Strong Jump-Traceability
The Computably Enumerable Case

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Preprint available: Cholak, Downey, and Greenberg, Strong Jump-Traceability I: the Computably Enumerable Case.
Reals with little *value* as oracles
Are there any? How low do they go? Are they all the same?

Try to understand the relation between reals with *low initial segment complexity* as measured by Kolmogorov complexity and reals with *low computational power* (as measured by the halting set relative to the real).

Example: Loveland showed the a real $\alpha$ is computable iff the sequence $C(\alpha \upharpoonright n) - C(n)$ is bounded, where $C$ is plain Kolmogorov complexity.
$K$-Trivial Reals
Reals with very low initial segment complexity

Definition
If the sequence $K(A \upharpoonright n) - K(n)$ is bounded then $A$ is $K$-trivial, where $K$ is prefix-free Kolmogorov complexity.

Theorem (Chatin, Downey, Hirschfeldt, Nies, Solovay, Stephan)
The $K$-trivial reals form a robust nontrivial ideal of low $\Delta^0_2$ degrees.
Cost Functions
How to build an $K$-trivial real. Or how do you prove your results.

Definition
The cost (or weight) of $x$ at stage $s$ is

$$c(x, s) = \sum_{x < n < s} 2^{-K_s(n)}.$$ 

Example: Define a computably enumerable set $A = \bigcup_s A_s$ by putting $x \in A_{s+1} - A_s$ if $W_{e,s} \cap A_s = \emptyset$, $x > 2e$, $x \in W_{e,s}$ and $c(x, s) < 2^{-(e+1)}$. Then $A$ is simple and $K$-trivial.
C.e. Traceability
Computationally Feeble

Definition

• A (c.e.) *trace* is an uniformly c.e. sequence $\langle T_x \rangle$ of finite sets. (Equivalently there is a computable function $g$ such that for all $x$, $T_x = W_g(x)$.)

• A trace *traces* a function $f$ if for all $x$, $f(x) \in T_x$.

• A function $h : \omega \to \omega \setminus \{0\}$ is an *order* if $h$ is computable, nondecreasing and $\lim_{s} h(s) = \infty$.

• The tracing *obeys* an order $h$ if for all $x$, $|T_x| \leq h(x)$.

• A degree $a$ is c.e. *traceable* if there is an order $h$ such that every $f \leq_T a$ can be traced by some trace obeying $h$.

Theorem (Zambella)

If $A$ is K-trivial then $\text{deg}(A)$ is c.e. traceable.
Jump Traceable

More Computationally Feeble

Definition

A is jump-traceable if there is some order $h$ and a c.e. trace $\langle T_x \rangle$ obeying $h$ and tracing $\{e\}^X(e)$ (if $\{e\}^X(e) \downarrow$) then $\{e\}^X(e) \in T_e$.

Theorem (Nies)

Jump-traceability and superlowness are the same on the c.e. sets. There are non $K$-trivial jump traceable sets.

Theorem (Nies, Figueira, and Stephan)

If $A$ is $K$-trivial, then $A$ is jump traceable with respect to an order roughly $h(n) = n \log n$. 

Will he finish in time? No way!
Yes!
Strongly Jump Traceable
Even More Computationally Feeble

**Definition**

A is *strongly jump-traceable* iff \( \{e\}^X(e) \) can be traced obeying any order.

**Theorem (Nies, Figueira, and Stephan)**

There are non-computable, strongly jump-traceable, computably enumerable reals. Strong jump-traceability is weaker than jump-traceability on the c.e. reals.

**Question (Nies and Miller)**

*Is the class of K-trivals exactly the class of strongly jump traceable reals? Is strongly jump traceability a combinatorial characterization of K-triviality?*
The c.e. strongly jump-traceable degrees form a proper subideal of the $K$-trivials.

**Theorem**

Every c.e. strongly jump-traceable set is $K$-trivial.

**Theorem**

There is a $K$-trivial c.e. set that is not strongly jump-traceable. Indeed it is not jump traceable with a bound of size roughly $\log \log n$.

**Theorem**

The c.e. strongly jump-traceable degrees form an ideal.

**Corollary (to the proof of the first theorem above)**

If a set $A$ is jump-traceable with respect to about $\sqrt{\log n}$ then it is $K$-trivial.
An hierarchy of jump-traceability?
Or a possible combinatorial characterization of the $K$-trivials.

$$\sqrt{\log n} < n \log n.$$ 

**Question**

*Is $A$ $K$-trivial iff for all orders $h$ with $\sum_{n \in \mathbb{N}} 2^{-h(n)} < \infty$, $A$ is jump traceable with order $h$?*