Every FIP degree computes a 1-generic

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An Example

- Let $X_{2n} = \{2m : m \geq n\}$ and $X_{2n+1} = \{2n + 1\}$.

- Then the evens have the finite intersection property, for every finite subset $F$ of the evens, $\bigcap_{i \in F} X_i \neq \emptyset$, and they are maximal with respect to the finite intersection property.

- Any sequence in any order of all the evens is a FIP sequence. Any singleton odd is also a FIP sequence. Eventually we will want to avoid finite FIP sequences.
An Example on a Tree

- $T \subseteq \omega^{<\omega}$ is a tree iff for all $\sigma, \tau$, if $\tau \in T$ and $\sigma < \tau$ then $\sigma \in T$.
- Trees are countable and hence can be considered as indexed by $\omega$ (such that if $\sigma < \tau$ then $i_\sigma < i_\tau$).
- Build $\{X_\sigma\}_{\sigma \in T}$ by adding $\tau \in X_\sigma$ iff $\sigma \preceq \tau$.
- $f \in \omega^{\omega}$ is a path though $T$ iff, for all $l$, $f \upharpoonright l \in T$.
- Each path $f$ gives rise to an increasing FIP sequence, $\{X_{f \upharpoonright l}\}_{l \in \omega}$.
- If $\sigma$ is a dead end in $T$ (no $\tau$ extending $\sigma$ in $T$) then $\{X_{\sigma \upharpoonright l}\}_{l < |\sigma|}$ is a finite FIP sequence.
- Eventually we will want to avoid computable FIP sequences in addition to finite FIP sequences. All computable trees with no computable paths have dead ends.
FIP Sequence

Definition
Consider a collection of sets \( \{X_i\}_{i \in \omega} \). Consider \( f \in \omega^\omega \cup \omega^{<\omega} \) a FIP sequence for this collection of sets if the sequence \( \{X_{f(n)}\}_{n \in \omega} \) has the finite intersection property and is maximal with respect to inclusion and the finite intersection property (so, for all \( i \notin \text{range}(f) \), the collection \( X_i, \{X_{f(n)}\}_{n \in \omega} \) does not have the finite intersection property).

The definition allows that \( f \) need not be in increasing order. Our results rely on this flexibility.
Questions

Question

Given a collection of sets \( \{X_i\}_{i \in \omega} \) how difficult is it to find a FIP sequence?

Definition

A FIP degree \( d \) is a Turing degree (think oracle) such that for all computable collection of sets \( \{X_i\}_{i \in \omega} \), \( d \) can computable a FIP sequence.

Question

What can a FIP degree compute?
Lemma (Zorn)

Let $P$ be a set partially ordered by $<$ such that every increasing chain (under $<$) has a upper bound in $P$. Then $P$ has a maximal element.

Let $P$ be the set of $Z \subseteq \omega$ such that $\{X_i\}_{i \in Z}$ has the finite intersection property. Order $P$ by inclusion. Given a chain, its union has the finite intersection property and is contained in $P$. Therefore, by Zorn’s Lemma, every $\{X_i\}_{i \in \omega}$ has a FIP sequence.

Lemma

The statement “every collection of sets has a FIP sequence” implies the Zorn’s Lemma.
Computing a FIP sequence

We will stagewise define $f$. For each $n$, add $X_n$ to what we have it has the finite intersection property with what we have, so there exists an $x \in X_n$ that is in all the other sets we have. Otherwise do not add it.

Lemma

For a computable collection of sets $\{X_i\}_{i \in \omega}$ an increasing FIP sequence can be computed from $0'$, an oracle that decides all “there exists” sentences, all $\Sigma_1$ sentences.

Corollary

$0'$ is a FIP degree.

Lemma

$0'$ is the least degree which can compute an increasing FIP sequence for every computable collection of sets $\{X_i\}_{i \in \omega}$. 
Cohen Forcing

Definition

$g \in \omega^\omega$ meets a filter $F \subseteq \omega^{<\omega}$ iff there is a $l$ such that $g \upharpoonright l \in F$. $g$ avoids $F$ iff there is a $l$ such that if $g \upharpoonright l \preceq \tau$ then $\tau \notin F$.

Consider a Cohen generic $g$ as the increasing union of elements of $\omega^{<\omega}$ meeting or avoiding a collection of filters. For example, filters of the form $F_f = \{ \sigma \in \omega^{<\omega} : \exists n[\sigma(n) \neq f(n)] \}$. This $g$ cannot equal any $f \in \omega^\omega$. The existence of $g$ is a step along the way to showing that it is possible for the Continuum Hypothesis to fail.
FIP generic

- Define $\sigma \in P \subseteq \omega^{<\omega}$ iff the collection $\{X_{\sigma(i)}\}$ has the finite intersection property. This is $\Sigma_1$ set of conditions. We can computably list out the elements of $P$.
- Consider filters $F_n = \{\sigma : \exists j[\sigma(j) = n]\}$.
- Let $g$ meet or avoid all $F_n$.

**Lemma**

$g$ is a FIP sequence.

**Proof.**

$\{X_{g(i)}\}_{i \in \omega}$ has the finite intersection property. If $g$ meets $F_n$ then $X_n$ is in $\{X_{g(i)}\}_{i \in \omega}$. If $g$ avoids $F_n$ then $X_n, \{X_{g(i)}\}_{i \in \omega}$ does not have the finite intersection property. $\{X_{g(i)}\}_{i \in \omega}$ is not necessarily in increasing order.
1-generic

Working in $\omega^{<\omega}$, $g$ is 1-generic iff $g$ meets or avoids all $\Sigma_1$ filters. While $0'$ can compute a 1-generic not all 1-generics can compute $0'$, they have less computational power.

Lemma
A 1-generic $g$ computes a FIP sequence for every $\{X_i\}_{i \in \omega}$.

Proof.
We can assume that there is not a finite FIP sequence. Then there is a computable, 1 – 1, onto map $h$ from $\omega^{<\omega}$ to $P$ such that if $\sigma < \tau$ then $h(\sigma) < h(\tau)$. Assume that $h(\sigma)$ exists let $h(\sigma^\hat{n})$ go to the $n$th element appearing in $P$ extending $h(\sigma)$. Such an element must exist or there is a finite FIP sequence. Now $F_n = \{\sigma : \exists j[h(\sigma)(j) = n]\}$ is $\Sigma_1$ filter. So, as above, $h(g)$ is a FIP sequence. □
Every FIP degree computes a 1-generic

Corollary
A 1-generic degree is a FIP degree.

Theorem
Every FIP degree computes a 1-generic.

Plan.
Must build a \( \{X_i\}_{i \in \omega} \) such that every FIP sequence (including nonincreasing FIP sequences) computes a 1-generic. So no finite or computable FIP sequences.
References


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