The c.e. sets disjoint from an c.e. set $A$

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The set up

- $A$ and $B$ and most mentioned (but not all) sets are always an infinite coinfinite c.e. set.
- $\mathcal{D}(A) = \{B | B \cap A = * \emptyset\}$, the sets disjoint from $A$.
- $\{D_0, D_1, D_2 \ldots\}$ generates $\mathcal{D}(A)$ iff, for all $i$, $D_i \in \mathcal{D}(A)$ and, for all $B \in \mathcal{D}(A)$, there is an $i$ such that $B \subseteq * \bigcup_{j<i} D_i$ ($B$ is covered by the union). The $D_i$ are the generators.

Theorem (Main Theorem, Take 1)

The possible generators for $\mathcal{D}(A)$ break up into 10 invariant types. Moreover, all types are realized when $A$ is a $\mathcal{D}$-maximal set.
• \( A \) is simple iff \( \mathcal{D}(A) \) is generated by \( \emptyset \). Write this as \( \mathcal{D}(A) = \{ \emptyset \} \).

• Simple is definable in the family of all c.e. sets with the language of inclusion, union, intersection, \( \emptyset \) and \( \omega \). We call this structure \( \mathcal{E} \). More on next slide.

• \( A \) and \( \hat{A} \) are in the same orbit and \( A \) is simple so is \( \hat{A} \).

• So invariant.
Why definable in $\mathcal{E}$?

- $R$ is computable (always!).
- $R$ computable iff there is a $W$ such that $R \sqcup W = \omega$. So if $R$ is computable then in $\mathcal{E}$, $\overline{R}$ is defined.
- Every infinite c.e. set has an infinite noncomputable subset. Fails for finite sets.
- $F$ is finite iff, for all $W \subseteq F$, $W$ is computable.
- $S$ is simple iff, for all $W$, if $W \cap S = \emptyset$ then $W$ is finite.
- $W_1 =^* W_2$ and $W_1 \subseteq^* W_2$ is definable in $\mathcal{E}$. 
Type 2

- A is Type 2 iff R generates $\mathcal{D}(A)$ iff $\mathcal{D}(A) = \{R\}$.
- $A \sqcup R$ is a trivial split of $A \sqcup R$. One of the halfes is computable.
- $A \sqcup R$ is simple iff $\mathcal{D}(A) = \{R\}$.
- Type 2 is invariant.

Lemma

$G$ are a set of generators for $A \sqcup R$ iff $G$ plus $R$ are a set of generators for $A$.

Trivial splits alter the generators. What about other kinds of splits?
Friedberg Splits

Definition

$A_0 \uplus A_1 = A$ is a *Friedberg split* of $A$ iff, for all $W$ (always c.e.), if $W - A$ is not a c.e. set either are $W - A_i$.

Theorem (Friedberg)

*Every noncomputable c.e. set $A$ has a Friedberg split. Moreover a code for the split can be found effectively in the code for $A$.*

Proof.

On board.
Friedberg splits and generators

Lemma

If $A_0 \sqcup A_1$ is a Friedberg split of $A$ and a set generators of $A$ is $G$ then a set of generators for $A_i$ is $G$ plus $A_i$. 

Proof.

Let $D \cap A_0 = \ast \emptyset$. So $D - A_0 = D$ is a c.e. set. Thus $D - A$ is a c.e. set. $D = (D - A) \sqcup (D \cap A_0) \sqcup (A \cap A_1)$. □

- Only for Friedberg splits! Take a nontrivial non-Friedberg split of a simple set. So there is a $W$ such that $W - A$ is not a c.e. set but $W - A_0$ is a c.e. set. This set is not covered by $A_1$.
- Converse fails. More in 2 slides.
- Friedberg splits alter the generators.
Type 3

- A is Type 3 iff there is a noncomputable $W$ such that $\mathcal{D}(A) = \{W\}$.
- Invariant.
- Friedberg splits of simple sets are Type 3.
- What about converse? Fails!
What are there other splits?

Nontrivial non-Friedberg splits (of simple sets)? YES!! Just wait. But ...

**Question**

*If* $f(e) = (e_0, e_1)$ *is a computable function such that if $W_e$ is not computable then $W_{e_0} \sqcup W_{e_1}$ is a nontrivial split of $W_e$. Then the split is always a Friedberg split of a noncomputable $W_e$.

As phased the answer is no. In near future we will address what the proper question should be and hopefully answer it.
Maximal sets and $\mathcal{D}$-maximal sets

Definition
$M$ is maximal iff, for all $W$, either $W \subseteq^* M$ or $W \cup M =^* \omega$ (and $M$ is not computable).

Definition
$A$ is $\mathcal{D}$-maximal iff, for all $W$, there is a $D \in \mathcal{D}(A)$ such that either $W \subseteq^* A \cup D$ or $W \cup A \cup D =^* \omega$ (and $A$ is not computable).

Theorem (Friedberg)
Maximal sets exists.
$\mathcal{D}$-maximal sets of Type 1

Maximal. By Sack and Martin there are complete and incomplete maximal sets.

**Theorem (Soare)**

*The maximal sets form an orbit.*
Splits of $\mathcal{D}$-maximal sets

Lemma

If $A_0 \sqcup A_1$ is $\mathcal{D}$-maximal and $A_0$ is not computable then $A_0$ is $\mathcal{D}$-maximal.

Proof.
On board.

Lemma (Cholak, Downey, and Herrmann following Downey and Stob)

Every nontrivial split of a $\mathcal{D}$-maximal set is a Friedberg split.

Proof.
On board
\(\mathcal{D}\)-maximal set of Type 2

So noncomputable halfes of trivial splits of maximal sets are only examples of \(\mathcal{D}\)-maximal sets of Type 2. There are incomplete and complete examples. By Soare’s result they also form an orbit.
hemi-\(P\)

**Definition**

\(A_0\) is hemi-\(P\) iff there a noncomputable \(A_1\) such that \(A_0 \uplus A_1\) has property \(P\).

If \(P\) is definable so is hemi-\(P\). Note that if \(P\) is definable then so are the Friedberg splits of the sets satisfying \(P\). In \(\mathcal{E}\) to say that "\(W_e - A\) is not a c.e. set" is there is not an c.e. set \(W\) such that \(W \cap A = \emptyset\) and \(W \cup A = W_e \cup A\).
\[D\text{-maximal set of Type 3}\]

- So halves of Friedberg splits of maximal sets are only examples of \(D\)-maximal sets of Type 3. There are incomplete and complete examples.
- Since all nontrivial splits of a maximal set are Friedberg, for a \(D\)-maximal set \(A\), \(A\) is hemimaximal iff \(A\) is Type 3.

**Theorem (Downey and Stob, Herrmann)**

*The hemimaximal sets form an orbit.* (Needs that all nontrivial splits are Friedberg.)

**Question**

*Is there a definable \(P\) such that the Friedberg splits are proper subclass of the nontrivial splits?*
Theorem (Main Theorem, Take 2)

The possible generators for \( D(A) \) break up into 10 invariant types. Moreover, all types are realized when \( A \) is a \( D \)-maximal set.

For \( D \)-maximal sets of Type 1, 2, and 3, all equal are definable orbits and contain complete and incomplete sets.
More Types

Lemma

If $D(A)$ be generated by $W_0, W_1, \ldots W_n$ then $D(A)$ is generated by their union.

Lemma

If $D(A)$ is partially generated by infinitely many computable sets then the same sets are covered an pairwise disjoint collection of computable sets.

Proof.

On board.
Infinitely many pairwise disjoint sets

Lemma
Assume that $\mathcal{D}(A)$ is generated by infinitely many pairwise disjoint sets then we can assume that either they all are computable (Type 4), or there is one noncomputable set (Type 5), or none of them are computable (Type 6).

Proof.
On board.

No longer elementary definable. But in $\mathcal{L}_{\omega_1,\omega}$. 
$\mathcal{D}$-maximal sets of Type 4, 5, and 6

A $\mathcal{D}$-maximal $A$ is

- Type 4 iff Herrmann
- Type 5 iff hemi-Herrmann
- Type 6 iff $A$ has an $A$-special lists.

All 3 are definable orbits containing complete and incomplete sets (Cholak, Herrmann, Downy and Cholak for first 2, Cholak and Harrington for last). Sets with $A$-special lists are still the only known example of a definable orbit which is not an orbit under $\Delta^0_3$ automorphisms.
Theorem (Main Theorem, Take 3)

The possible generators for $\mathcal{D}(A)$ break up into 10 invariant types. Moreover, all types are realized when $A$ is a $\mathcal{D}$-maximal set.

For $\mathcal{D}$-maximal sets of Type 1, 2, 3, 4, 5, and 6 all equal are definable orbits and contain complete and incomplete sets.

Why push this?
Reason 1

Theorem (Herrmann and Kummer)

*There is a split of simple (actually hhsimple) set which is $D$-maximal. Not a trivial or Friedberg split.*

Not of first six types. What kind of split?
Reason 2

Theorem (Cholak, Downey, Harrington)

There is a c.e. set $A$ such that the index set \{ $i \in \omega \mid W_i \cong A$\} is $\Sigma^1_1$-complete.

Theorem (Cholak and Harrington)

If $A$ is simple then the index set \{ $i \in \omega \mid W_i \cong A$\} is arithmetical.

$W_i$ and $A$ are in the same orbit iff $W_i \cong A$ iff $W_i$ and $A$ are automorphic.

Both groups of results have something to said but $D$-maximal sets. But neither group of results completely resolves how the $D$-maximal sets behave. Results like the those for first 6 types would resolve the issue. For $D$-maximal sets, it is enough to know where $D(A)$ is sent. Hope some structural property of $D(A)$ would arise and provide resolution. Still might! Just not there yet.
Question

Which c.e. sets are and are not automorphic to a complete set? Is it $\Sigma^1_1$ to decide if a set automorphic to a complete set?

Harrington and Soare showed there is a realizable definable property $Q(A)$ such that $A$ is not complete. Cholak and Harrington showed whether a simple set is automorphic to a complete set is arithmetical. But still not known exactly which simple sets are automorphic to complete sets. For example, atomless $\nu$-maximal sets. But known for hhsimple sets. So perhaps like the hhsimple sets, some structural property of $D(A)$, for the $D$-maximals sets, would arise and provide resolution.
Theorem (Main Theorem, Final Take)

The possible generators for $D(A)$ break up into 10 invariant types. Moreover, all types are realized when $A$ is a $D$-maximal set.

For $D$-maximal sets of Type 1, 2, 3, 4, 5, and 6 all equal are definable orbits and contain complete and incomplete sets.

The $D$-maximal sets of Type 7, 8, 9, or 10, break up into infinitely many orbits by defining a further invariant on each type and each example of the new invariant there are complete and incomplete examples which are not know to be automorphic.
C.e. sets as generators

Lemma
Assume that \( D(A) \) is partially generated by an infinite list of noncomputable c.e. sets, \( \{D_0, D_1, \ldots \} \). Then we can assume that these sets are pairwise disjoint or nested and \( D_{n+1} - D_n \) is not an c.e. set.
Last 4 types

Type 7 $\mathcal{D}(A) = \{D_0, R_0, R_1, \ldots\}$, where the $R_i$ are infinite pairwise disjoint computable c.e. sets (this is always the case) and $D$ is an infinite noncomputable c.e. set (and not of a lesser type).

Type 8 $\mathcal{D}(A) = \{D_0, D_1 \ldots, R_0, R_1, \ldots\}$, where the $D_i$ are infinite pairwise disjoint noncomputable c.e. sets (and not of a lesser type).

Type 9 $\mathcal{D}(A) = \{D_0, D_1 \ldots, R_0, R_1, \ldots\}$, where the $D_i$ are infinite nested noncomputable c.e. sets such that, for all $l$, $D_{l+1} - D_l$ is not an c.e. set, and and, for each $i$, there are infinitely many $j$ such that $R_j - D_i$ is infinite (and not of a lesser type).

Type 10 $\mathcal{D}(A) = \{D_0, D_1 \ldots\}$, where the $D_i$ are infinite nested noncomputable c.e. sets such that, for all $l$, $D_{l+1} - D_l$ is not an c.e. set (and not of a lesser type).
Definition

A coinfinite set $A$ is $r$-maximal iff no infinite computable splits $\overline{A}$ into two infinite pieces iff, for every computable set $R$, either $R \cap \overline{A} = \ast \emptyset$ ($R \subseteq \ast A$) or $\overline{R} \cap \overline{A} = \ast \emptyset$ (so $\overline{A} \subseteq \ast R$).

Supersets of $r$-maximal sets not almost equal to everything are $r$-maximal.

Lemma

Assume that $A$ is the half of a split of $r$-maximal set (so $A$ is not simple and hence cannot have Type 1) and $D(A)$ is not Type 2 or 3. Then $D(A)$ is Type 10.

The assumption not Type 2 or 3 is needed due trivial and Friedberg splits.
Atomless sets and $\mathcal{D}$-maximal Type 10 sets

**Definition**

$B$ is **atomless** iff $B$ does not have a maximal superset iff for every c.e. superset $C$, if $C \neq^* \omega$ then there is an $E$ such that $C \subset E \subset \omega$.

**Lemma**

*If $A$ is $\mathcal{D}$-maximal and of Type 10 then $A$ is half of a split of an atomless r-maximal set.*

For our proof $\mathcal{D}$-maximal is needed. What kind of split is this set? Not trivial or Friedberg.
Anti-Friedberg Splits

Definition
Let $A_0 \sqcup A_1$ be a nontrivial splitting of a $A$. We say $A_0 \sqcup A_1$ is an anti-Friedberg split of $A$ if for all $W$, either

1. there is a $D$ such that $D \cap A_0 = \emptyset$ and $W \cup D \cup A = \omega$
or
2. $W - A_0$ is c.e. set.

The order of the sets, $A_0$ and $A_1$, matters. An anti-Friedberg split which is not a Friedberg split is a properly anti-Friedberg split.

In the first case $W - A_0$ cannot be a c.e. set or $A_0$ would be computable.
Anti-Friedberg Splits and $\mathcal{D}$-maximal sets

**Lemma**

A is $\mathcal{D}$-maximal and $B$ is an infinite noncomputable c.e. set disjoint from $A$ iff $A \sqcup B$ is an anti-Friedberg split.

**Proof.**

$(\Rightarrow)$ For all $W$ there is a set $D$ disjoint from $A$ such that either $W \subseteq^* A \sqcup D$ (so $W - A = W \cap D$ is a c.e. set) or $W \cup A \cup D =^* \omega$.

$(\Leftarrow)$ For all $W$ there is a $D$ such that $D \cap A = \emptyset$ and either $W \cup D \cap A =^* \omega$ or $W - A = D$ is a c.e. set. In the latter case, $W \subseteq^* A \sqcup (W - A)$. □
Proper anti-Friedberg splits

Lemma

Let $A \sqcup B$ be an anti-Friedberg split. (So $A$ is $\mathcal{D}$-maximal). $A \sqcup B$ is $\mathcal{D}$-maximal iff $A \sqcup B$ is a Friedberg split.

Proof.

$(\Rightarrow)$ Earlier we showed all splits of a $\mathcal{D}$-maximal are trivial or Friedberg.

$(\Leftarrow)$ Consider $D$ disjoint from $A$. So $D - A = D$ is always a c.e. set. Since $A \sqcup B$ is a Friedberg split, $D - B = D - (A \sqcup B)$ has to also be a c.e. set. For all $W$, there is a $D$ disjoint from $A$ such that $W \subseteq^* A \sqcup B \sqcup (D - B)$ or $W \cup (A \sqcup B \sqcup (D - B))) =^* \emptyset$. So the union is also $\mathcal{D}$-maximal. \qed
More examples of proper anti-Friedberg splits

If $A$ is not simple but $D$-maximal then there is an computable set $R$ disjoint from $A$. Let $X$ a noncomputable c.e. subset of $R$. Then $R$ witnesses that $A \sqcup X$ is not $D$-maximal. Note that $R - (A \sqcup X)$ is not a c.e. set but $R - A = R$ is a c.e. set.
Anti-Friedberg splits and generators for $\mathcal{D}(A)$

Lemma

If $A_0 \sqcup A_1 = A$ is an anti-Friedberg split of a $\mathcal{D}$-maximal set $A$ then $\mathcal{D}(A_1)$ is generated by $A_0$ (so Type 3)

It is unclear what happens in the above lemma when $A$ is hh-simple.
Two questions

Question
If $A_0 \sqcup \hat{A}_1 = \hat{A}_0 \sqcup \hat{A}_1$ are anti-Friedberg splits then $A_0$ and $\hat{A}_0$ are automorphic?

Question
If $f(e) = (e_0, e_1)$ is a computable function such that if $W_e$ is not computable then $W_{e_0} \sqcup W_{e_1}$ is a nontrivial split of $W_e$. Then the split is always a Friedberg split of a noncomputable $W_e$.

As noted earlier, as phased the answer is no. In near future we will address what the proper question should be and hopefully answer it.