Rado Path Decomposition

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Monochromatic paths

Definition
Let $c : [\omega]^2 \to r$. A monochromatic path of color $j$ is an ordered listing (possibly finite or empty) of integers $a_0, a_1, a_2 \ldots$ such that, for all $i \geq 0$, if $a_{i+1}$ exists then $c(\{a_i, a_{i+1}\}) = j$.

An empty listing can be a path of any color. A singleton can be a path of any color. The color is determined for paths of more than one node. Paths might be finite or infinite.

As we will see 3 colors is more canonical than 2 colors.

The use of $\omega$ is might to imply there are no nonstandard integers.
Rado’s Theorem

Improving on a result of Edrős, Rado published a theorem which implies:

**Theorem (Rado Path Decomposition or RPD$_r$)**

Let $c : [\omega]^2 \to r$. Then, for each $j < r$, there is a monochromatic path of color $j$ such that these $r$ paths (as sets) partition $\omega$ (so they are pairwise disjoint sets and their union is everything).

Ramsey’s Theorem

Theorem (Ramsey’s Theorem or RT$_r^2$)

Let $c : [\omega]^2 \rightarrow r$. Then there is an infinite set $H$ such that $c([H]^2)$ is constant. $H$ is called homogenous.
Ultrafilter Definition

Definition
\( U \subseteq \mathcal{P}(\omega) \) is an ultrafilter iff

- \( \emptyset \notin U \),
- For all \( X \subseteq \omega \), either \( X \in U \) or \( \omega - X \in U \),
- \( U \) is closed under finite intersections and supersets.

For example, \( U = \{ X \subset \omega | 17 \in X \} \). This is principal ultrafilter, there is a least element, 17. The existence of an non-principal ultrafilter cannot be shown in ZF but they do exist with ZFC.

Our first proof use of a non-principal ultrafilter, \( U \). We will say that \( X \) is large iff \( X \in U \).

If \( X_0 \sqcup X_1 \sqcup \ldots \sqcup X_f = \omega \) (the \( X_i \)'s partition \( \omega \)) then exactly one of the \( X_i \)'s is large.
Let neighbors of $m$ with color $i$ be

$$N(m, i) = \{n : c(\{m, n\}) = i\}.$$ 

For every $m$, the sets $N(m, j)$ partition $\omega$, so for some unique $j < r$, $N(m, j)$ is large.

Let $A_j = \{m : N(m, j) \text{ is large}\}$. The $A_j$ partition $\omega$. Think of $m \in A_j$ as having color $j$. Each $m$ has a unique color.
Ultrafilter Proof, Part II

For any pair of points \( m < n \) in \( A_j \), \( N(m, j) \cap N(n, j) \) is large. So there are infinitely many \( v \in N(m, j) \cap N(n, j) \). For all such \( v \),
\[
c(m, v) = c(v, n) = j.
\]
Note that any such \( v \) is likely much larger than \( m \) and \( n \) and not necessarily the same color.

Stagewise build finite paths such that the current end of the path of color \( j \) has color \( j \) and at stage \( s \) (if \( s \) is not already in our finite paths) use a \( v \) like above to add \( s \) to the path of it’s color (the path and \( s \) have the same color).
One of the $A_j$ must be large. We can thin $A_j$ to get a homogenous set of color $j$. Given $a_i \in A_j$ choose $a_{i+1}$ in $A_j \cap \bigcap_{k \leq i} N(a_k, j)$. 

\[\text{Ramsey Theorem and our Ultrafilter Proof}\]
Cohesive Proof

Recall $N(m, i) = \{ n : c(\{m,n\}) = i \}$ is the neighbors of $m$ with color $i$. Let $C$ be cohesive w.r.t. to all $N(m, i)$, so $C$ is infinite and, for all $m, i$, either $C \subseteq^* N(m, i)$ or $C \subseteq^* \omega - N(m, i)$ ($\subseteq^*$ is containment modulo a finite set).

Now let a set $X$ is large iff $C \subseteq^* X$. The intersection of two large sets is large and, for all $m$, exactly one of the $N(m, j)$ is large. Hence we can repeat ultrafilter proofs of with this notion of largeness.
A careful analysis of the last proof shows that the path decomposition is computable in $C'$. Why the jump? Exactly one $N(m, j)$ is large (in our cohesive set $C$). It is $\Delta^C_2$ to determine which one. 

\[
(\exists k \forall l > k[l \in C \rightarrow l \in N(m, j)] \text{ and } \forall k \exists l[l \in C \cap N(m, j)].)
\]

By Jockusch and Stephan, $\mathbf{d}$ is PA over $0'$ iff there is a set $C$ cohesive for all computable sets such that $C' \leq_T \mathbf{d}$. If $\mathbf{d}$ is PA over $0'$ then for all computable graphs a path decomposition is computable in $\mathbf{d}$. 

**Question**

*Can this be improved? Can $\mathbf{d}$ have the same degree as $0'$? or below $0'$?*

It is possible to do better with $RT^2_r$. 

**PA over $0'$**
0′ is not enough

It is known that there is a computable linear order, \((\omega, <_L)\) of order type of \(\omega + \omega^*\) with no computable ascending or descending sequence. For \(x < y\), color the pair \((x, y)\) red iff \(x \leq_L y\). Blue otherwise. A computable red (blue) path gives rise to an ascending (descending) sequence.

While an ascending or descending sequence can be found below 0′, this idea can be greatly modified to show:

**Theorem**

There is a computable 2-coloring such that every path decomposition computes 0′.
Consider \((\tau_0, \tau_1 \ldots \tau_{r-1}, X)\) such that \(X\) is infinite, \(\tau_j\) is a finite path of color \(j\), and if \(\tau_j = \sigma \hat{m}\) then \(X \subseteq^* N(m, j)\) (so \(m\) has color \(j\) w.r.t. \(X\)) as our forcing conditions. A generic \(G\) for this forcing is a path decomposition. Forcing \(\Sigma^G_1\) statements (like does \(\Phi^G(w) \downarrow\)) is \(\Sigma^X_2\). So this forcing cannot be used for avoiding the cone above \(0'\).
Stable Colorings

A coloring $c$ is stable iff, for all $m$, $\lim_n c(m, n)$ exists. Hence $N(m, j)$ is either finite or cofinite. So $\omega$ is cohesive in this case. Fix a stable coloring and now let large mean almost all and repeat our ultrafilter proof.

Stable computable colorings have $\Delta^0_2$ path decompositions since determining a $m$’s color is $\Delta^0_2$. 
Stable Ramsey’s Theorem

Stable Ramsey’s Theorem, SRT\(^2\_2\), is just Ramsey’s Theorem (as stated above) restricted to stable colorings.

COH is the statement that a set cohesive for the computable sets exists.

COH + SRT\(^2\_2\) gives us RT\(^2\_2\): Let C be our cohesive set. Then \(c : [C]^2 \rightarrow r\) is stable. Now SRT\(^2\_2\) gives our homogenous set.

But a path decomposition for the coloring \(c : [C]^2 \rightarrow r\) does not help us find a path decomposition for the coloring \(c : [\omega]^2 \rightarrow r\). So COH does not help reduce the problem of handling RPD\(_r\).

**Question**

_In the above setting of Turing ideals, does SRT\(^2\_2\) imply RT\(^2\_2\)?_
Proof for 2-colorings of $K_M$

Assume the colors are RED and BLUE. Inductively assume we have two paths of color RED and BLUE. Let $x$ be the least integer not in any path. Let $x_r$ be the end of RED path and similarly with $x_b$.

If there is any RED path between $x_r$ and $x$ avoiding our partially constructed paths, add that path to the end of the RED path. (Since finite, this is a computable question.) Similarly for BLUE.

Otherwise look at the color of $(x_r, x_b)$. If this is RED add $x_b, x$ (in that order) to the end of the RED path and remove $x_b$ from the end of the BLUE path. So $x_b$ switches to RED. If this is BLUE add $x_r, x$ (in that order) to the end of the BLUE path and remove $x_r$ from the end of the RED path. Since there are only finitely many $x$’s we settle on our final paths.

This proof fails for $r = 3$. 
Pokrovskiy showed that the finite version of $\text{RPD}_r$ fails. That given $r > 2$ and $k$ there is a $M \geq k$ and an $r$-coloring of $[M]^2$ (this is just complete graph on $M$ nodes) which does not partition into $r$ many paths, one of each color. For $r = 3$, 3 paths is enough but two of them might have the same color. $r = 2$ is special and will be dealt with shortly.

The compactness proof of the finite version from the infinite version breaks down because the paths linking numbers below $M$ might also involve some very large numbers.
Path Decompositions for 2-colorings

Theorem
If \( c : [\omega]^2 \to 2 \) is computable then there is a \( \Delta^0_2 \) Path Decomposition and the proof is nonuniform.

Theorem
This non-uniformity cannot be removed.
A Key Observation about Switching

Assume $x_b$ switches to RED. Only the ends of the paths switch so if $x_b$ switches again back to BLUE $x$ also must switch back to BLUE but there no BLUE path from $x_b$ to $x$. If $x_b$ switches from the blue path to the red path, it cannot switch again.

If there are infinitely many BLUE and RED switches then both paths stabilized and are infinite. If there are only finitely many switches then again the path stabilized but one might be finite.

But otherwise our algorithm breaks down. We used this failure to create another algorithm which works within the environment of this failure. Hence the end result is nonuniform.
Questions

Question
Is there a computable non-stable 3-coloring \( c \) that every path decomposition is PA over \( 0' \)?

Question
Does \( \text{RPD}_r \) imply \( \text{RPD}_{r+1} \)? Is this computably true?

Take a path decomposition for an \( r + 1 \)-coloring. A coloring restricted to \( r \) of the paths is not necessarily a \( r \) coloring.