# THE DENSE SIMPLE SETS ARE ORBIT COMPLETE WITH RESPECT TO THE SIMPLE SETS

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ABSTRACT. We prove conjectures of Herrmann and Stob by showing that the dense simple sets are orbit complete w.r.t. the simple sets.

## 1. INTRODUCTION

We prove conjectures of Herrmann and Stob. We first learned of these conjectures when they appeared in "Open Questions in Recursion Theory" [Slaman, 1993]. The conjectures appeared there as part of Question 4.9, a two part question:

> 1. (Herrmann) The collection of hypersimple sets is orbit complete for simple sets. That is, every simple set is automorphic to a hypersimple set.
> 2. (Stob) The collection of dense simple sets is orbit complete. Here a set A is dense simple if the enumeration of the complement of A eventually dominates every total recursive function.

As suggested by the conjectures we make the following definition:

**Definition 1.** A collection $\mathcal{C}$ of computably enumerable sets is *orbit complete* w.r.t. a class of computably enumerable sets $\mathcal{D}$ if for every set $A \in \mathcal{D}$ there is a computably enumerable set $B \in \mathcal{C}$ and an automorphism $\Phi$ of the computably enumerable sets such that $\Phi(A) = B$.

A set $B$ is *dense simple* iff $B$ is computably enumerable and $p_{\overline{B}}$ dominates every computable total function. The dense simple sets are hypersimple and high. (For details see [Soare, 1987, V.2 and XI.1].) Thus the following theorem implies a positive answer to both conjectures.

**Theorem 2.** *The dense simple sets are orbit complete w.r.t. the simple sets.*

This theorem is related to a theorem of Harrington and Soare [Harrington and Soare, 1996] and independently Cholak [Cholak, 1995]. Restated,

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using the above definition and weakened, this theorem reads: the high computably enumerable sets are orbit complete w.r.t. to all noncomputable computably enumerable sets.

The idea behind these questions and theorems is that the more "information content" (in terms of the jump operation) the sets in a class $\mathcal{C}$ of computably enumerable sets have the more likely the class would be orbit complete w.r.t. any fixed class $\mathcal{D}$. Theorem 2 shows that dynamic properties of the sets in $\mathcal{C}$ (such as the domination of every computable function) along with information content play a role in determining whether $\mathcal{C}$ is orbit complete w.r.t. any fixed class $\mathcal{D}$.

Of course, the class $\mathcal{D}$ plays a role in determining whether a collection $\mathcal{C}$ is orbit complete w.r.t. $\mathcal{D}$. The relationship between $\mathcal{D}$, $\mathcal{C}$, the "information content" of these classes, and the dynamic properties of these classes remains to be worked out.

The rest of this paper contains the proof of the above theorem. Fix a computably enumerable simple set $A$. The proof will involve a construction of an automorphism $\Phi$ and a dense simple set $B$.

To get the desired automorphism we will use a modified version of the Automorphism Theorem [Harrington and Soare, 1996, Theorem 4.2]. We could have chosen to prove the theorem directly using either the methods of Harrington and Soare [1996] or Cholak [1995] but that would have taken far longer (this is how we initially thought of the proof). The fact that the proof is so short is surprising (at least to us) and helps support the view that one can use the Automorphism Theorem as a "black box" to create $\Delta_3$-automorphisms. On the other hand, the fact that we needed to slightly modify the Automorphism Theorem does not help support this view. So one could use this paper as a template on how to modify the Automorphism Theorem and its proof and still have it hold. This is not the first time the Automorphism Theorem and its proof were modified; this was done in Cholak and Downey [n.d.] but the modifications there were much more involved.

This paper is by no means self-contained. It heavily depends on Harrington and Soare [1996]. For the rest of this paper all the numbering of the form $x.y$ refers to Harrington and Soare [1996].

## 2. The Automorphism Theorem

We state an altered (for our purposes only) version of Harrington and Soare's Automorphism Theorem. We will assume the reader is familiar with the notation used in the statement of the theorem (again see Harrington and Soare [1996]). In this proof, we will need a new notion called *$\alpha$-approval*. We will *not* use the notion of *$\alpha$-witnesses*.

**Theorem 3.** *Assume that $A = U_0$ is a simple set. Suppose that the computably enumerable sets $\{U_\alpha\}_{\alpha\in T}$, $\{V_\alpha\}_{\alpha\in T}$, $\{\widehat{U}_\alpha\}_{\alpha\in T}$ and $\{\widehat{V}_\alpha\}_{\alpha\in T}$ are enumerated by the construction in Section 3 of [Harrington and Soare, 1996]) using Steps $1-6$, $\hat{1}-\hat{5}$ and 11 (in Sections 3 and 6 of [Harrington and Soare, 1996] except for some modifications below. Then the correspondence $U_\alpha \leftrightarrow \widehat{U}_\alpha$ and $\widehat{V}_\alpha \leftrightarrow V_\alpha$, $\alpha \subset f$, defines an automorphism of $\mathcal{E}$.*

Since we are not using $\alpha$-witnesses, if we ignore the needed modifications, the above theorem follows directly from the Automorphism Theorem. As we introduce the modifications we will try to argue why these changes do not affect the proof of the Automorphism Theorem. Except for the modifications the proof of the Automorphism Theorem and the above theorem are the same, line by line.

Let $B = \widehat{U}_\rho$, where $\rho = f \upharpoonright 1$. By the above theorem, $A$ and $B$ are $\Delta_3$-automorphic.

## 3. Making $B$ dense simple

We must show $B$ is dense simple. To do that it is enough to meet the following requirements.

$\mathcal{P}_e$ If $\varphi_e$ is total and increasing then $(\exists n_e)(\forall \widehat{x} \geq n_e)[p_{\overline{B}}(\widehat{x}) > \varphi_e(\widehat{x})]$.

In addition to their other duties, the nodes of length $5(e+1)$ in $T$ will be used to meet $\mathcal{P}_e$. First let us focus on one $\alpha$ such that $\alpha \subset f$, $|\alpha| = 5(e+1)$ and $\varphi_e$ is total and increasing and see what strategy $\alpha$ can use to meet $\mathcal{P}_e$. Recall from Lemma 5.8 that $\widehat{Y}_\alpha =^* \omega$ ($\widehat{Y}_\alpha$ is the set of hatted balls which reside at or below $\alpha$ at the end of the construction). It is easy to see that if $\widehat{x} \in \widehat{Y}_\alpha$ then there is stage $s$ such that $\widehat{x} \in \widehat{S}_{\alpha,s}$ ($\widehat{S}_{\alpha,s}$ is the set of hatted balls which reside at the node $\alpha$ at stage $s$ and all balls start at the top and work their way down level by level). Let $n_\alpha$ be such that for all $\widehat{x} \geq n_\alpha$, $\widehat{x} \in \widehat{Y}_\alpha$ and $\widehat{x}$ enters $\widehat{S}_\alpha$ for the first time at some stage $s > n_\alpha$.

**Notation 4.** Let $X$ and $Y$ be two (possibly infinite) sets. We say $X$ is *greater than* $Y$ iff there is an $x$ such that $X \upharpoonright x = Y \upharpoonright x$ and either $x \in (X - Y)$ or $X \upharpoonright x = X \subset Y$.

Since $\varphi_e$ is total, we can effectively find a computable set $C_\alpha \subseteq [n_\alpha, \infty)$ such that if $Z = [0, n_\alpha) \cup C_\alpha$ then for all $\widehat{x} \geq n_\alpha$, $p_Z(\widehat{x}) > \varphi_e(\widehat{x})$ and $C_\alpha$ is the greatest such set. (The $i^{\text{th}}$ element of $C_\alpha$ is $\max\{\varphi_e(i + n_\alpha), i + n_\alpha\}$.) We will $\alpha$-approve all elements of $C_\alpha$ and $\alpha$-disapprove all elements greater than $n_\alpha$ of $\overline{B}_\alpha$. If we can prove the following lemma then $\overline{B} \subseteq C_\alpha$ and $\mathcal{P}_e$ is met. (The $C_\alpha$ are an approximation to the set $\overline{B}$ and thus play different roles.)

**Lemma 5.** *If $\alpha \subset f$, $\widehat{x} \geq n_\alpha$, and $\widehat{x}$ is $\alpha$-disapproved, then $\widehat{x} \in B$.*

Other than showing the above lemma is true, there are three problems with this action: We cannot effectively determine $n_\alpha$. We must coordinate the action of all such $\alpha$. Whether $\varphi_e$ is total and increasing is a $\Pi_2$-question.

To get around these problems we will have to $\alpha$-approve and $\alpha$-disapprove balls stagewise. Once $\alpha$-approved (disapproved) always $\alpha$-approved (disapproved). We will require that

**Condition 6.** *If $\widehat{x} \geq k_\alpha$ enters $\widehat{S}_\beta$, where $\alpha = \beta^-$, at some stage $s$ then either $\widehat{x}$ is $\alpha$-approved or $\alpha$-disapproved. Furthermore if $\widehat{x}$ is $\alpha$-disapproved then $\widehat{x}$ must be in $B_s$.*

We will simply make this a requirement before the balls are allowed to move. This involves the addition of the above condition whenever balls are moved (Step 1 and Step 2 of the construction). In section 4 (of this paper), we will see why this does not affect the construction.

To deal with the first problem one might suggest encoding this information into the tree. But this is not legal. The best we could do is give $\alpha$ a “guess” for the value of $n_{\alpha^-}$ and this is not good enough. Luckily, we can approximate $n_\alpha$.

Let $t_s$ be the greatest stage less than $s$ such that $f_{t_s} <_L \alpha$ (or 0 if such a stage does not exists). If $\alpha \subset f$ then $\lim_s t_s = t$ exists and all balls greater than $t$ are $\alpha$-eligible (see Step 1.3). Let

$$Y_{<\alpha,s} = \cup\{Y_{\delta,s} : \delta <_L \alpha\}.$$

By Lemma 5.4, if $\alpha \subset f$ then $\cup\{Y_{<\alpha,s} : s \in \omega\}$ is finite. Let $t'$ be such that $\cup\{Y_{<\alpha,s} : s \in \omega\} = Y_{<\alpha,t'}$. By examining Step 2 and the proofs of Lemma 5.4 and 5.8, one realizes that if $\alpha \subset f$ then

$$n_\alpha = \max\big\{\{t, |\alpha^-|\} \cup \{k_\beta : \beta \subset \alpha\} \cup Y_{<\alpha,t'} \cup \{\cup\{S_{\gamma,t'} : \alpha \subseteq \gamma\}\}\big\}.$$

Let

$$k_s = \begin{cases} 0 & \text{if } s = 0 \text{ or } Y_{<\alpha,s} = Y_{<\alpha,s-1}, \\ \max\big\{Y_{<\alpha,s} \cup \{\cup\{S_{\gamma,s} : \alpha \subseteq \gamma\}\}\big\} & \text{otherwise.} \end{cases}$$

Let

$$n_{\alpha,s} = \max\big\{\{t_s, |\alpha^-|, k_s\} \cup \{k_\beta : \beta \subset \alpha\}\big\}.$$

If $\alpha \subset f$ then $\lim n_{\alpha,s} = n_\alpha$.

We will solve the second problem by forcing $C_{\alpha,s}$ to be a subset of

$$X_{\alpha,s} = \{\widehat{x} : \widehat{x} \text{ is not } \alpha\text{-disapproved (by stage } s), \widehat{x} \text{ is } \alpha^-\text{-approved (by stage } s) \text{ and } \widehat{x} \geq n_{\alpha,s}\}.$$

Following Remark 2.13 we can expand the tree to include other types of $\Delta^0_3$ information. So we can assume that $\alpha$ is equipped with a number $i_\alpha$

such that if $\alpha \subset f$ then $i_\alpha = 0$ iff $\varphi_e$ is total and increasing. Let $l_{\alpha,s}$ be the greatest number such that

$$(\forall n < l_{\alpha,s})[\varphi_s(n)\downarrow < \varphi_s(n+1)\downarrow].$$

If $\varphi_e$ is total and increasing then $\lim_s l_{\alpha,s} = \infty$.

There is one last problem we need to be concerned about. We must ensure that our pending implementation of Condition 6 does not cause any $\widehat{\alpha}$-states to became non well-resided. To do this we must be slightly careful about the balls which we $\alpha$-disapproved.

We will $\alpha$-approve or $\alpha$-disapprove balls at stages $s+1$ where $\alpha \subseteq f_{s+1}$ but not at every such stage. Let $s_{\alpha,s+1}$ be the last stage before $s+1$ where we $\alpha$-approved or $\alpha$-disapproved balls (if such a stage does not exist, let $s_{s+1} = 0$). We will $\alpha$-approve or $\alpha$-disapprove balls at stage $s+1$ only if there is a finite set of balls, $Z$, such that for each $\widehat{\nu} \in \widehat{\mathcal{M}}_\alpha$ some ball $\widehat{x} \in Z$ causes some entry of $\langle \widehat{\alpha}, \widehat{\nu} \rangle$ in the list $\widehat{\mathcal{L}}$ to be $\widehat{\alpha}$-marked since the stage $s_{\alpha,s+1}$ and it is legal (in terms of making $B$ dense simple) to $\alpha$-approve all the balls in $Z$. (The $\widehat{\alpha}$-marking is used to ensure that the well-visited $\widehat{\alpha}$-states are among the $\widehat{\alpha}$-entry states. See the paragraph under Remark 2.13 for more details.)

We should note that the ball $\widehat{x} \in Z$ which caused some entry of $\langle \widehat{\alpha}, \widehat{\nu} \rangle$ in the list $\widehat{\mathcal{L}}$ to be $\widehat{\alpha}$-marked may have changed its $\widehat{\alpha}$-state (from the state $\widehat{\nu}$) before it is $\alpha$-approved. However, our pending implementation of Condition 6 will not affect these balls at all and hence cannot cause $\widehat{\nu}$ to be non well-resided. Let $\widehat{\nu}$ be a non well-resided $\widehat{\alpha}$-state. Our implementation of Condition 6 will not cause $\widehat{\nu}$ to be non well-resided. However, we are NOT saying that if we ignored Condition 6 $\widehat{\nu}$ would be non well-resided. But either for all balls $x$ in state $\nu$ at some later stage there is a RED move to some other $\alpha$-state or for all $\alpha$-approved balls $\widehat{x}$ in state $\widehat{\nu}$ at some later stage there is a RED move to some other $\widehat{\alpha}$-state.

The end result of all this is that we can add the following substep to Step 11 of Section 3 of Harrington and Soare [1996] to $\alpha$-approve and $\alpha$-disapprove balls.

**Substep 11F** For every $\alpha \subseteq f_{s+1}$ do the following in increasing order of $\alpha$:
**Case 1**. If $\alpha = \lambda$, $|\alpha|$ is not divisible by 5, or $i_\alpha \neq 0$, then $\alpha$-approve all balls in $S_{\alpha,s+1}$.
**Case 2**. Otherwise. If there exists a set $Z$ such that $Z \subseteq X_{\alpha,s+1}$, $|Z| \leq l_{\alpha,s+1}$, all $\alpha$-approved balls are in $Z$, the interval $[0, n_{\alpha,s+1})$ is a subset of $Z$, for all $\widehat{\nu} \in \widehat{\mathcal{M}}_\alpha$, there is a ball $\widehat{x} \in Z$ such that $\widehat{x}$ caused some entry of $\langle \widehat{\alpha}, \widehat{\nu} \rangle$ on the list $\widehat{\mathcal{L}}$ to be $\widehat{\alpha}$-marked since the stage $s_{\alpha,s+1}$ and for all $n$ if $n_{\alpha,s+1} \leq n \leq |Z|$ then $p_Z(n) > \varphi_{e,s+1}(n)$, then let $C_{\alpha,s+1}$ be the greatest such set

(if no such set exists discontiune this Substep without $\alpha$-approving or $\alpha$-disapproving any balls). Let $\widehat{x}$ be such that either $\widehat{x} \in \widehat{S}_{\alpha,s+1} \cap X_{\alpha,s+1}$ or $\widehat{x} < n_{\alpha,s+1}$ and $\widehat{x}$ is neither $\alpha$-approved or $\alpha$-disapproved (by this stage). $\alpha$-approve $\widehat{x}$ if either $\widehat{x} \in C_{\alpha,s+1}$ or $\widehat{x} < n_{\alpha,s+1}$ and $\alpha$-disapprove $\widehat{x}$ otherwise.

Assuming all balls are either $\alpha$-approved or $\alpha$-disapproved, it is easy to see that this addition (by itself) does not interfere with the construction of the desired automorphism. Once $\alpha$-approved (disapproved) always $\alpha$-approved (disapproved). Assume $\alpha \subset f$. Then Step 11F will apply infinitely many times for $\alpha$. If $i_\alpha = 0$ then $\lim_s l_{\alpha,s+1} = \infty$ and, by Lemma 5.7 (iv), for each $\widehat{\nu} \in \widehat{\mathcal{M}}_\alpha$, there are infinitely many entries of the form $\langle \widehat{\alpha}, \widehat{\nu} \rangle$ on the list $\widehat{\mathcal{L}}$ and they are all $\widehat{\alpha}$-marked by infinitely many balls entering the $\widehat{\alpha}$-section. Hence we can show by induction on $\alpha \subset f$, using Condition 6, that all balls are either $\alpha$-approved or $\alpha$-disapproved, all $\alpha$-approved balls are $\alpha^-$-approved balls and there are infinitely many $\alpha$-approved balls. By Lemma 5, $\overline{B} \subseteq^* \{\widehat{x} : \widehat{x} \text{ is } \alpha\text{-approved}\}$. Hence to show $B$ is dense simple it is enough to show that the Lemma 5 holds and that Condition 6 does not affect the construction of the automorphism.

## 4. Setting up and implementing the forcing into $B$

The first part of this section is a very modified version of parts of Section 6 in Harrington and Soare [1996]. (There are no changes in the first definition. The second has a change in the first clause.)

**Definition 7.** (i) Let $\mathcal{W}_\alpha$ be that subset of $\mathcal{M}_\alpha$ which is generated by the following three clauses:

(1) $[\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle \;\&\; 0 \in \sigma_1] \implies \nu_1 \in \mathcal{W}_\alpha$.
(2) $(\exists \nu_2)[\nu_1 <_R \nu_2 \;\&\; \nu_2 \in \mathcal{W}_\alpha] \implies \nu_1 \in \mathcal{W}_\alpha$.
(3) $[\nu_1 \in \mathcal{B}_\alpha \;\&\; (\forall \nu_2 \in \mathcal{M}_\alpha)[\nu_1 <_B \nu_2 \implies \nu_2 \in \mathcal{W}_\alpha]] \implies \nu_1 \in \mathcal{W}_\alpha$.

(ii) Define $\mathcal{W}^\#_\alpha = \{\nu_1 : \nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle \in \mathcal{W}_\alpha \;\&\; 0 \notin \sigma_1\}$.
(iii) Define $\mathcal{V}_\alpha = \mathcal{M}_\alpha - \mathcal{W}_\alpha$.

$\mathcal{W}_\alpha$ consists of the $\alpha$-states $\nu_1 \in \mathcal{W}_\alpha$ for which RED has a *winning strategy* $F_\alpha$ to force any $x$ in the $\alpha$-state $\nu_1$ into $A$.

**Definition 8.** (i) A node $\alpha \in T$ is $\mathcal{C}$-*consistent* if $\alpha = \lambda$ or $\mathcal{V}_\alpha = \emptyset$.
(ii) A node $\alpha \in T$ is *consistent* if $\alpha$ is $\mathcal{M}$-consistent, $\mathcal{R}$-consistent and $\mathcal{C}$-consistent.

Step 6 is used to show nodes along the true path are $\mathcal{C}$-consistent. We use that step here. Using Step 6, the proof of Lemma 6.4 shows that if $\alpha$ is not $\mathcal{C}$-consistent (in the above sense) then for any $\nu_1 \in \mathcal{V}_\alpha$,

$$D_{\nu_1} = \{x : (\exists s > \upsilon_\alpha)[x \in S_{\alpha,s} \;\&\; \nu(\alpha, x, s) = \nu_1]\}$$

is an infinite computably enumerable subset of $\overline{A}$. Since $A$ is simple, we have proved the following:

**Lemma 9.** *If $A$ is simple and $\alpha \subset f$ then $\alpha$ is $\mathcal{C}$-consistent.*

Let $\widehat{\mathcal{W}}_\alpha = \{\widehat{\nu} : \nu \in \mathcal{W}_\alpha\}$. By duality, $\widehat{\mathcal{W}}_\alpha$ consists of the $\alpha$-states $\widehat{\nu}_1 \in \widehat{\mathcal{W}}_\alpha$ for which BLUE has a *winning strategy* $\widehat{F}_\alpha$ to force any $\widehat{x}$ in the $\alpha$-state $\widehat{\nu}_1$ into $B$. Assume $\nu(\alpha, \widehat{x}, s) = \widehat{\nu}_1$. If (1) (of Definition 7) holds (of $\nu_1$) then $\widehat{x} \in B_s$. If (2) (of Definition 7) holds then BLUE can change the $\alpha$-state of $\widehat{x}$ from $\widehat{\nu}_1$ to $\widehat{F}_\alpha(\widehat{\nu}_1) = \widehat{\nu}_2 >_B \widehat{\nu}_1$. If (3) (of Definition 7) holds then BLUE can wait for RED to change $\widehat{x}$ from $\alpha$-state $\widehat{\nu}_1$ to some $\widehat{\nu}_2 >_R \widehat{\nu}_1$ and then BLUE can apply $\widehat{F}_\alpha$ to $\widehat{\nu}_2$. Hence this strategy can be identified with a function,

$$\widehat{F}_\alpha : (\widehat{\mathcal{W}}^\#_\alpha - \widehat{\mathcal{R}}_\alpha) \to \widehat{\mathcal{W}}_\alpha \;\&\; (\forall \widehat{\nu}_1 \in \widehat{\mathcal{W}}^\#_\alpha - \widehat{\mathcal{R}}_\alpha)[\widehat{\nu}_1 <_B \widehat{F}_\alpha(\widehat{\nu}_1)].$$

(Careful with the hatting operation here. Recall that $\nu_1 <_R \nu_2$ iff $\widehat{\nu}_1 <_B \widehat{\nu}_2$. It might be helpful to examine Equation 53 in Harrington and Soare [1996] at this point.)

We will assume that BLUE's target function $\widehat{h}_\alpha$ agrees with the function $\widehat{F}_\alpha$,

$$(\forall \widehat{\nu} \in \widehat{\mathcal{W}}^\#_\alpha - \widehat{\mathcal{R}}_\alpha)[\widehat{\nu}_1 <_B \widehat{h}_\alpha(\widehat{\nu}_1) = \widehat{F}_\alpha(\widehat{\nu}_1)].$$

To do this we need to extend the domain of $\widehat{h}_\alpha$ from $\widehat{\mathcal{B}}_\alpha$ to $\widehat{\mathcal{W}}^\#_\alpha - \widehat{\mathcal{R}}_\alpha$. This new target function agrees with the old target function on $\widehat{\mathcal{W}}^\#_\alpha \cap \widehat{\mathcal{B}}_\alpha$ (see Equation 54 in Harrington and Soare [1996]). Hence for all $\widehat{x}$ if $\widehat{x} \in \widehat{S}_{\alpha,s}$ we can, if needed, force $\widehat{x}$ to enter $B$ without causing harm to the construction of the automorphism.

Now we know that it is possible to force these balls which are $\alpha$-disapproved into $B$ but we still must implement this action. Because of Condition 6, only (2) (of Definition 7) requires any action on our part (for more see the next paragraph). $\widehat{h}_\alpha$ is only applied in Step $\widehat{5}$. We need to expand the number of balls to which Step $\widehat{5}$ applies. We will do this by changing Condition $(\widehat{5.1})$ (this is the dual of Condition (5.1) in Step 5 of the construction (see Section 3) in Harrington and Soare [1996]) into the disjunction of two clauses (the first of which is the old condition):

$$\text{Either } \nu(\alpha, \widehat{x}, s) = \widehat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha \text{ or}$$

$$\widehat{x} \text{ is } \alpha\text{-disapproved (at stage } s) \text{ and } \widehat{h}_\alpha(\widehat{\nu}_0) >_B \widehat{\nu}_0$$

In addition, if $\widehat{x}$ is $\alpha$-disapproved at stage $s$ then we give Step 5 higher priority than Step 4; i.e. if both can apply at stage $s$ use Step 5.

Assume $\alpha \subset f$ and $\widehat{x} \in \widehat{S}_{\alpha,s}$ such that $\widehat{x}$ is $\alpha$-disapproved and $\widehat{x} \notin B_s$. $\nu(\alpha, \widehat{x}, s) = \widehat{\nu}_1 \in \widehat{\mathcal{W}}^\#_\alpha$. By Lemma 9, either Conditions 2 or 3 apply to $\nu_1$. If Condition 2 applies, then Step 5 will apply at some later stage and hence

$\nu(\alpha, \widehat{x}, t) >_R \widehat{\nu}_1$. If Condition 3 applies, Condition 6 forces us to wait for RED to change $\widehat{x}$'s $\alpha$-state before releasing the ball from $\widehat{S}_\alpha$. Since $\alpha \subset f$, there will be a stage $t$ such that $\nu(\alpha, \widehat{x}, t) >_B \widehat{\nu}_1$. Thus, by Condition 6, at some later stage $\widehat{x}$ must be in $B$ and hence Lemma 5 holds.

Condition 6 does not stop any balls from going further down in the machine. It may, however, delay the balls' departure. The forcing of $\alpha$-disapproved balls into $B$ may cause certain well-visited states to empty out into $B$ and hence these states are not seen further down in the machine. This requires us to make one last change from Harrington and Soare [1996]. We must redefine $\widehat{\mathcal{F}}^+_\beta$, where $\beta = \alpha^-$ as follows (the last clause is new):

$$\widehat{\mathcal{F}}^+_\beta = \{\nu : (\exists^\infty \widehat{x})(\exists s)[\widehat{x} \in \widehat{Y}^1_{\beta,s} \; \& \; \nu^+(\alpha, \widehat{x}, s) = \nu \\ \& \text{ either } \widehat{x} \text{ is } \beta\text{-approved or } \widehat{x} \in B_s.$$

$\widehat{\mathcal{F}}^+_\beta = \widehat{\mathcal{M}}_\alpha$, if $\alpha \subset f$. The above change limits $\widehat{\mathcal{F}}^+_\beta$ to those states which can legally appear (due to Condition 6) below $\beta$.

**Claim 10.** *The verification in Section 5 of Harrington and Soare [1996] goes though line by line except for reflecting the above modifications.*

We admit that it is difficult to present much (any?) evidence of this claim without going through the proof line by line. We do claim that any difficulties encountered when going through the proof line by line are addressed above. Thus we have proved Theorem 2.

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