On Friedberg Splits

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Computably Enumerable Sets

- \( W_e \) is the \( e \)th c.e. set under some nice acceptable uniform standard enumeration of all c.e. sets.
- \( W_{e,s} \subseteq \{0, 1, \ldots s\} \).
- A c.e. set \( R \) is computable iff \( \overline{R} \) is also a c.e. set.
- \( A_0, A_1 \) is a split of \( A \) iff \( A_0 \sqcup A_1 = A \) iff \( A_0 \cap A_1 = \emptyset \) and \( A_0 \cup A_1 = A \).
- Focus on splits of noncomputable c.e. sets into c.e. sets.
- If \( F \subseteq A \) is finite than \( F \sqcup (A - F) = A \).
- A split \( A_0, A_1 \) is trivial if \( A_0 \) or \( A_1 \) is computable.
Nontrivial Trivial Splits

Lemma

Every noncomputable c.e. set $A$ has an infinite computable subset $R$.

Then $A = R \sqcup (A \cap \overline{R})$.

Proof.

$A = a_0, a_1, a_2 \ldots$, in the order of enumeration with no repeats. Let $R = \{a_i | (\forall j \leq i)[a_i > a_j]\}$. $n \in R$ iff $n \in \{a_0, a_1 \ldots a_n\}$.

\qed
Myhill’s Question

Question

*Does every noncomputable c.e. set have a nontrivial split?*

Theorem (Friedberg)

*Yes!*

Myhill’s question appeared in the Journal of Symbolic Logic in June 1956, Volume 21, Number 2 on page 215 in the “Problems” section of the JSL. This question was the eighth problem appearing in this section. The question about the existence of maximal sets, also answered by Friedberg, was ninth.
Friedberg Splits

Definition
$A_0 \sqcup A_1 = A$ is a Friedberg Split of $A$ iff, for all $e$, if $W_e - A$ is not c.e. then $W_e - A_i$ are also not c.e.

Lemma
A Friedberg split of a noncomputable set is a nontrivial split.

Proof.
Assume $A_0$ is computable. So $\overline{A}_0$ is a c.e. set.
$\overline{A}_0 - A = \overline{A}_0 - A_1 = \overline{A}$. So this set is not a c.e. set. But then $\overline{A}_0 - A_0 = \overline{A}_0$ must not be c.e. set. Contradiction.
C.e. sets from the enumeration of $A$

- $W \setminus A = \{ x \mid (\exists s)(x \in W_s \& x \notin A_s) \}$. ($W$ and then maybe $A$.)
- $W \setminus A = (W \setminus A) \cap A$. ($W$ and then $A$.)
- $(W \setminus A) = (W - A) \sqcup (W \setminus A)$.
- $(W - A) = (W \setminus A) \sqcup \overline{(W \setminus A)}$
- So if $W - A$ is not a c.e. set then $W \setminus A$ is not computable and hence infinite.
Lemma
If $A_0 \sqcup A_1 = A$ and, for all $e$, if $W \setminus A$ is infinite then $W \setminus A_i$ is infinite, then $A_0, A_1$ is a Friedberg split of $A$.

Proof.
Assume $W - A$ is not a c.e. set but $W - A_0$ is a c.e. set. Let $X = W - A_0$. $X - A = W - A$ is not a c.e. set. So $X \setminus A$ is infinite. Therefore $X \setminus A_0$ is infinite. Contradiction.
Theorem (Friedberg)

*Every noncomputable set has a Friedberg Split.*

Proof.

Use a priority argument to meet the following

$$R_{e,i,k}: \quad W_e \setminus A \text{ is infinite} \Rightarrow (\exists x > k)[x \in A_i]$$

Corollary

*There is a computable total function $f(e) = \langle e_0, e_1 \rangle$ such that if $W_e$ is noncomputable then $W_{f(e_0)}, W_{f(e_1)}$ is a Friedberg split of $W_e$.*
The Motivating Questions

**Question**

*When does a c.e. set have a nontrivial nonFriedberg split?*

**Question**

*Is it possible to uniformly split all noncomputable c.e. sets into a nontrivial nonFriedberg split?*

Lots of results on splits of c.e. sets. See Downey, Stob, Splitting Theorems in Recursion Theory.
There are Nontrivial NonFriedberg Splits

- Let $R$ be an infinite, coinfinate computable set. Let $R_K$ be a noncomputable c.e. subset of $R$.
- Similarly let $\overline{R}_K$ be a noncomputable c.e. subset of $\overline{R}$.
- $R_K \sqcup \overline{R}_K = A$ is a nontrivial nonFriedberg split of $A$.
- $R - R_K$ is not a c.e. set but $\overline{R} - R_K = \overline{R}$ is a c.e. set.
- Here all 3 sets were built simultaneously. We need both $A$ and $R$ to construct the split.
**\( \mathcal{D} \)-hhsimple Sets**

**Definition**

- \( \mathcal{D}(A) = \{ B \mid B - A \text{ is a c.e. set} \} \).
- \( \mathcal{E} = \{ W \mid W \text{ is a c.e set} \} \) ordered by inclusion.
- \( A \) is **\( \mathcal{D} \)-hhsimple** iff \( \mathcal{E} \) modulo \( \mathcal{D}(A) \) is a Boolean Algebra.
- \( A \) is **\( \mathcal{D} \)-maximal** iff \( \mathcal{E} \) modulo \( \mathcal{D}(A) \) is a trivial Boolean Algebra, \( \{0, 1\} \).

**Lemma**

*\( A \) is **\( \mathcal{D} \)-maximal** iff for all \( W \), \( W - A \) is a c.e. set (so \( W \) is 0) or there is disjoint c.e. set \( D \) such that \( A \cup D \cup W = \omega \) (so \( W \) is 1).*
A More Difficult Example

Theorem
There is split $A_0, A_1$ of an $r$-maximal set $A$ such that the split is nontrivial and, for all $e$, either $W - A_0$ is a c.e. set or there is a $D$ with $D \cap A_0 = \emptyset$ and $A \cup D \cup W = \ast \omega$.

So $A_0$ is $D$-maximal but there are no restrictions on $A_1$.

Proof.
Sorry, some other talk. But again all 3 sets are built simultaneously.

□
Splits of $\mathcal{D}$-hhsimple Sets

**Theorem (Shavrukov)**

Let $A$ be $\mathcal{D}$-hhsimple but not $\mathcal{D}$-maximal. Then $A$ has a nontrivial nonFriedberg split.

**Proof.**

There are $X_0, X_1$ such that they are incomparable in $\mathcal{E}$ modulo $\mathcal{D}(A)$. We can assume that they are disjoint and $A \subset X_0 \sqcup X_1$. $X_i - A$ is not a c.e. set (otherwise $X_i$ is 0 and comparable to everything). So $X_i \cap A$ is not computable. $X_i - (X_i \cap A) = X_i$ is a c.e. set. Hence $X_0 \cap A, X_1 \cap A$ is a nontrivial nonFriedberg split. \qed
Definition (Kummer)

A is a **diagonal set** iff there is a numbering (likely unacceptable) of the partial computable functions, \( \{ \Xi_i | i \in \omega \} \), such that \( A = \{ i : \Xi_i(i) \downarrow \} \).

Acceptable numberings are more or less a computable permutation of the standard numbering. For unacceptable numberings any function such that \( \Xi_i = \varphi_{f(i)} \) is not computable. For example, Friedberg’s result that there is an enumeration of the c.e. sets without repeats.

**Theorem (Kummer)**

*A set \( A \) is diagonal iff \( A \) is not \( D \)-hh-simple.*

**Proof.**

Missing!!
Splits of Diagonal Sets

Theorem (Shavrukov)

Every diagonal set has a nontrivial nonFriedberg split.

Proof.

There exists an $s(n)$, a partial computable function, such that if $s(n) \downarrow$ then there exists an $x$ with $\Xi_n(x) = s(n)$. Let $Q_0 = \{n | s(n) = 0\}$ and $Q_1 = \{n | s(n) > 0\}$. Let $A \subseteq Q_0 \sqcup Q_1$. Let $A = (Q_0 \cap A) \sqcup (Q_1 \cap A)$. On the next slide we show that $Q_i - A$ is not a c.e. set. Then $Q_i \cap A$ is not computable and the split is not Friedberg since $Q_i - (Q_i^c \cap A) = Q_i$ is a c.e. set.

Needed to know the numbering which is not given uniformly.
$Q_i - A$ is not a c.e. set

Proof.
Assume otherwise. Let $h$ be a partial computable function such that if $h(n) \downarrow$ then $h(n) = i$ and $n \in (Q_i - A)$. Codes for the 3 finite functions, $\{\langle 0, i \rangle\}, \{\langle 1, i \rangle\}, \{\langle 0, i \rangle, \langle 1, i \rangle\}$ are all in $Q_i$. But at most 2 of these functions are in $A$. So $Q_i - A \neq \emptyset$. Let $m$ be the code for $h$. $m \in Q_i$. $m \in A$ iff $m \in Q_i - A$ iff $h(m) = \varphi_m(m) \uparrow$ iff $m \notin A$. Contradiction.
Lemma (Cholak, Downey, Herrmann)

All nontrivial splits of a $\mathcal{D}$-maximal set $A$ are Friedberg.

Proof.
Assume that $W - A$ is not a c.e. set. Then, since $A$ is $\mathcal{D}$-maximal, $W \cup A =^{*} \omega$. If $W - A_0$ is c.e. then $A_0 \sqcup ((W - A_0) \cup A_1) =^{*} \omega$. So $A_0$ is computable. Contradiction.

Theorem (Shavrukov)

All of $A$’s nontrivial splits are Friedberg iff $A$ is $\mathcal{D}$-maximal.
Question
When does a c.e. set have a nontrivial nonFriedberg split?
Shavrukov’s answer is when $A$ is not $D$-maximal and not computable.

Question
Is it possible to uniformly split all noncomputable c.e. sets into a nontrivial nonFriedberg split?
No.

Question
Is it possible to uniformly split all non $D$-maximal sets into a nontrivial nonFriedberg split?
Still no.
No Uniform Nontrivial NonFriedberg Splits

Theorem (Cholak)

For every computable $f$ there is an $e$ such that $W_e$ is not computable and if $f(e) = \langle e_0, e_1 \rangle$ then either

- $W_{e_0}, W_{e_1}$ is not a split of $W_e$,
- $W_{e_0}, W_{e_1}$ is a trivial split of $W_e$, or
- $W_{e_0}, W_{e_1}$ is a Friedberg split of $W_e$ and $W_e$ is not $\mathcal{D}$-maximal.
The Construction Viewed from 0''

Build $A = W_e$ via the recursion theorem. Assume that $f(e) = \langle e_0, e_1 \rangle$. Build infinite computable pairwise disjoint sets such that

$$\# \quad (\forall i)[W_i \subseteq \bigsqcup_{j \leq i} R_j \text{ or } W_i \cup A \cup \bigsqcup_{j \leq i} R_j =^* \omega]$$

Inside each $R_i$ try to build $A$ to be maximal via Friedberg’s maximal set construction. So $A$ is not computable. Assume that $W_{e_0} = A_0, W_{e_1} = A_1$ is a split (otherwise done). Now in $R_i$ ask is

$$\star \quad A_0 \cap R_i \text{ infinite?}$$

If no, then we want to focus the construction of $A$ at $R_i$. For $j < i$ dump every ball possible into $A$. For $j > i$, put no balls into $A$. So $A$ is only noncomputable inside $R_i$ and hence $A_0, A_1$ is a trivial split. Similarly, if $A_1 \cap R_i$ is finite.
Assume we have positive answers to \( \star \) for \( e_0 \) and \( e_1 \). So \( A \) is maximal inside each \( R_i \). The \( R_i \) modulo \( D(A) \) witness that \( A \) is not \( D \)-maximal. So \( A \) has a nontrivial nonFriedberg split. Locally inside each \( R_i \), our split \( A_0, A_1 \) is Friedberg. We must show globally that \( A_0, A_1 \) is a Friedberg split. Consider \( W_i \) and assume \( W_i - A \) is not a c.e. set. Now \( \# \) holds. If the first clause of \( \# \) holds, then \( W_i \) is handled locally inside \( R_j \) for \( j \leq i \) and \( W_i - A_l \) is not a c.e. set. Otherwise \( R_{i+1} - A \subseteq W_i \). This implies that \( (W_i - A_l) \cap R_{i+1} \) is not a c.e. set.