The Computably Enumerable Sets
The Past, the Present and the Future

Peter Cholak

University of Notre Dame
Department of Mathematics

Peter.Cholak.1@nd.edu

http://www.nd.edu/~cholak/papers/

Supported by NSF Grant DMS 02-45167

May, 2006
Main Goals

To understand orbits within the structure of c.e. (r.e.) sets under inclusion.

What are the good open interesting questions?

The story is rather complex.
Main Goals

To understand orbits within the structure of c.e. (r.e.) sets under inclusion.

What are the good open interesting questions?

The story is rather complex.
Main Goals

To understand orbits within the structure of c.e. (r.e.) sets under inclusion.

What are the good open interesting questions?

The story is rather complex.
The Computably Enumerable Sets, $\mathcal{E}$

- $W_e$ is the domain of the $e$th Turing machine.
- $(\{W_e : e \in \omega\}, \subseteq)$ are the c.e. (r.e.) sets under inclusion, $\mathcal{E}$.
- These sets are the same as the $\Sigma^0_1$ sets, 
  \[ \{x : (\mathbb{N}, +, \times, 0, 1) \models \varphi(x)\}, \text{ where } \varphi \text{ is } \Sigma^0_1. \]
- $W_{e,s}$ is the domain of the $e$th Turing machine at stage $s$.
- **Dynamic Properties**: How slow or fast can a c.e. set be enumerated with respect to another c.e. set? w.r.t. the standard enumeration of all c.e. sets?
- $0, 1, \cup$, and $\cap$ are definable from $\subseteq$ in $\mathcal{E}$.
- For safety all sets are c.e., infinite, and coinfinite unless otherwise noted.
The Computably Enumerable Sets, $\mathcal{E}$

- $W_e$ is the domain of the $e$th Turing machine.
- $\{W_e : e \in \omega\}$ are the c.e. (r.e.) sets under inclusion, $\mathcal{E}$.
- These sets are the same as the $\Sigma_1^0$ sets, $\{x : (\mathbb{N}, +, \times, 0, 1) \models \varphi(x)\}$, where $\varphi$ is $\Sigma_1^0$.
- $W_{e,s}$ is the domain of the $e$th Turing machine at stage $s$.
- **Dynamic Properties**: How slow or fast can a c.e. set be enumerated with respect to another c.e. set? w.r.t. the standard enumeration of all c.e. sets?
  - 0, 1, $\cup$, and $\cap$ are definable from $\subseteq$ in $\mathcal{E}$.
  - For safety all sets are c.e., infinite, and coinfinite unless otherwise noted.
The Computably Enumerable Sets, $\mathcal{E}$

- $W_e$ is the domain of the $e$th Turing machine.
- $(\{W_e : e \in \omega\}, \subseteq)$ are the c.e. (r.e.) sets under inclusion, $\mathcal{E}$.
- These sets are the same as the $\Sigma^0_1$ sets, $
\{x : (\mathbb{N}, +, \times, 0, 1) \models \varphi(x)\}$, where $\varphi$ is $\Sigma^0_1$.
- $W_{e,s}$ is the domain of the $e$th Turing machine at stage $s$.
- **Dynamic Properties**: How slow or fast can a c.e. set be enumerated with respect to another c.e. set? w.r.t. the standard enumeration of all c.e. sets?
- $0, 1, \cup, \text{ and } \cap$ are definable from $\subseteq$ in $\mathcal{E}$.
- For safety all sets are c.e., infinite, and coinfinite unless otherwise noted.
Automorphisms of $\mathcal{E}$

- $\Phi \in \text{Aut}(\mathcal{E})$ iff $\Phi(W)$ is a one to one, onto map taking c.e. sets to c.e. sets such that $W_e \subseteq W_{e'}$ iff $\Phi(W_e) \subseteq \Phi(W_{e'})$ and dually.

- For example, if $p$ is a computable permutation of $\mathbb{N}$ then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

- If $p$ is a permutation of $\mathbb{N}$ such that $p(W)$ is r.e. and dually then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

- $\Phi \in \text{Aut}(\mathcal{E})$ iff there is a permutation, $p$, of $\mathbb{N}$ such that $\Phi(W_e) = \widehat{W}_{p(e)}$ and $W_e \subseteq W_{e'}$ iff $\widehat{W}_{p(e)} \subseteq \widehat{W}_{p(e')}$ and dually.

- $A$ is automorphic to $\widehat{A}$ iff there is a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = \widehat{A}$.
Automorphisms of $\mathcal{E}$

- $\Phi \in \text{Aut}(\mathcal{E})$ iff $\Phi(W)$ is a one to one, onto map taking c.e. sets to c.e. sets such that $W_e \subseteq W_{e'}$ iff $\Phi(W_e) \subseteq \Phi(W_{e'})$ and dually.
- For example, if $p$ is a computable permutation of $\mathbb{N}$ then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.
- If $p$ is a permutation of $\mathbb{N}$ such that $p(W)$ is r.e. and dually then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.
- $\Phi \in \text{Aut}(\mathcal{E})$ iff there is a permutation, $p$, of $\mathbb{N}$ such that $\Phi(W_e) = \hat{W}_{p(e)}$ and $W_e \subseteq W_{e'}$ iff $\hat{W}_{p(e)} \subseteq \hat{W}_{p(e')}$ and dually.
- $A$ is automorphic to $\hat{A}$ iff there is a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = \hat{A}$. 
Automorphisms of $\mathcal{E}$

- $\Phi \in \text{Aut}(\mathcal{E})$ iff $\Phi(W)$ is a one to one, onto map taking c.e. sets to c.e. sets such that $W_e \subseteq W_{e'}$ iff $\Phi(W_e) \subseteq \Phi(W_{e'})$ and dually.

- For example, if $p$ is a computable permutation of $\mathbb{N}$ then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

- If $p$ is a permutation of $\mathbb{N}$ such that $p(W)$ is r.e. and dually then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

- $\Phi \in \text{Aut}(\mathcal{E})$ iff there is a permutation, $p$, of $\mathbb{N}$ such that $\Phi(W_e) = \hat{W}_{p(e)}$ and $W_e \subseteq W_{e'}$ iff $\hat{W}_{p(e)} \subseteq \hat{W}_{p(e')}$. and dually.

- $A$ is automorphic to $\hat{A}$ iff there is a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = \hat{A}$. 
Automorphisms of $\mathcal{E}$

- $\Phi \in \text{Aut}(\mathcal{E})$ iff $\Phi(W)$ is a one to one, onto map taking c.e. sets to c.e. sets such that $W_e \subseteq W_{e'}$ iff $\Phi(W_e) \subseteq \Phi(W_{e'})$ and dually.

- For example, if $p$ is a computable permutation of $\mathbb{N}$ then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

- If $p$ is a permutation of $\mathbb{N}$ such that $p(W)$ is r.e. and dually then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

- $\Phi \in \text{Aut}(\mathcal{E})$ iff there is a permutation, $p$, of $\mathbb{N}$ such that $\Phi(W_e) = \hat{W}_{p(e)}$ and $W_e \subseteq W_{e'}$ iff $\hat{W}_{p(e)} \subseteq \hat{W}_{p(e')}$ and dually.

- $A$ is automorphic to $\hat{A}$ iff there is a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = \hat{A}$.
Automorphisms of $\mathcal{E}$

- $\Phi \in \text{Aut}(\mathcal{E})$ iff $\Phi(W)$ is a one to one, onto map taking c.e. sets to c.e. sets such that $W_e \subseteq W_{e'}$ iff $\Phi(W_e) \subseteq \Phi(W_{e'})$ and dually.
- For example, if $p$ is a computable permutation of $\mathbb{N}$ then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.
- If $p$ is a permutation of $\mathbb{N}$ such that $p(W)$ is r.e. and dually then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.
- $\Phi \in \text{Aut}(\mathcal{E})$ iff there is a permutation, $p$, of $\mathbb{N}$ such that $\Phi(W_e) = \hat{W}_{p(e)}$ and $W_e \subseteq W_{e'}$ iff $\hat{W}_{p(e)} \subseteq \hat{W}_{p(e')}$ and dually.
- $A$ is automorphic to $\hat{A}$ iff there is a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = \hat{A}$.
The Turing Degrees

- $A \leq_T B$, $A$ is Turing reducible to $B$, iff a Turing machine with an oracle for $B$ computes $\chi_A$.
- $\equiv_T$ is the corresponding equivalence relation.
- The Turing degrees are the corresponding equivalence classes on $\mathcal{P}(\omega)$.
- A Turing degree $a$ is c.e. iff it contains a computably enumerable set.

**Turing Complexity:** Turing reducibility, Turing jump, and the related jump classes (low$_n$ and high$_n$ classes).
- $A$ is low$_n$ iff $A^{(n)} = \emptyset^{(n)}$ iff $\Delta^0_{n+1} = \Delta^A_{n+1}$.
- $A$ is high$_n$ iff $A^{(n)} = \emptyset^{(n+1)}$ iff $\Delta^0_{n+2} = \Delta^A_{n+1}$. 
The Turing Degrees

- \( A \leq_T B \), \( A \) is Turing reducible to \( B \), iff a Turing machine with an oracle for \( B \) computes \( \chi_A \).
- \( \equiv_T \) is the corresponding equivalence relation.
- The Turing degrees are the corresponding equivalence classes on \( P(\omega) \).
- A Turing degree \( a \) is c.e. iff it contains a computably enumerable set.
- **Turing Complexity**: Turing reducibility, Turing jump, and the related jump classes (low_\( n \) and high_\( n \) classes).
  - \( A \) is low_\( n \) iff \( A^{(n)} = \emptyset^{(n)} \) iff \( \Delta_0^{n+1} = \Delta^A_{n+1} \).
  - \( A \) is high_\( n \) iff \( A^{(n)} = \emptyset^{(n+1)} \) iff \( \Delta_0^{n+2} = \Delta^A_{n+1} \).
Core Themes

- **Automorphisms** and **Orbits** of $E$.
- **Definability**: If two sets are not in the same orbit there is some definable difference. So definability within a fixed structure; within arithmetic; (principal) types; elementary definability; within $L_{\omega_1,\omega}$, within $L_{\omega_1,\omega_1}$.
- **Dynamic Properties**.
- **Turing Complexity**.
Post’s Program. Find an incomplete, noncomputable computably enumerable degree. Solved by Friedberg 57 and Muchnik 56.

Post’s Program. Find a “set-theoretic” or “thinness” property of a computably enumerable set which guarantees incompleteness.
Main Open Questions

Question (Completeness)

Which c.e. sets are automorphic to complete sets?

Question (Cone Avoidance)

Given an incomplete c.e. degree $d$ and an incomplete c.e. set $A$, is there an $\hat{A}$ automorphic to $A$ such that $d \nleq_T \hat{A}$?
The Creative Sets

Theorem (Myhill)
A c.e. set is creative iff it is 1-complete.

Theorem (Harrington)
Being creative is definable in $\mathcal{E}$.

Corollary
The creative sets form an orbit; the simplest automorphism of $\mathcal{E}$ is a computable permutation of $\omega$, a 1-reduction.
Orbits of Complete Sets

Theorem (Harrington)

*The creative sets are the only orbit of $E$ which remains an orbit when we restrict the allowable automorphisms to computable permutations.*

Theorem (Harrington and Soare)

*There is another definable orbit containing just complete sets.*
Orbits of Complete Sets

Theorem (Harrington)

*The creative sets are the only orbit of $E$ which remains an orbit when we restrict the allowable automorphisms to computable permutations.*

Theorem (Harrington and Soare)

*There is another definable orbit containing just complete sets.*
The Maximal Sets

Definition

$M$ is maximal if for all $W$ either $W \subseteq^* M$ or $M \cup W =^* \omega$.

Theorem (Friedberg)

Maximal sets exist.

Proof.

Make $\overline{M}$ cohesive. Use markers $\Gamma_e$ to mark the $e$th element of $\overline{M}_s$. Let $\sigma(e, x, s) = \{i \leq e : x \in W_{i,s}\}$; the $e$th state of $x$ at stage $s$. Allow $\Gamma_e$ to pull at any stage to increase its $e$-state.

Corollary

Let $M$ be maximal. If $M \cup W =^* \omega$ then there is a computable set $R_W$ such that $R_W \subseteq^* M$ and $W \cup R_W = \omega$. So $W = R_W \sqcup (W \cap R_W)$. 
Building Automorphisms

Theorem (Soare)

If $A$ and $\hat{A}$ are both noncomputable then there is an isomorphism, $\Phi$, between the c.e. subsets of $A$, $\mathcal{E}(A)$, to the c.e. subsets of $\hat{A}$ sending computable subsets of $A$ to computable subsets of $\hat{A}$ and dually.

False Proof.

Let $p$ be a computable 1-1 map from $A$ to $\hat{A}$. Let $\Phi(W) = p(W)$. □
Orbits of Maximal Sets

**Theorem (Soare 74)**

The maximal sets form an orbit.

**Proof.**

Define if \( W \subseteq^* M \) then let \( \Psi(W) = \Phi(W) \). If \( M \cup W =^* \omega \) then let \( \Psi(W) = \Phi(R_W) \cup \Phi(R_W \cap W) \).

**Theorem (Martin)**

A c.e. degree \( h \) is high iff there is a maximal set \( M \) such that \( M \in h \).
Theorem (Soare 74)

The maximal sets form an orbit.

Proof.
Define if $W \subseteq^* M$ then let $\Psi(W) = \Phi(W)$. If $M \cup W =^* \omega$ then let $\Psi(W) = \Phi(R_W) \sqcup \Phi(R_W \cap W)$. □

Theorem (Martin)

A c.e. degree $h$ is high iff there is a maximal set $M$ such that $M \in h$. 

Orbits of Maximal Sets
Friedberg Splits of Maximal Sets

**Definition**

$S_1 \sqcup S_2 = M$ is a Friedberg split of $M$, for all $W$, if $W \setminus M$ is infinite then $W \setminus S_i$ is also.

**Theorem (Downey and Stob)**

*The Friedberg splits of the maximal sets form an definable orbit.*

**Theorem (Downey and Stob)**

*All high degrees are hemimaximal degrees. All prompt degrees are hemimaximal. There is a low degree which is not a hemimaximal degree.*
Friedberg Splits of Maximal Sets

Definition
$S_1 \sqcup S_2 = M$ is a Friedberg split of $M$, for all $W$, if $W \setminus M$ is infinite then $W \setminus S_i$ is also.

Theorem (Downey and Stob)
The Friedberg splits of the maximal sets form an definable orbit.

Theorem (Downey and Stob)
All high degrees are hemimaximal degrees. All prompt degrees are hemimaximal. There is a low degree which is not a hemimaximal degree.
Almost Prompt Sets

Definition
Let $X^e_n$ be the $e$th $n$-r.e. set. $A$ is almost prompt iff there is a computable nondecreasing function $p(s)$ such that for all $e$ and $n$ if $X^e_n = A$ then $(\exists x)(\exists s)[x \in X^e_n, s \text{ and } x \in A_{p(s)}]$.

Theorem (Maass, Shore, Stob)
Theorem (Maass, Shore, Stob)
There is a definable property $P(A)$ which implies $A$ is prompt and furthermore for all prompt degree, $d$, there is set $A$ such that $P(A)$ and $A \in d$.

Theorem (Harrington and Soare)
All almost prompt sets are automorphic to a complete set.
Tardy Sets

**Definition**

*D* is *2-tardy* iff for every computable nondecreasing function *p(s)* there is an *e* such that *X^2_e = D* and *(∀x)(∀s)[if x ∈ X^2_e,s then x ∉ D_p(s)]*.

**Definition**

*D* is *codable* iff for all *A* there is an *Â* in the orbit of *A* such that *D ≤_T Â*.

**Theorem (Harrington and Soare)**

There are *E* definable properties *Q(D)* and *P(D, C)* such that

- *Q(D)* implies that *D* is 2-tardy,
- if there is an *C* such that *P(D, C)* and *D* is 2-tardy then *Q(D)* (and *D* is high),
- *X* is codable iff there is a *D* such that *X ≤_T D* and *Q(D)*.
Tardy Sets

Definition

$D$ is 2-	extit{tardy} iff for every computable nondecreasing function $p(s)$ there is an $e$ such that $X_e^2 = \overline{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^2 \text{ then } x \notin D_{p(s)}]$.

Definition

$D$ is 	extit{codable} iff for all $A$ there is an $\hat{A}$ in the orbit of $A$ such that $D \leq_T \hat{A}$.

Theorem (Harrington and Soare)

There are $\mathcal{E}$ definable properties $Q(D)$ and $P(D, C)$ such that

- $Q(D)$ implies that $D$ is 2-tardy,
- if there is an $C$ such that $P(D, C)$ and $D$ is 2-tardy then $Q(D)$ (and $D$ is high),
- $X$ is codable iff there is a $D$ such that $X \leq_T D$ and $Q(D)$.
Questions about Tardiness

Question

How do the following sets of degrees compare:

- the hemimaximal degrees,
- the tardy degrees,
- for each \( n \), \( \{d : \text{there is a } n\text{-tardy } D \text{ such that } d \leq_T D \} \),
- \( \{d : \text{there is a } 2\text{-tardy } D \text{ such that } Q(D) \text{ and } d \leq_T D \} \),
- \( \{d : \text{there is a } A \in d \text{ which is not automorphic to a complete set} \} \).

Theorem (Harrington and Soare)

There is a maximal 2-tardy set.

Question

Is there a nonhigh 2-tardy set which is automorphic to a complete set?
Invariant Classes

Definition
A class $\mathcal{D}$ of degrees is \textit{invariant} if there is a class $S$ of (c.e.) sets such that

1. $d \in \mathcal{D}$ implies there is a $W$ in $S$ and $d$.
2. $W \in S$ implies $\deg(W) \in \mathcal{D}$ and
3. $S$ is closed under automorphic images (but need not be one orbit).

Corollary
\textit{The high degrees are invariant by a single orbit.}

Corollary
\textit{The prompt degrees are invariant by a single orbit.}
More Invariant degree classes

Theorem (Shoenfield 76)
*The nonlow\(_2\) degrees are invariant.*

Theorem (Harrington and Soare)
- *The nonlow degrees are not invariant.*
- *There is properly low\(_2\) degree \(d\) such that if \(A \in d\) then \(A\) is automorphic to a low set.*
- *There is a low\(_2\) set which is not automorphic to a low set.*

Theorem (Cholak and Harrington 02)
*For \(n \geq 2\), nonlow\(_n\) and high\(_n\) degrees are invariant.*
Questions on Jump Classes and Degrees

Question (Can single jumps be coded into $E$?)

Let $J$ be C.E.A. in $0'$ but not of degree $0''$. Is there a degree $a$ such that $a' \equiv_T J$ and, for all $A \in a$, there is an $\hat{A}$ with $A$ automorphic to $\hat{A}$ and $\hat{A}' <_T a'$ or $\hat{A}' |_{T a'}$?

Question (Can a single Turing degree be coded into $E$?)

Is there a degree $d$ and an incomplete set $A$ such that, for all $\hat{A}$ automorphic to $A$, $d \leq \hat{A}$? $A \in d$?
More Single Orbit Invariant Classes?

Theorem (Downey and Harrington – No fat orbit)

There is a property $S(A)$, a prompt low degree $d_1$, a prompt high degree $d_2$ greater than $d_1$, and tardy high degree e such that for all $E \leq_T e$, $\neg S(E)$ and if $d_1 \leq_T D \leq_T d_2$ then $S(D)$. 
Orbits Containing Prompt (Tardy) High Sets

Question

Let $A$ be incomplete. If the orbit of $A$ contains a set of high prompt degree must the orbit of $A$ contain a set from all high prompt degrees?

Question

If the orbit of $A$ contains a set of high tardy degree must the orbit of $A$ contain a set from all high tardy degrees?

A positive answer to both questions would answer the cone avoidance question. But not the completeness question.

Question

For every degree $a$ there is a set $A \in a$ whose orbit contains every high degree.
Questions on Invariant Degree Classes

**Question**

*Are the 2-tardy degrees invariant? The degrees containing sets not automorphic to a complete set?*

**Question**

*There are the array non-computable degrees invariant?*

**Question**

*Is there is an anc degree \( d \) such that if \( A \in d \) then \( A \) is automorphic an non-anc set?*
On the Complexity of the Orbits

Look at the index set of all \( \hat{A} \) in the orbit of \( A \) with the hopes of finding some answers. The index set of such \( \hat{A} \) is in \( \Sigma_1^1 \).

Theorem (Cholak and Harrington)
If \( A \) is hhsimple then \( \{ e : W_e \text{ is automorphic to } A \} \) is \( \Sigma_5^0 \).

Theorem (Cholak and Harrington)
If \( A \) is simple then \( \{ e : W_e \text{ is automorphic to } A \} \) is \( \Sigma_8^0 \).

Question
Is every low\(_2\) simple set automorphic to a complete set? At least this is an arithmetic question.
On the Complexity of the Orbits

Look at the index set of all $\hat{A}$ in the orbit of $A$ with the hopes of finding some answers. The index set of such $\hat{A}$ is in $\Sigma^1_1$.

**Theorem (Cholak and Harrington)**

*If $A$ is hhsimple then* $\{e : W_e \text{ is automorphic to } A\}$ *is $\Sigma^0_5$.*

**Theorem (Cholak and Harrington)**

*If $A$ is simple then* $\{e : W_e \text{ is automorphic to } A\}$ *is $\Sigma^0_8$.*

**Question**

*Is every low$_2$ simple set automorphic to a complete set? At least this is an arithmetic question.*
On the Complexity of the Orbits

Look at the index set of all $\hat{A}$ in the orbit of $A$ with the hopes of finding some answers. The index set of such $\hat{A}$ is in $\Sigma_1^1$.

Theorem (Cholak and Harrington)

If $A$ is hhsimple then $\{e : W_e \text{ is automorphic to } A\}$ is $\Sigma_5^0$.

Theorem (Cholak and Harrington)

If $A$ is simple then $\{e : W_e \text{ is automorphic to } A\}$ is $\Sigma_8^0$.

Question

Is every low$_2$ simple set automorphic to a complete set? At least this is an arithmetic question.
On the Complexity of the Orbits

Look at the index set of all \( \hat{A} \) in the orbit of \( A \) with the hopes of finding some answers. The index set of such \( \hat{A} \) is in \( \Sigma^1_1 \).

**Theorem (Cholak and Harrington)**

*If \( A \) is hhsimple then \( \{ e : W_e \text{ is automorphic to } A \} \) is \( \Sigma^0_5 \).*

**Theorem (Cholak and Harrington)**

*If \( A \) is simple then \( \{ e : W_e \text{ is automorphic to } A \} \) is \( \Sigma^0_8 \).*

**Question**

*Is every low\(_2\) simple set automorphic to a complete set? At least this is an arithmetic question.*
\[ \Sigma^{1}_{1} \text{-completeness} \]

**Theorem (Cholak, Downey, and Harrington)**

*There is an set \( A \) such that the index set \( \{ i : W_i \text{ is automorphic to } A \} \) is \( \Sigma^{1}_{1} \)-complete.*

**Corollary**

- Not all orbits are elementarily definable.
- No arithmetic description of all orbits.
- Scott rank of \( \mathcal{E} \) is \( \omega^{CK}_{1} + 1 \).
- (of the proof) For all \( \alpha \geq 8 \), there is a properly \( \Delta^{0}_{\alpha} \) orbit.
There is an set $A$ such that the index set
\[ \{ i : W_i \text{ is automorphic to } A \} \text{ is } \Sigma^1_1 \text{-complete.} \]

**Corollary**

- Not all orbits are elementarily definable.
- No arithmetic description of all orbits.
- Scott rank of $\mathcal{E}$ is $\omega^{CK}_1 + 1$.
- (of the proof) For all $\alpha \geq 8$, there is a properly $\Delta^0_\alpha$ orbit.
The isomorphism problem for computable infinitely branching trees which is known to be $\Sigma^1_1$-complete. Embed this problem into orbits.

Given a tree $T$ build $A_T$ using ideas from maximal sets, Friedberg splits, and cohesiveness arguments and a jazzed up version of Soare’s theorem on isomorphisms to code a tree $T$ into the orbit of $A_T$. 
$T$ is an invariant which determines the orbit of $A_T$. Only works for some $A$; those in the orbit of some $A_T$.

**Question**

*Can we find an “invariant” (in terms of trees?) which works for all $A$?*

**Conjecture**

*The degrees of the possible $A_T$ are the hemimaximal degrees.*
$T$ is an invariant which determines the orbit of $A_T$. Only works for some $A$; those in the orbit of some $A_T$.

**Question**

*Can we find an “invariant” (in terms of trees?) which works for all $A$?*

**Conjecture**

*The degrees of the possible $A_T$ are the hemimaximal degrees.*
Main Open Questions, Again

Question (Completeness)
Which c.e. sets are automorphic to complete sets?

Question (Cone Avoidance)
Given an incomplete c.e. degree $d$ and an incomplete c.e. set $A$, is there an $\hat{A}$ automorphic to $A$ such that $d \not\leq_T \hat{A}$?

Question
Are these arithmetical questions?
Main Open Questions, Again

Question (Completeness)
Which c.e. sets are automorphic to complete sets?

Question (Cone Avoidance)
Given an incomplete c.e. degree $d$ and an incomplete c.e. set $A$, is there an $\hat{A}$ automorphic to $A$ such that $d \not\leq_T \hat{A}$?

Question
Are these arithmetical questions?