c$ computes $0'$. 
$RT^2_{3,2}$ admits strong cone avoidance: Inductive part

As we step though this be aware of the various uses of (strong) cone avoidance.

Lemma

Fix a set $C$, a set $A \not\leq T C$, and a coloring $f : [\omega]^2 \to 3$. There is an infinite set $X$ and color $i_0$ such that $A \not\leq T X \oplus C$ and

$$(\forall x \in X)(\exists t \in \omega)(\forall y \in X \cap (t, \infty))[f(x, y) = i_0]$$

Proof.

Let $R_{x,i} = \{y|f(x, y) = i\}$. Apply strong cone avoidance of COH to get $X_0$ such that $X_0 \not\leq T X \oplus C$ and

$$(\forall x \in X_0)(\exists i < 3)(\exists t \in \omega)(\forall y \in X_0 \cap (t, \infty))[f(x, y) = i].$$

Let $c$ be the 3-coloring on $X_0$ induced by $f$. Now use strong cone avoidance of $RT^1_{3,1}$ to get $X$. □
\( RT_{3,2}^2 \) admits strong cone avoidance: Forcing part

Same setup as above lemma. Apply above lemma and assume \( i_0 = 2 \). Using Mathis-like forcing there are generic \( G_i \) such that, for both \( i, f([G_i]^2) \subseteq \{i, 2\} \), and, for some \( i, A \not\subseteq_T G_i \oplus C \). Conditions are \((F_0, F_1, X), F_i \text{ finite, max}(F_i) < \min(X), X \subseteq X_0, X \text{ infinite, and } A \not\subseteq_T X \oplus C \). Meet for all \( e_0, e_1 \)

\[
\Phi_{e_0}^{G_0 \oplus C} \neq A \text{ or } \Phi_{e_1}^{G_1 \oplus C} \neq A
\]

The rest on the board.
Announcements

**Computability Theory and Applications, Waterloo, Canada, June 4 – 8, 2018.** Some Speakers: Damir Dzhafarov, Bjørn Kjos-Hanssen, Keng Meng (Selwyn) Ng, Joseph Miller, Jan Reimann, Richard Shore, and Linda Brown Westrick. Organizers: Laurent Bienvenu, Peter Cholak, Barbara Csima (Chair), and Matthew Harrison-Trainor. [https://cta-waterloo.sciencesconf.org](https://cta-waterloo.sciencesconf.org)

**Workshop on Ramsey Theory and Computability, Rome, Italy, July 9 – 13, 2018.** Some Speakers: Lorenzo Carlucci, Mauro Di Nasso, Jeff Hirst, Nicola Galesi, Wei Li, Arno Pauly, Paul Shafer, Silvia Steila, and Henry Towner. Program Committee: Lorenzo Carlucci, Peter Cholak (Chair), Damir Dzhafarov, Denis Hirschfeldt, and Ludovic Patey. [https://www3.nd.edu/~cholak/Colosseum.html](https://www3.nd.edu/~cholak/Colosseum.html)