

SOME RECENT RESEARCH DIRECTIONS IN THE COMPUTABLY ENUMERABLE SETS

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ABSTRACT. As suggested by the title, this paper is a survey of recent results and questions on the collection of computably enumerable sets under inclusion. This is not a broad survey but one focused on the author's and a few others' current research.

There are many equivalent ways to define a computably enumerable or c.e. set. The one that we prefer is the domain of a Turing machine or the set of balls accepted by a Turing machine. Perhaps this definition is the main reason that this paper is included in this volume and the corresponding talk in the “Incomputable” conference. The c.e. sets are also the sets which are Σ_1^0 definable in arithmetic.

There is a computable or effective listing, $\{M_e | e \in \omega\}$, of all Turing machines. This gives us a listing of all c.e. sets, x in W_e at stage s iff M_e with input x accepts by stage s . This enumeration of all c.e. sets is very dynamic. We can think of balls x as flowing from one c.e. set into another. Since they are sets, we can partially order them by inclusion, \subseteq and consider them as model, $\mathcal{E} = \langle \{W_e | e \in \omega\}, \subseteq \rangle$. All sets (not just c.e. sets) are partially ordered by Turing reducibility, where $A \leq_T B$ iff there is a Turing machine that can compute A given an oracle for B .

Broadly, our goal is to study the structure \mathcal{E} and learn what we can about the interactions between definability (in the language of inclusion \subseteq), the dynamic properties of c.e. sets and their Turing degrees. A very rich relationship between these three notions has been discovered over the years. We cannot hope to completely cover this history in this short paper. But, we hope that we will cover enough of it to show the reader that the interplay between these three notions on c.e. sets is, and will continue to be, an very interesting subject of research.

We are assuming that the reader has a background in computability theory as found in the first few chapters of Soare [26]. All unknown notation also follows [26].

1. FRIEDBERG SPLITS

The first result in this vein was Friedberg [15], every noncomputable c.e. set has a Friedberg split. Let us first understand the result then explore why we feel this result relates to the interplay of definability, Turing degrees and dynamic properties of c.e. sets.

Definition 1.1. $A_0 \sqcup A_1 = A$ is a *Friedberg split* of A iff, for all W (all sets in this paper are always c.e.), if $W - A$ is not a c.e. set neither are $W - A_i$.

The following definition depends on the chosen enumeration of all c.e. sets. We use the enumeration given to us in the second paragraph of this paper, $x \in W_{e,s}$ iff M_e with input x accepts by stage s , but with the convention that if $x \in W_{e,s}$ then $e, x < s$ and, for all stages s , there is at most one pair e, x where x enters W_e at stage s . Some details on how we can effectively achieve this type of enumeration can be found in Soare [26, Exercise I.3.11]. Moreover, when given a c.e. set, we are given the index of this c.e. set in terms of our enumeration of all c.e. sets. At times we will have to appeal to Kleene's Recursion Theorem to get this index.

Definition 1.2. For c.e. sets $A = W_e$ and $B = W_i$,

$$A \setminus B = \{x \mid \exists s [x \in (W_{e,s} - W_{i,s})]\}$$

and $A \searrow B = A \setminus B \cap B$.

By the above definition, $A \setminus B$ is a c.e. set. $A \setminus B$ is the set of balls that enter A before they enter B . If $x \in A \setminus B$ then x may or may not enter B and if x does enter B , it only does so after x enters A (in terms of our enumeration). Since the intersection of two c.e. sets is c.e., $A \searrow B$ is a c.e. set. $A \searrow B$ is the c.e. set of balls x that first enter A and then enter B (under the above enumeration).

Note that $W \setminus A = (W - A) \sqcup (W \searrow A)$ (\sqcup is the disjoint union). Since $W \setminus A$ is a c.e. set, if $W - A$ is not a c.e. set then $W \searrow A$ must be infinite. (This happens for all enumerations.) Hence infinitely many balls from W must flow into A .

Lemma 1.3 (Friedberg). *Assume $A = A_0 \sqcup A_1$, and, for all e , if $W_e \searrow A$ is infinite then both $W_e \searrow A_0$ and $W_e \searrow A_1$ are infinite. Then $A_0 \sqcup A_1$ is a Friedberg split of A . Moreover if A is not computable neither are A_0 and A_1 .*

Proof. Assume that $W - A$ is not a c.e. set but $X = W - A_0$ is a c.e. set. $X - A = W - A$ is not a c.e. set. So $X \searrow A$ is infinite and therefore $X \searrow A_0$ is infinite. Contradiction.

If A_0 is computable then $X = \overline{A_0}$ is a c.e. set and if A is not computable then $X - A$ cannot be a c.e. set. So use the same reasoning as above to show $X \searrow A_0$ is infinite for a contradiction. \square

Friedberg more or less invented the priority method to split every c.e. set into two disjoint c.e. sets while meeting the hypothesis of the above lemma. The main idea of Friedberg's construction is when a ball x enters A at stage s to add it to one of A_0 or A_1 but which set x enters is determined by priority. Let

$\mathcal{P}_{e,i,k}$ if $W_e \searrow A$ is infinite then $|W_e \searrow A_i| \geq k$.

We say x meets $\mathcal{P}_{e,i,k}$ at stage s if $|W_e \searrow A_i| < k$ by stage $s - 1$ and if we add x to A_i then $|W_e \searrow A_i| \geq k$ by stage s . Find the highest $\langle e, i, k \rangle$ that x can meet and add x to A_i at stage s . It is not hard to show that all the $\mathcal{P}_{e,i,k}$ are met.

It is clear that the existence of a Friedberg split is very dynamic. Let's see why it is also a definable property. But, first, we need to understand what we can say about \mathcal{E} with inclusion. We are not going to go through the details but we can define union, intersection, disjoint union, the empty set and the whole set. We can say that a set is complemented. A very early result shows that if A and \overline{A} are both c.e. then A is computable. So it is definable if a c.e. set is computable. Inside every computable set we can repeat the construction of the halting set. So a c.e. set X is finite iff every subset of X is computable. Hence $W - A$ is a c.e. set iff there is a c.e. set X disjoint from A such that $W \cup A = X \sqcup A$. So saying that $A_0 \sqcup A_1 = A$ is a Friedberg split and A is not computable is definable.

Friedberg's result answers a question of Myhill, "Is every non-recursive, recursively enumerable set the union of two disjoint non-recursive, recursively enumerable sets?" The question of Myhill was asked in print in the Journal of Symbolic Logic in June 1956, Volume 21, Number 2 on page 215 in the "Problems" section of the JSL. This question was the eighth problem appearing in this section. The question about the existence of maximal sets, also answered by Friedberg, was ninth. This author does not know how many questions were asked or when this section was dropped. Myhill also reviewed Friedberg [15] for the AMS, but the review left no clues why he asked the question in the first place.

The big question in computability theory in the 1950's was "Does there exist an incomplete noncomputable c.e. set"? Kleene and Post [20] showed that there are a pair of incomparable Turing degrees below $\mathbf{0}'$. We feel that after Kleene-Post, Myhill's question is very natural.

So we can claim that the existence of a Friedberg split for every c.e. set A fits into our theme, the interplay of definability, dynamic properties and Turing degree on the c.e. sets.

1.0.1. *Recent Work and Questions on Friedberg Splits.* Given a c.e. set one can uniformly find a Friedberg split. It is known that there are other types of splits. One wonders if any of these non-Friedberg splits can be done uniformly. It is also known that for some c.e. sets the only nontrivial splits ($A = A_0 \sqcup A_1$ and the A_0 and A_1 are not computable) are Friedberg. So one cannot hope to get a uniform procedure which always provides a nontrivial non-Friedberg split of every noncomputable c.e. set. But it would be nice to find a computable function $f(e) = \langle e_0, e_1 \rangle$ such that, for all e , if W_e is noncomputable then $W_{e_0} \sqcup W_{e_1} = W_e$ is a nontrivial split of W_e and, for every c.e. set A , if A has a nontrivial non-Friedberg split and $A = W_e$ (so W_e is any enumeration of A), and then $W_{e_0} \sqcup W_{e_1} = W_e$ is a nontrivial non-Friedberg split. So, if A has a nontrivial non-Friedberg split and W_e is any enumeration of A , f always gives out a nontrivial non-Friedberg split. In work yet to appear, the author has shown that such a computable f cannot exist.

Let \mathcal{P} be a property in \mathcal{E} . We say that A is *hemi- \mathcal{P}* iff there are c.e. sets B and C such that $A \sqcup B = C$ and C has \mathcal{P} . We can also define *Friedberg- \mathcal{P}* iff there are c.e. sets B and C such that $A \sqcup B = C$ is a Friedberg split and C has \mathcal{P} . If \mathcal{P} is definable then *hemi- \mathcal{P}* and *Friedberg- \mathcal{P}* are also definable. One can get lots of mileage from the *hemi- \mathcal{P}* , see Downey and Stob [10] and Downey and Stob [11]. Most of these results are about properties \mathcal{P} where every nontrivial split of a set with \mathcal{P} is Friedberg. We feel that one should be using *Friedberg- \mathcal{P}* rather than *hemi- \mathcal{P}* . To that end we ask the following:

Question 1.4. *Is there a definable \mathcal{P} such that the Friedberg splits are a proper subclass of the nontrivial splits?*

We feel that the Friedberg splits are very special and they should not be able to always cover all the nontrivial splits of every definable property.

2. ALL ORBITS NICE? NO!

As we mentioned earlier, Friedberg also constructed a maximal set answering another question of Myhill. A maximal set, M , is a c.e. set such that for every superset X either $X =^* M$ ($=^*$ is equal modulo finite) or $W =^* \omega$. Being maximal is definable. Friedberg's construction of a maximal set is very dynamic. Martin [23] showed that all

maximal sets must be high. A further result of Martin [23] shows a c.e. degree is high iff it contains a maximal set. A remarkable result of Soare [24] shows that the maximal sets form an orbit, even an orbit under automorphisms computable from $\mathbf{0}''$ or Δ_3^0 -automorphisms.

The result of Soare gives rise to the question are all orbits as nice as the orbit of the maximal sets? We can go more into the formality of the question but that was dealt with already in another survey paper, Cholak, Downey, and Harrington [6]. To tell if two c.e. sets, A and B , are in the same orbit, it is enough to show if there is an automorphism Φ of \mathcal{E} taking the one to the other, $\Phi(A) = B$ (we write this as A is *automorphic* to B). Hence it is Σ_1^1 to tell if two sets are in the same orbit. The following theorem says that is the best that we can do and hence not all orbits are as nice as the orbits of maximal sets. The theorem has a number of interesting corollaries.

Theorem 2.1 (Cholak, Downey, and Harrington [7]). *There is a c.e. set A such that the index set $\{i : W_i \approx A\}$ is Σ_1^1 -complete.*

Corollary 2.2 (Cholak et al. [7]). *Not all orbits are elementarily definable; there is no arithmetic description of all orbits of \mathcal{E} .*

Corollary 2.3 (Cholak et al. [7]). *The Scott rank of \mathcal{E} is $\omega_1^{\text{CK}} + 1$.*

Theorem 2.4 (Cholak et al. [7]). *For all finite $\alpha > 8$ there is a properly Δ_α^0 orbit.*

These results were completely explored in the survey, [6]. So we will focus on some more recent work. In the work leading to the above theorems, Cholak and Harrington also showed that:

Theorem 2.5 ([4]). *Two simple sets are automorphic iff they are Δ_6^0 automorphic. A set A is simple iff for every (c.e.) set B if $A \cap B$ is empty then B is finite.*

Recently Harrington improved this result to show:

Theorem 2.6 (Harrington 2012, private email). *The complexity of the $\mathcal{L}_{\omega_1, \omega}$ formula describing the orbit of any simple set is very low (close to 6).*

That leads us to make the following conjecture:

Conjecture 2.7. *We can build the above orbits in Theorem 2.4 to have complexity close to α in terms of the $\mathcal{L}_{\omega_1, \omega}$ formula describing the orbit.*

3. COMPLETE SETS

Perhaps the biggest questions on the c.e. sets are the following:

Question 3.1 (Completeness). *Which c.e. sets are automorphic to complete sets?*

Motivation for this question dates back to Post. Post was trying to use properties of the complement of a c.e. set to show that the set was not complete. In the structure \mathcal{E} all the sets in the same orbit have the same definable properties.

By Harrington and Soare [16], [17], and [18], we know that not every c.e. set is automorphic to a complete set and, furthermore, there is a dichotomy between the “prompt” sets and the “tardy” (nonprompt) sets with the “prompt” sets being automorphic to complete sets. We will explore this dichotomy in more detail, but more definitions are needed:

Definition 3.2. $X = (W_{e_1} - W_{e_2}) \cup (W_{e_3} - W_{e_4}) \cup \dots (W_{e_{2n-1}} - W_{e_{2n}})$ iff X is $2n$ -c.e. and X is $2n + 1$ -c.e. iff $X = Y \cup W_e$, where Y is $2n$ -c.e.

Definition 3.3. Let X_e^n be the e th n -c.e. set. A is *almost prompt* iff there is a computable nondecreasing function $p(s)$ such that for all e and n if $X_e^n = \bar{A}$ then $(\exists x)(\exists s)[x \in X_{e,s}^n$ and $x \in A_{p(s)}]$.

Theorem 3.4 (Harrington and Soare [18]). *Each almost prompt sets are automorphic to some complete set.*

Definition 3.5. D is *2-tardy* iff for every computable nondecreasing function $p(s)$ there is an e such that $X_e^2 = \bar{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^2$ then $x \notin D_{p(s)}]$

Theorem 3.6 (Harrington and Soare [17]). *There are \mathcal{E} definable properties $Q(D)$ and $P(D, C)$ such that*

- (1) $Q(D)$ implies that D is 2-tardy and hence the orbit of D does not contain a complete set.
- (2) for D , if there is a C such that $P(D, C)$ and D is 2-tardy then $Q(D)$ (and D is high).

The 2-tardy sets are not almost prompt and the fact they are not almost prompt is witnessed by $e = 2$. It would be nice if the above theorem implied that being 2-tardy was definable. But it says with an extra definable condition being 2-tardy is definable.

Harrington and Soare [17] ask if each 3-tardy set is computable by some 2-tardy set. They also ask if all low_2 simple sets are almost prompt (this is the case if A is low). With Gerdes and Lange, Cholak answered these negatively:

Theorem 3.7 (Cholak, Gerdes, and Lange [8]). *There exists a properly 3-tardy B such that there is no 2-tardy A such that $B \leq_T A$. Moreover, B can be built below any prompt degree.*

Theorem 3.8 (Cholak, Gerdes, and Lange [8]). *There is a low_2 , simple, 2-tardy set.*

Moreover, with Gerdes and Lange, Cholak showed that there are definable (first-order) properties $Q_n(A)$ such that if $Q_n(A)$ then A is n -tardy and there is a properly n -tardy set A such that $Q_n(A)$ holds. Thus the collection of all c.e. sets not automorphic to a complete set breaks up into infinitely many orbits.

But, even with the work above, the main question about completeness and a few others remain open. These open questions are of a more degree-theoretic flavor. The main still open questions are:

Question 3.9 (Completeness). *Which c.e. sets are automorphic to complete sets?*

Question 3.10 (Cone Avoidance). *Given an incomplete c.e. degree \mathbf{d} and an incomplete c.e. set A , is there an \hat{A} automorphic to A such that $\mathbf{d} \not\leq_T \hat{A}$?*

It is unclear whether these questions have concrete answers. Thus the following seems reasonable.

Question 3.11. *Are these arithmetical questions?*

Let us consider how we might approach these questions. One possible attempt would be to modify the proof of Theorem 2.1 to add degree-theoretic concerns. Since the coding comes from how A interacts with the sets disjoint from it, we should have reasonable degree-theoretic control over A . The best we have been able to do so far is alter Theorem 2.1 so that the set constructed has hemimaximal degree and everything in its orbit also has hemimaximal degree. However, what is open is whether the orbit of any set constructed via Theorem 2.1 must contain a representative of every hemimaximal degree or only have hemimaximal degrees. If the infinite join of hemimaximal degrees is hemimaximal then the degrees of the sets in these orbits only contain the hemimaximal degrees. But, it is open whether the infinite join of hemimaximal degrees is hemimaximal.

3.1. Tardy Sets. As mentioned above, there are some recent results on n -tardy and very tardy sets (a set is very tardy iff it is not almost prompt). But there are several open questions related to this work. For example, is there a (first-order) property Q_∞ so that if $Q_\infty(A)$ holds, then A is very tardy (or n -tardy, for some n). Could we define Q_∞ such that $Q_n(A) \implies Q_\infty(A)$? How do hemi- Q and Q_3 compare? But the big open questions here are the following:

Question 3.12. *Is the set B constructed in Theorem 3.7 automorphic to a complete set? If not, does $Q_3(B)$ hold?*

It would be very interesting if both of the above questions have a negative answer.

Not a lot about the degree theoretic properties of the n -tardies is known. The main question here is whether Theorem 3.7 can be improved to n other than 2.

Question 3.13. *For which n are there $n + 1$ tardies which are not computed by n -tardies?*

But there are many other approachable questions. For example, how do the following sets of degrees compare:

- the hemimaximal degrees?
- the tardy degrees?
- for each n , $\{\mathbf{d} : \text{there is an } n\text{-tardy } D \text{ such that } \mathbf{d} \leq_T D\}$?
- $\{\mathbf{d} : \text{there is a } 2\text{-tardy } D \text{ such that } Q(D) \text{ and } \mathbf{d} \leq_T D\}$?
- $\{\mathbf{d} : \text{there is an } A \in \mathbf{d} \text{ which is not automorphic to a complete set}\}$?

Does every almost prompt set compute a 3-tardy? Or a very tardy? Harrington and Soare [16] show there is a maximal 2-tardy set. So there are 2-tardy sets which are automorphic to complete sets. Is there a nonhigh, nonhemimaximal, 2-tardy set which is automorphic to a complete set?

3.2. Cone Avoidance, Question 3.10. The above prompt vs. tardy dichotomy gives rise to a reasonable way to address Question 3.10. An old result of Cholak [1] and, independently, Harrington and Soare [18], says that every c.e. set is automorphic to a high set. Hence, a positive answer to both the following questions would answer the cone avoidance question but not the completeness question. These questions seem reasonable as we know how to work with high degrees and automorphisms, see [1],

Question 3.14. *Let A be incomplete. If the orbit of A contains a set of high prompt degree, must the orbit of A contain a set from all high prompt degrees?*

Question 3.15. *If the orbit of A contains a set of high tardy degree, must the orbit of A contain a set from all high tardy degrees?*

Similarly we know how to work with prompt degrees and automorphisms, see Cholak, Downey, and Stob [5] and Harrington and Soare [18]. We should be able to combine the two. No one has yet explored how to work with automorphisms and tardy degrees.

4. \mathcal{D} -MAXIMAL SETS

In the above sections we have mentioned maximal and hemimaximal sets several times. It turns out that maximal and hemimaximal sets are both \mathcal{D} -maximal.

Definition 4.1. $\mathcal{D}(A) = \{B : \exists W (B \subseteq A \cup W \text{ and } W \cap A = \emptyset)\}$ under inclusion. Let $\mathcal{E}_{\mathcal{D}(A)}$ be \mathcal{E} modulo $\mathcal{D}(A)$.

$\mathcal{D}(A)$ is the ideal of c.e. sets of the form $\tilde{A} \sqcup \tilde{D}$ where $\tilde{A} \subseteq A$ and $\tilde{D} \cap A = \emptyset$.

Definition 4.2. A is \mathcal{D} -hhsimple iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Σ_3^0 Boolean algebra. A is \mathcal{D} -maximal iff $\mathcal{E}_{\mathcal{D}(A)}$ is the trivial Boolean algebra iff for all c.e. sets B there is a c.e. set, D , disjoint from A , such that either $B \subset A \cup D$ or $B \cup D \cup A = \omega$.

Maximal sets and hemimaximal sets are \mathcal{D} -maximal. Plus, there are many other examples of \mathcal{D} -maximal sets. In fact, with the exception of the creative sets, all known elementary definable orbits are orbits of \mathcal{D} -maximal sets. In the lead up to Theorem 2.1, Cholak and Harrington were able to show:

Theorem 4.3 ([3]). *If A is \mathcal{D} -hhsimple and A and \hat{A} are in the same orbit then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta_3^0} \mathcal{E}_{\mathcal{D}(\hat{A})}$.*

So it is an arithmetic question to ask if the orbit of a \mathcal{D} -maximal set contains a complete set. But the question remains does the orbit of every \mathcal{D} -maximal set contain a complete set? It was hoped that the structural properties of \mathcal{D} -maximal sets would be sufficient to allow us to answer this question.

Cholak, Gerdes, and Lange [9] have completed a classification of all \mathcal{D} -maximal sets. The idea is to look at how $\mathcal{D}(A)$ is generated. For example, for a hemimaximal set A_0 , $\mathcal{D}(A_0)$ is generated by A_1 , where $A_0 \sqcup A_1$ is maximal. There are ten different ways that $\mathcal{D}(A)$ can be generated. Seven were previously known and all these orbits contain complete and incomplete sets. Work from Herrmann and Kummer [19] shows that these seven types are not enough to provide a characterization of all \mathcal{D} -maximal sets. Cholak, Gerdes, and Lange construct three more types and show that these ten types provide a characterization of all \mathcal{D} -maximal sets. We have constructed three new types of \mathcal{D} -maximal sets; for example, a \mathcal{D} -maximal set where $\mathcal{D}(A)$ is generated by infinitely many not disjoint c.e. sets. We show these three types plus another split into infinitely many different orbits. We can build examples of these sets which are incomplete or complete. But, it is open if

each such orbit contains a complete set. So, the structural properties of \mathcal{D} -maximal sets was not enough to determine if each \mathcal{D} -maximal set is automorphic to a complete set.

It is possible that one could provide a similar characterization of the \mathcal{D} -hhsimple sets. One should fix a Σ_3^0 Boolean algebra, \mathcal{B} , and characterize the \mathcal{D} -hhsimple sets, A , where $\mathcal{E}_{\mathcal{D}(A)} \cong \mathcal{B}$. It would be surprising if, for some \mathcal{B} , the characterization would allow us to determine if every orbit of these sets contains a complete set.

5. LOWNESS

Following his result that the maximal sets form an orbit, Soare[25] showed that the low sets resemble computable sets. A set A is low_n iff $\mathbf{0}^{(n)} \equiv_T A^{(n)}$. We know that noncomputable low sets cannot have a computable set in their orbit, so, the best that Soare was able to do is the following:

Definition 5.1. $\mathcal{L}(A)$ are the c.e. supersets of A under inclusion. \mathcal{F} is the filter of finite sets. $\mathcal{L}^*(A)$ is $\mathcal{L}(A)$ modulo \mathcal{F} .

Theorem 5.2 (Soare [25]). *If A is low then $\mathcal{L}^*(A) \approx \mathcal{L}^*(\emptyset)$.*

In 1990, Soare conjectured that this can be improved to low_2 . Since then there have been a number of related results but this conjecture remains open. To move forward some definitions are needed:

Definition 5.3. A is *semilow* iff $\{i|W_i \cap A \neq \emptyset\}$ is computable from $\mathbf{0}'$. A is *semilow_{1.5}* iff $\{i|W_i \cap A \text{ is finite}\} \leq_1 \mathbf{0}''$. A is *semilow₂* iff $\{i|W_i \cap A \text{ is finite}\}$ is computable from $\mathbf{0}''$.

Semilow implies $\text{semilow}_{1.5}$ implies semilow_2 , if A is low then \bar{A} is semilow, and low_2 implies semilow_2 (details can be found in Maass [22] and Cholak [1]). Soare [25] actually showed that if \bar{A} is semilow then $\mathcal{L}^*(A) \approx \mathcal{L}^*(\emptyset)$. Maass [22] improved this to when \bar{A} is $\text{semilow}_{1.5}$.

In Maass's proof $\text{semilow}_{1.5}$ ness is used in two ways: A c.e. set, W , is *well-resided outside A* iff $W \cap \bar{A}$ is infinite. $\text{Semilow}_{1.5}$ makes determining which sets are well-resided outside A a Π_2^0 question. The second use of $\text{semilow}_{1.5}$ was to capture finitely many elements of $W \cap \bar{A}$. For that Maass showed that $\text{semilow}_{1.5}$ implies the *outer splitting property*:

Definition 5.4. A has the *outer splitting property* iff there are computable functions f, h such that, for all e , $W_e = W_{f(e)} \sqcup W_{h(e)}$, $W_{f(e)} \cap \bar{A}$ is finite, and if $W_e \cap \bar{A}$ is infinite then $W_{f(e)} \cap \bar{A}$ is nonempty.

Cholak used these ideas to show that:

Theorem 5.5 (Cholak [1]). *If A has the outer splitting property and \overline{A} is semilow₂ then $\mathcal{L}^*(A) \approx \mathcal{L}^*(\emptyset)$.*

It is known that there is a low₂ set which does not have the outer splitting property, see Downey, Jockusch, and Schupp [12, Theorem 4.6]. So to prove that if A is low₂ then $\mathcal{L}^*(A) \approx \mathcal{L}^*(\emptyset)$ will need a different technique. However, Lachlan [21] showed that every low₂ set has a maximal superset using the technique of *true stages*. Perhaps the true stages technique can be used to show Soare's conjecture.

Recently there has been a result of Epstein.

Theorem 5.6 (Epstein [13] and [14]). *There is a properly low₂ degree \mathbf{d} such that if $A \leq_T \mathbf{d}$ then A is automorphic to a low set.*

Epstein's result shows that there is no collection of c.e. sets which is closed under automorphisms and contains at least one set of every non-low degree. Related results were discussed in Cholak and Harrington [2].

This theorem does have a nice yet unmentioned corollary: The collection of all sets A such that \overline{A} is semilow (these sets are called *speedable*) is not definable. By Downey, Jockusch, and Schupp [12, Theorem 4.5], every nonlow c.e. degree contains a set A such that \overline{A} is not semilow_{1.5} and hence not semilow. So there is such a set A in \mathbf{d} . A is automorphic to a low set \hat{A} . Since \hat{A} is low, $\overline{\hat{A}}$ is semilow.

Epstein's result leads us wonder if the above results can be improved as follows:

Conjecture 5.7 (Soare). *Every semilow set is (effectively) automorphic to a low set.*

Conjecture 5.8 (Cholak and Epstein). *Every set A such that A has the outer splitting property and \overline{A} is semilow₂ is automorphic to a low₂ set.*

Cholak and Epstein are currently working on a proof of the latter conjecture and some related results. Hopefully, a draft will be available soon.

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