Computable Mathias genericity

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On Mathias generic sets.

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Mathias conditions

Definition.

1 A (computable Mathias) pre-condition is a pair \((D, E)\) such that \(D\) is a finite set, \(E\) is a computable set, and \(\max D < \min E\).

2 \((D, E)\) is a (computable Mathias) condition if \(E\) is infinite.

3 A pre-condition \((D', E')\) extends \((D, E)\), written \((D', E') \leq (D, E)\), if \(D \subseteq D' \subseteq D \cup E\) and \(E' \subseteq E\).

4 A set \(S\) satisfies \((D, E)\) if \(D \subseteq S \subseteq D \cup E\).

Named after Mathias’s use of it in set theory, but used earlier by Soare and others in computability theory.

Useful in studying Ramsey’s theorem and related properties. In computability, used in various arguments about \(\text{RT}_2^2\).
Mathias generics

A set $S$ meets a set $\mathcal{C}$ of conditions if it satisfies some condition in $\mathcal{C}$. $S$ avoids $\mathcal{C}$ of conditions if it meets the conditions with no extension in $\mathcal{C}$.

Definition.

1. A $\Sigma^0_n$ set of conditions is a $\Sigma^0_n$-definable set of pre-conditions, each of which is a condition.

2. A set $G$ is (Mathias) $n$-generic if it meets or avoids every $\Sigma^0_n$ set of conditions.

3. A set $G$ is weakly (Mathias) $n$-generic if it meets every dense $\Sigma^0_n$ set of conditions.
Computable setting

Definition. An index for a pre-condition \((D, E)\) is a pair \((d, e)\) \(\in \omega^2\) such that \(d\) is the canonical index of \(D\) and \(E = \{x \in \omega : \varphi_e(x) \downarrow = 1\}\).

The set of all (indices for) pre-conditions is \(\Pi_1^0\), but this has a computable subset containing an index for every pre-condition.

Even working over this set, the set of all (indices for) conditions is \(\Pi_2^0\).

Definition. A set \(G\) is strongly (Mathias) \(n\)-generic if it meets or avoids every \(\Sigma_n^0\)-definable set of pre-conditions.

Proposition (Cholak, Dzhafarov, Hirst). A set is strongly \(n\)-generic if and only if it is \(\max\{n, 3\}\)-generic.

Without further comment, \(n\) below will always be a number \(\geq 3\).
Comparison with Cohen generics

Computability of Cohen generics studied by Jockusch, Kurtz, and others.

Similarities.

1 Implications: \( n\)-generic \( \implies \) weakly \( n\)-generic \( \implies \) \((n−1)\)-generic.
2 There exists an \( n\)-generic \( G \leq_T \emptyset^{(n)} \).
3 Every weakly \( n\)-generic set is hyperimmune relative to \( \emptyset^{(n−1)} \).

Dissimilarities.

1 Every weakly Mathias \( n\)-generic set \( G \) is cohesive. Hence, if \( G = G_0 \oplus G_1 \) then either \( G_0 =^* \emptyset \) or \( G_1 =^* \emptyset \).
2 If \( G \) is Mathias 3-generic then \( G' \geq \emptyset'' \).

Thus, no Mathias \( n\)-generic can be Cohen 1-generic, and no Cohen 2-generic can even compute a Mathias 3-generic.
Jump properties

It is a well-known result of Jockusch that if $G$ is Cohen $n$-generic then $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$. In particular, every Cohen generic set has $\text{GL}_1$ degree.

Theorem (Cholak, Dzhafarov, Hirst). If $G$ is Mathias $n$-generic, then:

1. $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$;
2. $G$ has $\text{GH}_1$ degree, i.e., $G' \equiv_T (G \oplus \emptyset')'$.

Corollary. If $G$ is Mathias $n$-generic then it has $\overline{\text{GL}}_1$ degree. So $G$ cannot have Cohen 1-generic degree, but $G$ computes a Cohen 1-generic.
Complexity of the forcing relation

Let \( L_1^* \) be the language of first-order arithmetic, with a special set variable, \( X \), and the epsilon relation, \( \in \). Let \( \varphi(X) \) be a formula of \( L_1^* \).

We can define the forcing relation \((D, E) \models \varphi(G)\) inductively such that forcing implies truth:

**Proposition (Cholak, Dzhafarov, Hirst).** If \( \varphi \) is \( \Sigma_n \), and if \( G \) is \( n \)-generic and satisfies some \((D, E)\) that forces \( \varphi(G) \), then \( \varphi(G) \) holds.

**Lemma (Cholak, Dzhafarov, Hirst).**

1. If \( \varphi \) is \( \Sigma_0 \), then the relation \((D, E) \models \varphi(G)\) is computable.
2. If \( \varphi \) is \( \Pi_1^0 \), \( \Sigma_1^0 \), or \( \Sigma_2^0 \), then so is the relation \((D, E) \models \varphi(G)\).
3. If \( \varphi \) is \( \Pi_n^0 \) for some \( n \geq 2 \), then the relation \((D, E) \models \varphi(G)\) is \( \Pi_{n+1}^0 \).
4. If \( \varphi \) is \( \Sigma_n^0 \) for some \( n \geq 3 \), then the relation \((D, E) \models \varphi(G)\) is \( \Sigma_{n+1}^0 \).
Computing from Mathias generics

So far: Cohen $n$-generics do not compute Mathias $n$-generics, but Mathias $n$-generics compute Cohen 1-generics.

This raises the following question:

**Question.** Does every Mathias $n$-generic computes a Cohen $n$-generic?

**Theorem (Cholak, Dzhafarov, Hirst).** If $G$ is Mathias $n$-generic and $B \leq_T \emptyset^{(n-1)}$ is bi-immune, then $G \oplus B$ computes a Cohen $n$-generic.

**Theorem (Slaman).** If $G$ is Mathias $n$-generic then $G$ computes a Cohen $n$-generic.
Questions

What is the reverse mathematical content of the principle asserting the existence, for every $X$, of an $n$-generic set for $X$-computable Mathias forcing? It is $\Pi^1_1$ conservative over $\text{RCA}_0$ but not over $\text{BΣ}_2^0$.

Shore has asked if there are any interesting degrees realizing properties of the form $d^j = (d^k \lor 0^l)^m$. The Cohen and Mathias generics realize two such properties. Do generics for other forcing notions realize others?