Iterated Relative Recursive Enumerability[1]

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ABSTRACT. A result of Soare and Stob asserts that for any non-recursive r.e. set C, there exists a r.e.[C] set A such that $A \oplus C$ is not of r.e. degree. A set Y is called [of] m-REA (m-REA[C]) [degree] iff it is [Turing equivalent to] the result of applying m-many iterated 'hops' to the empty set (to C), where a hop is any function of the form $X \mapsto X \oplus W_e^X$. The cited result is the special case $m = 0, n = 1$ of our **Theorem**. For $m = 0, 1$, and any $(m+1)$-REA set C, if C is not of m-REA degree, then for all n there exists a n-r.e.[C] set A such that $A \oplus C$ is not of $(m+n)$-REA degree.

1. Introduction and Summary

This paper is in the recent tradition of studying sets (of natural numbers) and (Turing) degrees which although not recursively enumerable (r.e.) are closely related to r.e. sets and degrees. Our starting point is the following pair of results.

1.1 Theorem. (Cooper-Epstein-Lachlan, unpublished) *There exists a 2-r.e. set which is not of r.e. degree.*

1.2 Theorem. [SoSt] *For any non-recursive r.e. set C, there exists an REA[C] set which is not of r.e. degree.*

We recall that a set A is 2-r.e. (or d.r.e) iff there exist r.e. sets A_0 and A_1 such that A is their set difference, $A = A_0 \sim A_1$. A set A is r.e.[C] (***recursively enumerable in*** C) iff for some r.e. set U,

$$A = U^C = \{\, x : \exists y \,\langle\, x, C \mid y \,\rangle \in U \,\}.$$

Here $C \mid y$ denotes the number which codes the initial segment of the characteristic function of C of length y. We say that $\langle\, x, C \mid y \,\rangle$ is an ***axiom*** that is satisfied by C and witnesses that $x \in A$. A is REA[C] (***recursively enumerable in and above*** C) iff A is r.e.[C] and also $C \leq_T A$. Note that if A is r.e.[C], then the recursive join $A \oplus C$ is REA[C]. Hence, up to degree, every REA[C] set is of the form $C \oplus U^C$ for some r.e. set U.

We shall review in the next section the proofs of these theorems. The basic idea is that in either case the set A may "change its mind" twice about the membership status of any number x; this flexibility enables a diagonalization procedure for constructing A not Turing equivalent with any r.e. set.

Our goal here is to establish some natural generalizations of these results. The notion of a set which may "change its mind" any finite number of times is well-known; we give the relativized version:

1.3 Definition. For any sets A and C and any n,

(i) A is 0-r.e.[C] iff $A \leq_T C$;

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[1] German speakers should not be unduely influenced by the acronym for this title

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(ii) A is $(n+1)$-r.e.$[C]$ iff $A = A_0 \sim B$ for some r.e.$[C]$ set A_0 and some n-r.e.$[C]$ set B;

(iii) A is n-r.e. iff A is n-r.e.$[\emptyset]$.

For this and other properties defined below, a degree has the property iff it contains at least one set with the property. Now a first candidate for a generalization of Theorems 1.1 and 1.2 is

1.4 Theorem. *For all n*

(i) *there exists an $(n+1)$-r.e. set which is not of n-r.e. degree;*

(ii) *for any non-recursive r.e. set C, there exists a set A which is n-r.e.$[C]$ such that $A \oplus C$ is not of n-r.e. degree.*

A strengthening of part (i) is proved in [JoSh2] and will be discussed further below; the analogous strengthening of part (ii) will follow from some of our results below. In both cases the strengthening consists in replacing n-r.e. by n-REA:

1.5 Definition. For any sets A and C and all n,

(i) A is 0-REA$[C]$ iff $A \equiv_T C$;

(ii) A is $(n+1)$-REA$[C]$ iff A is REA$[B]$ for some B which is n-REA$[C]$;

(iii) A is n-REA iff A is n-REA$[\emptyset]$.

This notion was introduced and studied by Jockusch and Shore [JoSh2], who proved in particular

1.6 Theorem. [JoSh2, Theorem 1.4] *For all sets A and C and all n,*

(i) *if A is n-r.e.$[C]$, then $A \oplus C$ is of n-REA$[C]$ degree;*

(ii) *there exists an $(n+1)$-r.e.$[C]$ set A such that $A \oplus C$ is not of n-REA$[C]$ degree.*

These clearly imply part (i) of Theorem 1.4 and suggest the following strengthening of (ii), which we shall prove in Section 3:

1.7 Theorem. *For any non-recursive r.e. set C and any n, there exists an n-r.e.$[C]$ set A such that $A \oplus C$ is not of n-REA degree. Hence, there exists a set which is n-REA$[C]$ but not of n-REA degree.*

Of course, Theorem 1.2 is exactly the case $n = 1$ of this. Soare [So, p.116] introduced the term ***hop*** for any mapping of the form $C \mapsto C \oplus U^C$ for an r.e. set U. The Turing jump $C \mapsto C'$ is (up to degree) a hop and for every hop, $C \oplus U^C \leq_T C'$. If we represent a given hop by $\stackrel{\forall}{\longrightarrow}$ and one we construct by $\stackrel{\exists}{\longrightarrow}$, then the second clause of this theorem may be represented by the diagram

$$\mathbf{0} \stackrel{\forall}{\longrightarrow} C \stackrel{\exists}{\longrightarrow} D_0 \stackrel{\exists}{\longrightarrow} \cdots \stackrel{\exists}{\longrightarrow} D_{n-1} = A.$$

In words, for any C which is reachable from $\mathbf{0}$ by one hop but not fewer, there exists a set A reachable from C in n hops but not reachable from $\mathbf{0}$ in n hops. This perspective suggests the following question — if for some $m > 0$, C is reachable from $\mathbf{0}$ in $m+1$ hops but not in m hops, is there a set reachable from C in n hops which is not reachable from $\mathbf{0}$ in $m+n$ hops. More precisely,

1.8 Conjecture. *For any set C and any m and n, if C is $(m+1)$-REA but not of m-REA degree, then there exists an n-r.e.[C] set A such that $A \oplus C$ is not of $(m+n)$-REA degree. Hence there exists a set which is n-REA[C], hence $(m+n+1)$-REA, but not of $(m+n)$-REA degree.*

The second clause may be represented by the diagram:

$$\mathbf{0} \overset{\forall}{\longrightarrow} B_0 \overset{\forall}{\longrightarrow} \cdots \overset{\forall}{\longrightarrow} B_{m-1} \overset{\forall}{\longrightarrow} C \overset{\exists}{\longrightarrow} D_0 \overset{\exists}{\longrightarrow} \cdots \overset{\exists}{\longrightarrow} D_{n-1} = A.$$

At one point in the work on this paper we believed that we had proved this conjecture and announced it in [Ch-Hi]. Unfortunately, our "proof" contained a gap that we have been unable to fill and we are currently able to prove only

1.9 Main Theorem. *The Conjecture holds for all n for $m = 0$ and $m = 1$.*

We shall refer to Case (m, n) of the Conjecture with the obvious meaning. Note that Theorem 1.2 is the case $(0, 1)$, Theorem 1.7 comprises all cases $(0, n)$ and that all cases $(m, 0)$ are trivial. Where not otherwise specified, our notation conforms with that of [So].

2. Background

To facilitate understanding of the somewhat complex proofs in the following sections, we sketch here in a compatible notation and style proofs of some of the known results mentioned in Section 1. We begin with a

Proof (of Theorem 1.1). To construct a 2-r.e. set $A = A_0 \sim A_1$ which is not of r.e. degree, it suffices to satisfy all requirements of the form

$$(1) \qquad A \neq \Phi^E \qquad \text{or} \qquad E \neq \Psi^A,$$

where Φ and Ψ are any recursive functionals and E is any r.e. set. We describe a strategy to satisfy a single instance of (1) while imposing at most finitely much restraint; it is then a standard exercise in the finite injury priority method to combine these strategies for all instances of (1).

Suppose that (1) fails, so that in particular for any fixed number x

$$A(x) = \Phi^E(x) \qquad \text{and} \qquad E \mid u(x) = \Psi^A \mid u(x),$$

where $u(x)$ is the E-use of the computation $\Phi^E(x)$. Let $v(x)$ denote the maximum of $x+1$ and the A-use of $\Psi^A \mid u(x)$ — that is, the combined A-use of $\Psi^A(y)$ for $y \leq x$. The key to the strategy is that the potential failure of (1) can be recognized at a finite stage of the construction and evasive action taken to avoid it. Let $\phi(s,x)$ denote the condition

$$A_s(x) = \Phi_s^{E_s}(x) \qquad \text{and} \qquad E_s \mid u(s,x) = \Psi_s^{A_s} \mid u(s,x),$$

where $u(s,x)$ is the E_s-use of the computation $\Phi_s^{E_s}(x)$; Φ_s, Ψ_s and E_s refer to standard enumerations, and A_s is the s-th stage of the set we are constructing: $A_s = A_{0,s} \sim A_{1,s}$. Let $v(s,x)$ denote the maximum of $x+1$ and the A_s-use of the computations $\Psi_s^{A_s} \mid u(s,x)$. By increasing them if necessary, we may assume that $u(s,x)$ and $v(s,x)$ are monotone non-decreasing with respect to both s and x.

We say that $\phi(s,x)$ holds ***correctly*** iff $\phi(s,x)$ holds, $E_s \mid u(s,x) = E \mid u(s,x)$, and $A_s \mid v(s,x) = A \mid v(s,x)$. Since A_0, A_1, and E are r.e. sets, if (1) fails, then for all sufficiently large s we have $E_s \mid u(x) = E \mid u(x)$, and $A_s \mid v(x) = A \mid v(x)$, from which it follows that $u(s,x) = u(x)$, $v(s,x) = v(x)$ and $\phi(s,x)$ holds correctly. Thus,

(1.1) (***Recognition***) if (1) fails, then for all x and all sufficiently large s, $\phi(s,x)$ holds correctly.

Note that $\phi(s,x)$ alone is a recursive condition, whereas its correctness is not. In the construction we shall search for stages where $\phi(s,x)$ holds without regard for correctness; in fact, it is crucial that some instances are *not* correct.

The strategy now goes as follows. Choose $x \notin A_0, A_1$ and wait for a stage s_0 such that $\phi(s_0,x)$. If there is none, then (1) is satisfied by (1.1); otherwise enumerate x into A_{0,s_0+1} and restrain $A_{s_0+1} \mid v(s_0,x)$ — that is, ensure that no $x' < v(s_0,x)$ is enumerated into either A_0 or A_1 at any stage $s > s_0+1$ at which this restraint is in effect. Recall that $x < v(s_0,x)$. Now wait for a stage $s_1 > s_0$ such that $\phi(s_1,x)$. If there is none, then (1) is satisfied by (1.1); otherwise enumerate x into A_{1,s_1+1} and continue the restraint of $A \mid v(s_0,x)$. Let $u_0 = u(s_0,x)$ and $v_0 = v(s_0,x)$. Since

$$\Phi_{s_0}^{E_{s_0}}(x) = A_{s_0}(x) = 0 \neq 1 = A_{s_1}(x) = \Phi_{s_1}^{E_{s_1}}(x),$$

it follows that $E_{s_0} \mid u_0 \neq E_{s_1} \mid u_0$. But for all $t > s_1$ clearly $A_t(x) = 0 = A_{s_0}(x)$, whence by the restraint imposed, $A_t \mid v_0 = A_{s_0} \mid v_0$ so that

$$E_t \mid u_0 \neq E_{s_0} \mid u_0 = \Psi_{s_0}^{A_{s_0}} \mid u_0 = \Psi_t^{A_t} \mid u_0.$$

The inequality holds because E is an r.e. set; once it changes on an initial segment it will never revert to its previous value. Now we have also that for all $t > s_1$,

$$E_t \mid u(t,x) \neq \Psi_t^{A_t} \mid u(t,x),$$

so that $\phi(t,x)$ does not hold and again (1) is satisfied by (1.1). □

The main difficulty with the generalization to Theorem 1.6(ii) (here with $C = \emptyset$) lies in finding the appropriate condition $\phi(s,x)$ such that the analogue of (1.1) holds. For example, to construct a 3-r.e. set $A = A_0 \sim (A_1 \sim A_2)$ not of 2-REA degree it is necessary to satisfy requirements

(1) $$A \neq \Phi^{E \oplus W^E} \qquad \text{or} \qquad E \neq \Psi^A \qquad \text{or} \qquad W^E \neq \Theta^A$$

for all recursive functionals Φ, Ψ, and Θ and all r.e. sets E and W. In order to recognize the potential failure of (1) at a stage s for the purpose of taking action to avoid this failure, we need appropriate approximations to all the sets and functionals involved. The key feature in the proof of Theorem 1.1 above is that under the hypothesis that (1) (of that proof) fails, the standard approximations A_s, E_s, $\Phi_s^{E_s}$ and $\Psi_s^{A_s}$ all converge on initial segments to A, E, Φ^E and Ψ^A, respectively. In the generalized context, however, the standard approximation $W_s^{E_s}$ does not in general converge to W^E, since it may happen that for some y there are infinitely many s and z_s such that $\langle y, E_s \mid z_s \rangle \in W_s$, so $y \in W_s^{E_s}$, but for all such s, $E_s \mid z_s \neq E \mid z_s$ and $y \notin W^E$.

The solution to this problem found by Jockusch and Shore is to use $\Theta_s^{A_s}$ itself as the approximation to W^E and to include in $\phi(s, x)$ conditions which ensure that there is sufficient coherence between the approximation and the set approximated. We give below the details for the general case.

2.1 Theorem. [JoSh2, Proposition 1.7] *For all n, there exists an $(n+1)$-r.e. set which is not of n-REA degree.*

Proof. Any n-REA set E^n is determined by a sequence $E^1, W_1, \ldots, W_{n-1}$ of r.e. sets. For $1 \leq i < n$, we define recursively

$$F^1 = E^1, \qquad F^{i+1} = W_i^{E^i}, \qquad \text{and} \qquad E^{i+1} = E^i \oplus F^{i+1}.$$

We aim to construct r.e. sets $A_0, \ldots, A_n$ such that

$$A = A_0 \sim (A_1 \sim (A_2 \sim \cdots \sim (A_{n-1} \sim A_n) \cdots))$$

satisfies all requirements of the form

$$(1) \qquad A \neq \Phi^{E^n} \qquad \text{or} \qquad \bigvee_{1 \leq i \leq n} \left[F^i \neq \Theta_i^A \right].$$

Suppose that (1) fails so that for each x,

$$A(x) = \Phi^{E^n}(x) \qquad \text{and} \qquad \bigwedge_{1 \leq i \leq n} \left[F^i \mid w^i(x) = \Theta_i^A \mid w^i(x) \right],$$

where $u^n(x)$ is the E^n-use of the computation $\Phi^{E^n}(x)$, and recursively, for $1 \leq i < n$, $u^i(x)$ and $w^i(x)$ are chosen minimal such that for all y,

$$\begin{aligned} 2y < u^{i+1}(x) &\implies y < u^i(x) \\ 2y + 1 < u^{i+1}(x) &\implies y < w^{i+1}(x) \\ y \in F^{i+1} \mid w^{i+1}(x) &\implies (\exists z < u^i(x)) \, \langle y, E^i \mid z \rangle \in W_i \\ w^1(x) &= u^1(x). \end{aligned}$$

These parameters, together with the analogous ones at stage s, are chosen precisely to guarantee the Recognition and Positive Change properties below. Let $v(x)$ be the maximum of $x + 1$ and the total A-use for all computations for which A is an oracle.

For any s, let E^1_s, $W_{i,s}$ and $\Theta^{A_s}_{i,s}$ denote the standard approximation to E^1, W_i and Θ^A_i at stage s, respectively. For $1 \le i < n$, set

$$F^i_s = \Theta^{A_s}_{i,s} \qquad \text{and} \qquad E^{i+1}_s = E^i_s \oplus F^{i+1}_s.$$

This is, of course, an abuse of notation, since $\Theta^{A_s}_{i,s}(y)$ is defined for only finitely many y so F^i_s is not a characteristic function. In practice we shall always refer to $F^i_s \mid w$, for some w, and we interpret this usage to imply that $\Theta^{A_s}_{i,s}(y)$ is defined for all $y < w$.

Let $\phi(s,x)$ denote the condition that

$$A_s(x) = \Phi^{E^n_s}_s(x),$$

and for $1 \le i \le n$ there exist $u^i(s,x)$, $v(s,x)$, and $w^i(s,x)$, such that $u^n(s,x)$ is the E^n_s-use of this computation and

$$\begin{aligned} 2y < u^{i+1}(s,x) &\implies y < u^i(s,x) \\ 2y+1 < u^{i+1}(s,x) &\implies y < w^{i+1}(s,x) \\ y \in F^{i+1}_s \mid w^{i+1}(s,x) &\iff (\exists z < u^i(s,x))\ \langle\, y, E^i_s \mid z\,\rangle \in W_{i,s} \\ F^1_s \mid w^1(s,x) &= E^1_s \mid w^1(s,x) \\ w^1(s,x) &= u^1(s,x), \end{aligned}$$

and $v(s,x)$ is the maximum of $x+1$ and the total A_s-use for all computations for which A_s is an oracle. As in the preceding proof we may assume that all of these functions are monotone non-decreasing with respect to s and x. We say that $\phi(s,x)$ holds ***correctly*** iff $\phi(s,x)$ holds; for $1 \le i \le n$, $E^i_s \mid u^i(s,x) = E^i \mid u^i(s,x)$; and $A_s \mid v(s,x) = A \mid v(s,x)$.

We need first to establish two key properties of these approximations:

(1.1) (***Recognition***) if (1) fails, then for all x and all sufficiently large stages s, $\phi(s,x)$ holds correctly;

(1.2) (***Positive Change***) for all x and all $s < t$ such that both $\phi(s,x)$ and $\phi(t,x)$,

(a) for $1 \le i < n$,

$$E^i_s \mid u^i(s,x) = E^i_t \mid u^i(s,x) \implies F^{i+1}_s \mid w^{i+1}(s,x) \subseteq F^{i+1}_t \mid w^{i+1}(s,x);$$

(b) for $1 \le i \le n$,

$$E^i_s \mid u^i(s,x) \ne E^i_t \mid u^i(s,x) \implies \text{for some } 1 \le j \le i, \quad F^j_s \mid w^j(s,x) \subset F^j_t \mid w^j(s,x).$$

To prove (1.1), fix x and suppose that (1) fails at x with $u^i(x)$, $v(x)$, and $w^i(x)$ defined as above. Choose s^1 large enough so that for all $s \ge s^1$,

$$A_s \mid v(x) = A \mid v(x) = \Phi^{E^n}_s(x) \qquad \text{and} \qquad F^1_s \mid u^1(x) = E^1_s \mid u^1(x) = E^1 \mid u^1(x).$$

This is possible since these approximations converge. Let $w^1(s,x) = u^1(s,x) = u^1(x)$. Next choose $s^2 \geq s^1$ such that for all $s \geq s^2$, $F^2_s \mid w^2(x) = F^2 \mid w^2(x)$, and for all y,

$$y \in F^2 \mid w^2(x) \quad \Longrightarrow \quad (\exists z < u^1(x)) \left[\langle\, y, E^1 \mid z \,\rangle \in W_{1,s} \right].$$

Then $E^2_s \mid u^2(x) = E^2 \mid u^2(x)$, and if we set $w^2(s,x) = w^2(x)$, we have for all y,

$$\begin{aligned} y \in F^2_s \mid w^2(s,x) \quad &\Longleftrightarrow \quad y \in F^2 \mid w^2(x) \\ &\Longleftrightarrow \quad (\exists z < u^1(x))\langle\, y, E^1 \mid z \,\rangle \in W_1 \\ &\Longleftrightarrow \quad (\exists z < u^1(s,x))\langle\, y, E^1_s \mid z \,\rangle \in W_{1,s}, \end{aligned}$$

as required for $\phi(s,x)$ to hold.

Continuing recursively, we obtain finally s^n such that for all $s \geq s^n$, $E^n_s \mid u^n(x) = E^n \mid u^n(x)$, so

$$A_s(x) = A(x) = \Phi^{E^n}(x) = \Phi^{E^n_s}_s(x),$$

and thus $\phi(s,x)$ holds correctly.

(1.2)(a) is immediate from the definition of $\phi(s,x)$: if $y \in F^{i+1}_s \mid w^{i+1}(s,x)$, then for some $z < u^i(s,x)$, $\langle\, y, E^i_s \mid z \,\rangle \in W_{i,s} \subseteq W_{i,t}$. By hypothesis, $E^i_s \mid z = E^i_t \mid z$, so also $y \in F^{i+1}_t$.

(1.2)(b) is immediate for $i = 1$ and we proceed by induction; assume the result for i and $E^{i+1}_s \mid u^{i+1}(s,x) \neq E^{i+1}_t \mid u^{i+1}(s,x)$. If $E^i_s \mid u^i(s,x) \neq E^i_t \mid u^i(s,x)$, then the conclusion for some $j \leq i$ follows from the induction hypothesis. Otherwise, by (a), $F^{i+1}_s \mid w^{i+1}(s,x) \subseteq F^{i+1}_t \mid w^{i+1}(s,x)$ and for some y with $2y+1 < u^{i+1}(s,x)$ we have $F^{i+1}_s(y) \neq F^{i+1}_t(y)$. Thus $y < w^i(s,x)$ and this inclusion is proper, so the conclusion holds for $j = i+1$.

Now, the strategy for satisfying a single requirement (1) while imposing finite restraint is as follows. Choose a witness x which belongs to none of the sets $A_0, \ldots, A_n$ and wait for a stage s_0 such that $\phi(s_0,x)$. If there is none, (1) is satisfied by (1.1); otherwise, enumerate x into A_{0,s_0+1}, restrain $A_{s_0+1} \mid v(s_0,x)$, and wait for a stage $s_1 > s_0$ such that $\phi(s_1,x)$. If there is none, (1) is satisfied by (1.1); otherwise enumerate x into A_{1,s_1+1}, restrain $A_{s_1+1} \mid v(s_1,x)$, and wait for a stage $s_2 > s_1$ such that $\phi(s_2,x)$. If (1) is not satisfied at any stage of this process by (1.1), then we generate a sequence $s_0 < s_1 < \ldots < s_n$ such that for $j \leq n$, $\phi(s_j,x)$.

Let $u^i_j = u^i(s_j,x)$, $v_j = v(s_j,x)$, and $w^i_j = w^i(s_j,x)$. We establish first the following fact: for all $1 \leq i < n$ and $j < (n-1)$,

$$(*) \qquad E^{i+1}_{s_j} \mid u^{i+1}_j \neq E^{i+1}_{s_{j+1}} \mid u^{i+1}_j \quad \Longrightarrow \quad \text{for some } h \leq i, \quad E^h_{s_{j+1}} \mid u^h_{j+1} \neq E^h_{s_{j+2}} \mid u^h_{j+1}.$$

By the hypothesis of this implication and (1.2)(b), for some $h \leq i$, $F^{h+1}_{s_j} \mid w^{h+1}_j \subset F^{h+1}_{s_{j+1}} \mid w^{h+1}_j$. On the other hand, since enumerating x in both A_j and A_{j+1} has no net effect, by the restraint imposed we have $A_{s_j} \mid v_j = A_{s_{j+2}} \mid v_j$, so

$$F^{h+1}_{s_j} \mid w^{h+1}_j = \Theta^{A_{s_j}}_{h+1,s_j} \mid w^{h+1}_j = \Theta^{A_{s_{j+2}}}_{h+1,s_{j+2}} \mid w^{h+1}_j = F^{h+1}_{s_{j+2}} \mid w^{h+1}_j.$$

and it follows that

$$F^{h+1}_{s_{j+1}} \mid w^{h+1}_j \not\subseteq F^{h+1}_{s_{j+2}} \mid w^{h+1}_j \qquad \text{hence also} \qquad F^{h+1}_{s_{j+1}} \mid w^{h+1}_{j+1} \not\subseteq F^{h+1}_{s_{j+2}} \mid w^{h+1}_{j+1}.$$

The desired conclusion follows by (1.2)(a).

Now

$$\Phi^{E^n_{s_0}}_{s_0}(x) = A_{s_0}(x) = 0 \neq 1 = A_{s_1}(x) = \Phi^{E^n_{s_1}}_{s_1}(x),$$

and thus $E^n_{s_0} \mid u^n_0 \neq E^n_{s_1} \mid u^n_0$. Using $(*)$ it follows by induction that there exist $n = h_0 > \cdots > h_{n-1}$ such that for $j < n$, $E^{h_j}_{s_j} \mid u^{h_j}_j \neq E^{h_j}_{s_{j+1}} \mid u^{h_j}_j$, so in fact $h_j = n - j$ and in particular, $E^1_{s_{n-1}} \mid u^1_{n-1} \neq E^1_{s_n} \mid u^1_{n-1}$. But then for all $t > s_n$ we have $A_t \mid v_n = A_{s_{n-1}} \mid v_n$ and thus

$$E^1_t \mid u^1_{n-1} \neq E^1_{s_{n-1}} \mid u^1_{n-1} = \Theta^{A_{s_{n-1}}}_{1,s_{n-1}} \mid u^1_{n-1} = \Theta^{A_t}_{1,t} \mid u^1_{n-1},$$

and hence $\phi(t,x)$ does not hold and (1) is satisfied by (1.1). □

Theorem 1.6(i) is less germane to our methods here, but the following short proof seems not to have appeared in print, so we include it.

2.2 Lemma. *For any sets A and C and all n, if A is $(n+1)$-r.e.[C], then $A \oplus C$ is of REA[$D \oplus C$] degree for some n-r.e.[C] set D.*

Proof. Let A be $(n+1)$-r.e.[C], so $A = A_0 \sim B$ for some r.e.[C] set A_0 and some n-r.e.[C] set B. Fix a C-recursive function f such that $A_0 = \mathsf{Im}(f)$ and let $D = f^{-1}(B)$. Easily D is n-r.e.[C] and it suffices to observe that

$$D \leq_T A, \qquad \text{since for all } x, \quad x \in D \iff f(x) \notin A;$$

and

$$A \text{ is r.e.}[D], \qquad \text{since for all } x, \quad x \in A \iff \exists y\,[y \notin D \wedge f(y) = x]. \ \square$$

2.3 Corollary. *For any sets A and C and all n, if A is n-r.e.[C], then $A \oplus C$ is of n-REA[C] degree.*

Proof. We proceed by induction. For $n = 0$, the result is clear. If A is $(n+1)$-r.e.[C], choose D as in the Lemma. By the induction hypothesis, $D \oplus C \equiv_T E$ for some n-REA[C] set E, and $A \oplus C$ is of REA[E]-degree, hence of $(n+1)$-REA[C] degree. □

Before we turn to an exposition of the proof of Theorem 1.2, we have to dispose of a technical point, which is nonetheless of great importance in that proof and others to follow. It concerns the fact that the results relative to an arbitrary non-recursive set are necessarily non-uniform — that is, an index for the set constructed cannot be computed recursively from the indices of the given sets. To formulate this precisely, fix an enumeration $\langle W_e : e \in \omega \rangle$ of the r.e. sets and set $X^{\langle e \rangle} = X \oplus W^X_e$, $X^{\langle\, \rangle} = X$, and

$$X^{\langle e_0,\ldots,e_m \rangle} = \left(X^{\langle e_0,\ldots,e_{m-1} \rangle}\right)^{\langle e_m \rangle}.$$

We call a sequence $(e_0, \ldots, e_m)$ of (indices of) $m+1$ many hops ***non-degenerate*** iff $\emptyset^{\langle e_0,\ldots,e_m \rangle}$ is not of m-REA degree. The second clause of our Conjecture 1.8 may be reformulated as follows: for all m, n, and $b_0, \ldots, b_m$, if $(b_0, \ldots, b_m)$ is non-degenerate, then there exist $d_0, \ldots, d_{n-1}$ such that $(b_0, \ldots, b_m, d_0, \ldots, d_{n-1})$ is non-degenerate. Then we have the following generalization of [SoSt, Corollary 4.3].

2.4 Proposition. *For all m and n, there do not exist recursive functions $g_0, \ldots, g_n$ such that for all $b_0, \ldots, b_m$, if $(b_0, \ldots, b_m)$ is non-degenerate, then also*

$$(b_0, \ldots, b_m, g_0(b_0, \ldots, b_m), \ldots, g_n(b_0, \ldots, b_m))$$

is non-degenerate.

Proof. By the relativized version of [JoSh1, Theorem 3.1] there exists a recursive function h such that for all sets X and indices e,

$$X <_T X^{\langle h(e) \rangle} \quad \text{and} \quad X^{\langle h(e), e \rangle} \equiv_T X'.$$

Now, given m, n, and recursive $g_0, \ldots, g_n$, let $b_0 = \cdots = b_{m-1}$ be indices for the Turing jump and choose b_m by the Recursion Theorem such that for all X,

$$W^X_{h(g_0(b_0, \ldots, b_{m-1}, b_m))} = W^X_{b_m}.$$

Then by the first property of h, for any X, $X <_T X^{\langle b_m \rangle}$, so

$$\emptyset^{\langle b_0, \ldots, b_{m-1} \rangle} = \emptyset^{(m)} <_T \emptyset^{\langle b_0, \ldots, b_{m-1}, b_m \rangle},$$

and thus $\emptyset^{\langle b_0, \ldots, b_{m-1}, b_m \rangle}$ is not of m-REA degree — that is, $(b_0, \ldots, b_m)$ is non-degenerate. But by the second property of h,

$$\emptyset^{\langle b_0, \ldots, b_{m-1}, b_m, g_0(b_0, \ldots, b_m) \rangle} \equiv_T (\emptyset^{\langle b_0, \ldots, b_{m-1} \rangle})',$$

is of $m+1$-REA degree, from which it follows that

$$\emptyset^{\langle b_0, \ldots, b_m, g_0(b_0, \ldots, b_m), \ldots, g_n(b_0, \ldots, b_m) \rangle}$$

is of $m+n+1$-REA degree — that is,

$$(b_0, \ldots, b_m, g_0(b_0, \ldots, b_m), \ldots, g_n(b_0, \ldots, b_m))$$

is degenerate. □

Finally, we sketch

Proof (of Theorem 1.2). Let C be a given non-recursive r.e. set. We aim to construct an r.e. set U, such that if $A = U^C$, then $A \oplus C$ is not of r.e. degree. In view of the preceding proposition, we shall in fact construct two r.e. sets U_0 and U_1 such that $A_k = U_k^C$ $(k = 0, 1)$ satisfy the family of requirements

$$\textbf{(1)} \qquad \bigvee_{k=0,1} \left[A_k \neq \Phi_k^{E_k} \quad \text{or} \quad E_k \neq \Psi_k^{A_k \oplus C} \right],$$

for all sextuples $(\Phi_0, \Phi_1, \Psi_0, \Psi_1, E_0, E_1)$ of four recursive functionals and two r.e. sets. It follows that at least one of the sets $A_0 \oplus C$, $A_1 \oplus C$ is not of r.e. degree as desired; the non-uniformity arises in that we can not determine effectively which one.

If (1) fails, then for any x,

$$\bigwedge_{k=0,1}\left[A_k \upharpoonright x+1 = \Phi_k^{E_k} \upharpoonright x+1 \qquad \text{and} \qquad E_k \upharpoonright u(x) = \Psi_k^{A_k \oplus C} \upharpoonright u(x)\right],$$

where $u(x)$ is the combined use of $\Phi_0^{E_0}(y)$ and $\Phi_1^{E_1}(y)$ for $y \le x$. The difficulty in recognizing this potential failure is somewhat similar to that in Theorem 2.1 — the standard approximations for the r.e. sets E_k and C converge, but the approximations $A_{k,s} = U_{k,s}^{C_s}$ do not generally converge to A_k. In this case we shall invoke a standard device, the method of ***true stages***. Fix an enumeration $\langle C_s : s \in \omega \rangle$ of C such that $C_{s+1} \sim C_s \neq \emptyset$ for all s and let C_{s+1} denote the smallest element of this set. A stage s is C-***true*** iff $C_s \upharpoonright c_s = C \upharpoonright c_s$. It is clear that there are infinitely many C-true stages and we shall arrange the construction to ensure that at any C-true stage s, we have $A_{k,s} \subseteq A_k$ so that for any v, for sufficiently large C-true stages, $A_{k,s} \upharpoonright v = A_k \upharpoonright v$ — in other words, $A_{k,s}$ converges to A_k on C-true stages.

Let $\phi(x,s)$ denote the condition

$$\bigwedge_{k=0,1}\left[A_{k,s} \upharpoonright x+1 = \Phi_{k,s}^{E_{k,s}} \upharpoonright x+1 \quad \text{and} \quad E_{k,s} \upharpoonright u(s,x) = \Psi_{k,s}^{A_{k,s} \oplus C_s} \upharpoonright u(s,x)\right],$$

where $u(s,x)$ is the combined use of the computations $\Phi_{k,s}^{E_{k,s}}(y)$ for $k = 0,1$ and $y \le x$. We assume that $u(s,x)$ is monotone non-decreasing in both s and x. Note that we now require agreement of $A_{k,s}$ and $\Phi_{k,s}^{E_{k,s}}$ for all $y \le x$ instead of merely at x; this is a minor change required by the use of multiple witnesses as described below. Let $v(s,x)$ denote the maximum of $x+1$ and the $A_{k,s}$ and C_s uses in these computations and say that $\phi(s,x)$ holds ***correctly*** iff for $k = 0,1$, $E_{k,s} \upharpoonright u(s,x) = E_k \upharpoonright u(s,x)$, $A_{k,s} \upharpoonright v(s,x) = A_k \upharpoonright v(s,x)$, and $C_s \upharpoonright v(s,x) = C \upharpoonright v(s,x)$. Then much as above, we have

(1.1) (***Recognition***) if (1) fails, then for all x and all sufficiently large C-true stages s, $\phi(s,x)$ holds correctly.

The mechanism for ensuring convergence on C-true stages is simple: at each stage $t+1$ of the construction we enumerate into $U_{k,t+1}$ certain ***axioms*** $\langle x, C_t \upharpoonright v \rangle$, with $v \le c_{t+1}$; thus $x \in A_{k,t+1}$. If $c_s < v$ for some $s > t$, then $C_s \upharpoonright v \neq C_t \upharpoonright v$, so generally $x \notin A_{k,s}$. If s is C-true and $x \in A_{k,s}$, then $v \le c_s$, so $C_t \upharpoonright v = C_s \upharpoonright v = C \upharpoonright v$, and thus also $x \in A_k$.

The strategy for satisfying a single instance of (1) is complicated in two ways (relative to Theorem 1.1) by the replacement of the 2-r.e. set A by $A \oplus C$ with $A = U^C$. First, the set $A \oplus C$ is subject to unpredictable changes due to changes in C. Second, at the point in the proof of Theorem 1.1 when we enumerate x into A_1 to remove it from A, we can now only wait for a change in some $C \upharpoonright v$, which may or may not ever occur, to remove x from $A = U^C$.

The solution to both of these problems consists in assigning to (1) not a single witness but a (potentially) infinite sequence $\langle x_l : l \in \omega \rangle$ of witnesses. Success with any one of the witnesses x_l suffices to satisfy (1) and we show that failure of all the witnesses leads to an algorithm for computing C, contrary to its assumed non-recursiveness. Roughly, the failure of a witness x_{l-1} is due to the fact that the corresponding $C \upharpoonright v_l$ does not change to remove x_{l-1} from A and thus $C \upharpoonright v_l$ has its final value at a predictable stage.

Before giving the formal construction, we describe it informally. We define the sequence of witnesses as we go along; x_l will be a candidate for enumeration into $A_{\pi(l)}$, where $\pi(l) = 0$, if l is even; 1, if l is odd. Let $x_0 = 0$. We take no action until we arrive at a stage s_0 for which $\phi(s_0, x_0)$ holds. If there is no such stage, then (1) holds by (1.1) (since there are infinitely many C-true stages); otherwise we define x_1 larger than any number mentioned so far, in particular, $x_1 > v_0 := v(s_0, x_0)$, restrain $A_{0,s_0+1} \mid v_0$, and wait for a stage $s_1 > s_0$ at which $\phi(s_1, x_1)$ holds. If there is none, then again (1) holds by (1.1); otherwise enumerate x_0 into A_{0,s_1+1} with use $v_1 = v(s_1, x_1)$ — that is, enumerate the axiom $\langle x_0, C_{s_1} \mid v_1 \rangle$ into U_{0,s_1+1} — restrain $A_{1,s_1+1} \mid v_1$, and choose $x_2 > v_1$. Similarly, we wait for a stage $s_2 > s_1$ at which $\phi(s_2, x_2)$ holes, enumerate x_1 into A_{1,s_2+1} with use v_2, restrain $A_{0,s_2+1} \mid v_2$, and choose $x_3 > v_2$. At s_3, x_2 is enumerated into A_0, and so on.

The following diagram may be helpful in following this argument.

0		s_0	s_0+1		s_1	s_1+1
$\rho=0$	. .	$\rho=0$	$\rho=1$	. .	$\rho=1$	$\rho=2$
		$\phi(s_0,x_0)$			$\phi(s_1,x_1)$	$\neg\,\phi(s_1+1,x_1)$
$x_0=0$			x_1 def			x_2 def
						$x_0 \in A_0$

	s_2	s_2+1		s_l	s_l+1	
. .	$\rho=2$	$\rho=3$	. . .	$\rho=l$	$\rho=l+1$	. . .
	$\phi(s_2,x_2)$	$\neg\,\phi(s_2+1,x_2)$		$\phi(s_l,x_l)$	$\neg\,\phi(s_l+1,x_l)$	
		x_3 def			x_{l+1} def	
		$x_1 \in A_1$			$x_{l-1} \in A_{\pi(l-1)}$	

Each instance of $\phi(s_l, x_l)$ is dependent on the value of $C_{s_l} \mid v_l$; if this is not the correct value $C \mid v_l$, the agreement recorded by $\phi(s_l, x_l)$ may be destroyed at some later stage. Let us assume temporarily that this never happens, although in the end we shall show that it must happen.

Let $u_l = u(s_l, x_l)$; by our conventions, u_l is monotone non-decreasing in l. Since

$$A_{0,s_0} \mid v_0 = A_{0,s_1} \mid v_0, \quad \phi(s_0, x_0), \quad \text{and} \quad \phi(s_1, x_1),$$

we have

$$E_{0,s_0} \mid u_0 = \Psi_{0,s_0}^{A_{0,s_0} \oplus C_{s_0}} \mid u_0 = \Psi_{0,s_1}^{A_{0,s_1} \oplus C_{s_1}} \mid u_0 = E_{0,s_1} \mid u_0,$$

and $u(s_1, x_0) = u_0$. On the other hand, since $\phi(s_1, x_1)$, $\phi(s_2, x_2)$, and $x_0 < x_1 < x_2$, we have

$$\Phi_0^{E_{0,s_1}}(x_0) = A_{0,s_1}(x_0) = 0 \neq 1 = A_{0,s_2}(x_0) = \Phi_{0,s_2}^{E_{0,s_2}}(x_0).$$

Hence $E_{0,s_1} \mid u_0 \neq E_{0,s_2} \mid u_0$, and if we set

$$t_0 = \text{least } t > s_0[E_{0,s_0} \mid u_0 \neq E_{0,t} \mid u_0],$$

then $s_1 < t_0 \leq s_2$. Furthermore,

$$s_1 = \text{least } s < t_0[\phi(s, x_1) \wedge C_s \mid v_1 = C_{t_0} \mid v_1].$$

This means that from s_0 we can effectively determine s_1. Extending this argument we obtain an algorithm for the function $l \mapsto s_l$, which is therefore recursive. But clearly $v_l > l$, so by our temporary assumption that $C_{s_l} \mid v_l = C \mid v_l$ we have $C(l) = C_{s_l}(l)$ and we conclude, contrary to hypothesis, that C is recursive.

Thus our temporary hypothesis is untenable, and it is exactly this fact that guarantees that (1) is satisfied. To see how this works, suppose that for $j \leq l$, $C_{s_j} \mid v_j = C \mid v_j$ but $C_{s_{l+1}} \mid v_{l+1} \neq C \mid v_{l+1}$. Since x_l is enumerated into $A_{\pi(l),s_{l+1}+1}$ with the axiom $\langle x_l, C_{s_{l+1}} \mid v_{l+1} \rangle$, if at some later $t > s_{l+1}$, $C_t \mid v_{l+1} \neq C_{s_{l+1}} \mid v_{l+1}$, then $x_l \notin A_{\pi(l),t}$ so by the restraint imposed, $A_{\pi(l),t} \mid v_l = A_{\pi(l),s_l} \mid v_l$. Now if also $E_{\pi(l),t} \mid u_l \neq E_{\pi(l),s_l} \mid u_l$, then for all $s \geq t$ we have

$$E_{\pi(l),s} \mid u_l \neq E_{\pi(l),s_l} \mid u_l = \Psi_{\pi(l),s_l}^{A_{\pi(l),s_l} \oplus C_{s_l}} \mid u_l = \Psi_{\pi(l),s}^{A_{\pi(l),s} \oplus C_s} \mid u_l,$$

so $\phi(s, x_l)$ fails and (1) is satisfied by (1.1). On the other hand, if $E_{\pi(l),t} \mid u_l = E_{\pi(l),s_l} \mid u_l$, then there may be a stage $s'_{l+1} \geq t$ such that $\phi(s'_{l+1}, x_{l+1})$ with use $v'_{l+1} = v(s'_{l+1}, x_{l+1})$ — whence also $\phi(s'_{l+1}, x_l)$. At this point we re-enumerate x_l into $A_{\pi(l),s'_{l+1}+1}$ with use v'_{l+1}, redefine $x_{l+2} > v'_{l+1}$, and wait for a stage s_{l+2} such that $\phi(s_{l+2}, x_{l+2})$. Again $C_{s'_{l+1}} \mid v'_{l+1}$ may or may not be correct. Thus there are three possibilities:

(a) for some l, from some point on, $\phi(s, x_l)$ never holds;

(b) for some l, for infinitely many s, $\phi(s, x_l)$ holds with use $v(s, x_l)$, but $C_s \mid v(s, x_l) \neq C \mid v(s, x_l)$;

(c) for all l there is eventually a stage $\overline{s}_l$ such that $\phi(\overline{s}_l, x_l)$ with use $\overline{v}_l = v(\overline{s}_l, x_l)$ and $C_{\overline{s}_l} \mid \overline{v}_l = C \mid \overline{v}_l$.

In cases (a) and (b), requirement (1) is satisfied by (1.1); note that in case (b) it is the correctness of $\phi(s, x_1)$ which is infinitely often violated. By a variant of the argument given above, case (c) cannot hold, and thus (1) is satisfied.

The rest of the machinery of the proof consists in bookkeeping to keep track of the situation at stage s. We set $\rho(s) = l$ if at stage s are defined witnesses $x_j(s)$ for $j \leq l$ and uses $u_j(s)$ and $v_j(s)$ for $j < l$. Here $x_j(s)$ is the current value of x_j and the uses are those of the most recent instance of $\phi(\cdot, x_j)$. At stage s we take action only if either **Case 1**: $c_{s+1} < v_j(s)$ for some $j < l$, or **Case 2**: $\phi(s, x_l(s))$ via computations with use $\leq c_{s+1}$. If neither ever occurs, then $x_l = x_l(s)$ is a successful witness. If Case 1 occurs, then the earlier agreements up to $x_j(s), \ldots, x_l(s)$ have been injured and we reduce our list of potential witnesses to $x_0(s+1), \ldots, x_j(s+1)$; the witnesses $x_{j+1}(s), \ldots, x_l(s)$ are discarded with new values possibly to be chosen at later stages. If Case 2 occurs, then we set $u_l(s+1)$ and $v_l(s+1)$ to be the associated uses, choose a new witness $x_{l+1}(s+1)$ larger than any number mentioned so far, and (if $l > 0$) enumerate $x_{l-1}(s)$ into $A_{\pi(l-1),s+1}$ with use $v_l(s+1)$. The formal proof that (1) is satisfied now proceeds by establishing the following two facts:

(1.3) If (1) fails, then $\liminf_s \rho(s) = \infty$.

(1.4) If $\liminf_s \rho(s) = \infty$, then C is recursive, contrary to hypothesis.

Now if we set

$$s_l = \text{least } s\, (\forall t > s)[\rho(t) \geq l+1],$$

then the condition $\liminf_s \rho(s) = \infty$ is exactly the condition that all s_l exist, which is essentially the temporary hypothesis of our sketch above. Note that when s_l exists, $C \mid v_l$ has its final value at stage s_l, and to compute C it suffices to compute the function $l \mapsto s_l$.

We now give the precise construction and a series of lemmas which formalize the preceding argument. Set $\rho(0) = 0$ and $x_0(0) = x_0 = 0$. At stage $s+1$ we have one of three cases as follows. Any parameter not mentioned is assigned the same value at $s+1$ as at s.

Case 1. If $\rho(s) > 0$ and for some (least) $k < \rho(s)$, $c_{s+1} < v_k(s)$, then $\rho(s+1) := k$; $u_j(s+1)$, and $v_j(s+1)$ are undefined for all $j \geq k$; and $x_j(s+1)$ is undefined for all $j > k$;

Case 2. otherwise, if $\rho(s) = l$ and $\phi(s, x_l(s))$ holds, with associated uses $u(s, x_l(s))$ and $v(s, x_l(s)) \leq c_{s+1}$, then

(a) $\rho(s+1) := l+1$, $u_l(s+1) = u(s, x_l(s))$, and $v_l(s+1) = v(s, x_l(s))$;
(b) if $l > 0$, then $x_{l-1}(s)$ is enumerated into $A_{\pi(l-1),s+1}$ with use $v_l(s+1)$;
(c) $x_{l+1}(s+1)$ is chosen greater than any number used so far;

Case 3. otherwise, $\rho(s+1) := \rho(s)$.

We note first

(2) For all s, i, and j,

(a) $x_j(s)$ is defined for (exactly) $j \leq \rho(s)$; $u_j(s)$ and $v_j(s)$ are defined for (exactly) $j \leq \rho(s) - 1$;

(b) $i < j \leq \rho(s) \implies v_i(s) \leq x_j(s)$;

(c) $j < \rho(s+1) \implies v_j(s) \leq c_{s+1}$;

(d) for (exactly) $j \leq \rho(s) - 2$, $x_j(s) \in A_{\pi(j),s}$ with use $\leq c_s$;

(e) for all $t < s$, if $x_j(t)$ is defined but $x_j(s)$ is either undefined or $\neq x_j(t)$, then for all $s' \geq s$, $x_j(t) \notin A_{\pi(j),s'}$. In particular, this holds when $\rho(s) < j \leq \rho(t)$.

Proof. Parts (a)–(d) are straightforward to verify by induction. for (e), suppose that $x_j(t)$ is enumerated into $A_{\pi(j),t}$ with use $v_{j+1}(t)$. In either of the cases of the hypothesis, for some t' with $t \leq t' < s$, x_j becomes undefined at stage $t'+1$ because $c_{t'+1} < v_j(t')$. If $v_{j+1}(t') = v_{j+1}(t)$, then also $c_{t'+1} < v_{j+1}(t)$ and $x_j(t)$ is removed from $A_{\pi(j),t'+1}$; otherwise it was already removed at an earlier stage. Now the number $x_j(t)$ will never again be used as a witness so is never again enumerated into $A_{\pi(j)}$. □

(3) For each $l \geq 0$, if s_l exists, then

(a) $\rho(s_l) = l$, $\phi(s_l, x_l)$, and $\rho(s_l+1) = l+1$;

(b) for all $s > s_l$, $\rho(s) \geq l+1$;

(c) for all $j \le l$, $x_{j+1}(t)$, $u_j(t)$ and $v_j(t)$ have the same values for all $t > s_l$ (which we denote by x_{j+1}, u_j, and v_j);

(d) $C_{s_l} \mid v_l = C \mid v_l$;

(e) for all $j \le (l-2)$, $\quad x_j \in A_{\pi(j),s}$ for all $s \ge s_l$;

(f) if $l \ge 1$, then $x_{l-1} \notin A_{\pi(l-1),s_l}$, but for all $s > s_l$, $x_{l-1} \in A_{\pi(l-1),s}$.

Proof. (a) is immediate from the minimality of s_l and (b) is merely a restatement of the definition. Now by the condition $v_l(s+1) \le c_{s+1}$ in Case 2, $C_{s_l} \mid v_l(s+1) = C_{s_l+1} \mid v_l(s+1)$. Then for (c) and (d) we prove by induction that for all $t > s_l$, $x_{j+1}(t)$, $u_j(t)$, and $v_j(t)$ have the same values as at stage $s_l + 1$ and that $C_t \mid v_l(t) = C_{t+1} \mid v_l(t)$. The key point is that Case 1 will never apply at any stage $t+1 > s_l + 1$ with $k \le l$, since then $\rho(t+1) = k \le l$ contrary to the definition of s_l. Parts (e) and (f) follow from (a), (b), (d) and (2)(e). $\square$

We have also the converse to (3)(d)

(4) For any l and s, if $\rho(s) \ge l+1$ and $C_s \mid v_l(s) = C \mid v_l(s)$, then s_l exists and $s > s_l$.

Proof. Here, again, the point is that Case 1 will never apply at any stage $t \ge s+1$ with $k \le l$ $\square$

(5) For each $l \ge 0$, if s_{l+1} exists, then $A_{\pi(l),s_{l+1}} \mid v_l = A_{\pi(l),s_l} \mid v_l$.

Proof. It suffices to verify that for all j and all $t \le s_{l+1}$ such that $x_j(t) < v_l$ we have

$$A_{\pi(l),s_l}(x_j(t)) = A_{\pi(l),s_{l+1}}(x_j(t)).$$

For $h > l+1$, this follows from (2)(e): $x_j(t)$ belongs to neither set. No value $x_{l+1}(t)$ belongs to either set because $\pi(l+1) \ne \pi(l)$. For $j \le l$, $x_j(s) = x_j := x_j(s_l)$ for all $s \ge s_l$, so again by (2)(e), if for some $t < s_l$, $x_j(t) \ne x_j$, then $x_j(t)$ belongs to neither set. Hence we are left with the the numbers x_j for $j \le l$. Of these, only those such that $\pi(j) = \pi(l)$ — that is, $h \equiv l \pmod 2$ — belong to either set and by (3)(e) and (f), for such j,

$$h \le l-2 \quad \Longrightarrow \quad (\forall s \ge s_l) x_j \in A_{\pi(l),s}$$
$$x_l \notin A_{\pi(l),s_l} \quad \text{and} \quad x_l \notin A_{\pi(l),s_{l+1}}. \ \square$$

(6) For each $l \ge 0$,

(a) if s_{l+1} exists, then, $E_{\pi(l),s_{l+1}} \mid u_l = E_{\pi(l),s_l} \mid u_l$;

(b) if s_{l+2} exists, then, $(\exists s > s_l)[E_{\pi(l),s} \mid u_l \ne E_{\pi(l),s_l} \mid u_l]$, and if t_l denotes the least such s, then $s_{l+1} < t_l$.

Proof. For (a), since both $\phi(s_{l+1}, x_l)$ and $\phi(s_l, x_l)$ hold, by (5) and the definition of v_l,

$$E_{\pi(l),s_{l+1}} \mid u_l = \Psi_{\pi(l),s_{l+1}}^{A_{\pi(l),s_{l+1}} \oplus C_{s_{l+1}}} \mid u_l = \Psi_{\pi(l),s_l}^{A_{\pi(l),s_l} \oplus C_{s_l}} \mid u_l = E_{\pi(l),s_l} \mid u_l.$$

For (b), since both $\phi(s_l, x_l)$ and $\phi(s_{l+2}, x_l)$, if equality holds for $s = s_{l+2}$, we have the contradiction

$$0 = A_{\pi(l),s_l}(x_l) = \Phi_{\pi(l),s_l}^{E_{\pi(l),s_l}}(x_l) = \Phi_{\pi(l),s_{l+2}}^{E_{\pi(l),s_{l+2}}}(x_l) = A_{\pi(l),s_{l+2}}(x_l) = 1.$$

The last clause is now immediate from (a). $\square$

We can now establish (1.3) and (1.4).

(1.3) If (1) fails, then $\liminf_s \rho(s) = \infty$.

Proof. Suppose that (1) fails but $\liminf_s \rho(s) = l < \infty$. Thus s_{l-1} exists, but for infinitely many s, $\rho(s) = l$ (so s_l does not exist). For $s > s_{l-1}$, $x_l(s)$ has the constant value x_l. By (1.1) there exists $\bar{s} > s_{l-1}$ such that $\phi(\bar{s}, x_l)$ holds correctly. Then either $\rho(\bar{s}) \geq l+1$ and $\rho(\bar{s}+1) \geq l+1$, or $\rho(\bar{s}) = l$, Case 2 holds at stage $\bar{s}$ and $\rho(\bar{s}+1) = l+1$. In either case, by (4), s_l exists, contrary to hypothesis. □

(1.4) Only finitely many s_l exist; in other words, $\liminf_s \rho(s) < \infty$.

Proof. Suppose to the contrary that all s_l exist. Then by (6)(b) and (4) for all l, t_l exists and

$$s_{l+1} = \text{least } s < t_l \, [\phi(s, x_{l+1}) \wedge C_s \mid v_{l+1}(s) = C_{t_l} \mid v_{l+1}(t_l)].$$

Since t_l and x_{l+1} are recursively calculable from s_l, this shows that the function $l \mapsto s_l$ is recursive. But then since $l < v_l$, $C(l) = C_{s_l}(l)$ and C is recursive, contrary to hypothesis. □

The remainder of the proof consists in combining the strategies for all requirements (1) on a tree. The ***outcome*** of a requirement (1) is the value l of $\liminf_s \rho(s)$. We shall not take space here to do this in detail for Theorem 1.2, as the techniques are well illustrated by the Theorem of the next section.

3. The Main Theorem

In this section we shall give the proof of the cases $(1, n)$ of the Conjecture; the cases $(0, n)$ have a similar but slightly simpler proof which is easily derivable from the one here. That is, we prove

3.1 Theorem. *For any set C and any n, if C is 2-REA but not of r.e. degree, then there exists an n-r.e.[C] set A such that $A \oplus C$ is not of $(n+1)$-REA degree.*

Proof. Fix $n \geq 0$ and let C be a fixed 2-REA set, say $C = B \oplus V^B$ for r.e. sets V and B, such that C is not of r.e. degree. By Proposition 2.4 we must construct at least two n-r.e.[C] sets A_0 and A_1; in fact, for reasons which will become clear during the proof we construct $(n+2)$ such sets A_k for $k < n+2$. For each $k < n+2$ we attempt to ensure that $A_k \oplus C$ is not Turing equivalent to any $(n+1)$-REA set E_k^{n+1}, which is characterized in terms of r.e. sets $E_k^1, W_{k,1}, \ldots, W_{k,n}$ by setting for $1 \leq i \leq n$,

$$F_k^1 = E_k^1, \qquad F_k^{i+1} = W_{k,i}^{E_k^i}, \qquad E_k^{i+1} = E_k^i \oplus F_k^{i+1}.$$

Each A_k will be of the form

$$A_k = A_{k,0} \sim (A_{k,1} \sim (A_{k,2} \sim \cdots \sim (A_{k,n-2} \sim A_{k,n-1}) \cdots)),$$

where each $A_{k,j}$ is C-r.e. To guarantee that at least one such A_k is not of $(n+1)$-REA degree, we aim to satisfy all requirements of the form

$$(1) \qquad \bigvee_{k<n+2} \left[A_k \neq \Phi_k^{E_k^{n+1}} \quad \text{or} \quad \bigvee_{1 \leq i \leq n+1} \left[F_k^i \neq \Theta_{k,i}^{A_k \oplus C} \right] \right].$$

We consider first one such requirement. If it fails, then for all x,

$$\bigwedge_{k<n+2}\left[A_k \restriction x+1 = \Phi_k^{E_k^{n+1}} \restriction x+1 \qquad \text{and} \qquad \bigwedge_{1\le i\le n+1}\left[F_k^i \restriction w^i(x) = \Theta_{k,i}^{A_k\oplus C} \restriction w^i(x)\right]\right],$$

where $u^{n+1}(x)$ is the combined E_k^{n+1}-uses of the computations $\Phi_k^{E_k^{n+1}}(y)$ for $y \le x$ and $k < n+2$, and for $1 \le i \le n$, $u^i(x)$ and $w^i(x)$ are chosen minimal such that for all y and all $k < n+2$,

$$\begin{aligned} 2y < u^{i+1}(x) &\implies y < u^i(x) \\ 2y+1 < u^{i+1}(x) &\implies y < w^{i+1}(x) \\ y \in F_k^{i+1} \restriction w^{i+1}(x) &\implies (\exists z < u^i(x))\, \langle\, y, E_k^i \restriction z\,\rangle \in W_i \\ w^1(x) &= u^1(x). \end{aligned}$$

Let $v(x)$ be the maximum of $x+1$ and the A_k and C-uses for all computations for which $A_k \oplus C$ is an oracle.

As in the proof of Theorem 1.2 we shall construct each C-r.e. set $A_{k,j}$ as $U_{k,j}^C$. However, we note that it suffices for $U_{k,j}$ to be B-r.e. rather than outright r.e. This fact allow us to use B freely as an oracle during the construction; in particular we may assume that C is given via a B-recursive enumeration $\langle\, C_s : s \in \omega\,\rangle$ and in effect treat C as an r.e. set. We define c_s and the set of C-true stages relative to this enumeration as before.

Let $A_{k,j,s} = U_{k,j,s}^{C_s}$, $E_{k,s}^1$, $W_{k,i,s}$, and $\Theta_{k,i,s}^{A_{k,s}\oplus C_s}$ be the standard approximations, and for $1 \le i \le n$, set

$$F_{k,s}^i = \Theta_{k,i,s}^{A_{k,s}\oplus C_s} \qquad \text{and} \qquad E_{k,s}^{i+1} = E_{k,s}^i \oplus F_{k,s}^{i+1},$$

with the convention as before that when we write $F_{k,s}^i \restriction w$, we imply that $\Theta_{k,i,s}^{A_{k,s}\oplus C_s}(y)$ is defined for all $y < w$. Let $\phi(s,x)$ denote the condition that for all $k < n+2$

$$A_{k,s} \restriction x+1 = \Phi_{k,s}^{E_{k,s}^{n+1}} \restriction x+1,$$

and for $1 \le i \le n+1$ there exist $u^i(s,x)$, $v(s,x)$, and $w^i(s,x)$ all monotone non-decreasing in both arguments such that $u^{n+1}(s,x)$ is the combined $E_{k,s}^{n+1}$-use of these computations for $y \le x$, and for $k < n+2$ and all y,

$$\begin{aligned} 2y < u^{i+1}(s,x) &\implies y < u^i(s,x) \\ 2y+1 < u^{i+1}(s,x) &\implies y < w^{i+1}(s,x) \\ y \in F_{k,s}^{i+1} \restriction w^{i+1}(s,x) &\iff (\exists z < u^i(s,x))\, [\langle\, y, E_{k,s}^i \restriction z\,\rangle \in W_{k,i,s}] \\ F_{k,s}^1 \restriction w^1(s,x) &= E_{k,s}^1 \restriction w^1(s,x) \\ w^1(s,x) &= u^1(s,x), \end{aligned}$$

and $v(s,x)$ is the maximum of $x+1$ and the combined $A_{k,s}$ and C_s-uses for all $k < n+2$ and all computations for which $A_{k,s} \oplus C_s$ is an oracle. We say that $\phi(s,x)$ holds ***correctly*** iff $\phi(s,x)$ holds and for $k < n+2$ and $1 \le i \le n+1$, $E_{k,s}^i \restriction u^i(s,x) = E_k^i \restriction u^i(s,x)$ and $A_{k,s} \restriction v(s,x) = A_k \restriction v(s,x)$.

The Recognition and Positive Change properties may be proved exactly as before:

(1.1) (***Recognition***) if (1) fails, then for all x and all sufficiently large C-true stages s, $\phi(s,x)$ holds correctly;

(1.2) (***Positive Change***) for all x, all $k < n+2$, and all $s < t$ such that both $\phi(s,x)$ and $\phi(t,x)$,

(a) for $1 \le i \le n$,

$$E^i_{k,s} \restriction u^i(s,x) = E^i_{k,t} \restriction u^i(s,x) \implies F^{i+1}_{k,s} \restriction w^{i+1}(s,x) \subseteq F^{i+1}_{k,t} \restriction w^{i+1}(s,x);$$

(b) for $1 \le i \le n+1$,

$$E^i_{k,s} \restriction u^i(s,x) \neq E^i_{k,t} \restriction u^i(s,x) \implies \text{for some } 1 \le j \le i, \quad F^j_{k,s} \restriction w^j(s,x) \subset F^j_{k,t} \restriction w^j(s,x).$$

Before giving the formal construction of sets A_k which satisfy a single instance of (1) we describe it informally. As in the proof of Theorem 1.2 we use a potentially infinite sequence $\langle x_l : l \in \omega \rangle$ of witnesses; x_l is a witness for $A_{\pi(l)}$, where $\pi(l) = l \bmod (n+2)$. For this sketch let us assume that $n = 2$ so that we are constructing 2-r.e[C] sets $A_k = A_{k,0} \sim A_{k,1}$ $(k < 4)$ with the goal that for at least one k, $A_k \oplus C$ is not of 3-REA degree — in particular, $A_k \oplus C$ is not Turing equivalent to E^3_k. Our overall strategy is to show that if (1) fails, then C is Turing equivalent to $B \oplus E^1_0 \oplus \cdots \oplus E^1_3$ and is hence of r.e. degree, contrary to hypothesis.

Let $x_0 = 0$ and wait for a stage s_0 such that $\phi(s_0, x_0)$ holds. In contrast to the proof of Theorem 1.2 we immediately enumerate x_0 into $A_{0,0,s_0+1}$ with use $v_0 = v(s_0, x_0)$ and as before choose $x_1 > v_0$, restrain $A_0 \restriction v_0$, and wait for s_1 such that $\phi(s_1, x_1)$. We again make the (untenable) assuption that for each l, $C_{s_l} \restriction v_l = C \restriction v_l$, or more generally that this is true for some eventual s_l for each x_l. At stage s_1 we enumerate x_1 into $A_{1,0,s_1+1}$ with use v_1 but we do *not* enumerate x_0 into $A_{0,1}$; this delay corresponds to the fact that in the proof of Theorem 1.2 we made no enumeration at stage s_0. At stage s_2 where $\phi(s_2, x_2)$ holds with use $v_2 = v(s_2, x_2)$ we enumerate x_0 into $A_{0,1,s_2+1}$ and x_2 into $A_{2,0,s_2+1}$, each with use v_2. At stage s_3 where $\phi(s_3, x_3)$ holds we enumerate x_1 into $A_{1,1,s_3+1}$ and x_3 into $A_{3,0,s_3+1}$ with use v_3. Finally, at stage s_4 we return to A_0 by enumerating x_4 into $A_{0,0,s_4+1}$.

Let $u^i_j = u^i(s_j, x_j)$. As in the proof of Theorem 2.1, $E^3_{0,s_0} \restriction u^3_0 \neq E^3_{0,s_1} \restriction u^3_0$. By the Positive Change property (1.2)(b), for some $1 \le h \le 3$, $F^h_{0,s_0} \restriction w^h_0 \subset F^h_{0,s_1} \restriction w^h_0$. Since x_0 is removed from A_0 at stage $s_2 + 1$, we have $A_{0,s_3} \restriction v_0 = A_{0,s_0} \restriction v_0$ and by $\phi(s_3, x_0)$,

$$F^h_{0,s_0} \restriction w^h_0 = \Theta^{A_{0,s_0}}_{0,h,s_0} \restriction w^h_0 = \Theta^{A_{0,s_3}}_{0,h,s_3} \restriction w^h_0 = F^h_{0,s_3} \restriction w^h_0,$$

so

$$F^h_{0,s_1} \restriction w^h_0 \not\subseteq F^h_{0,s_3} \restriction w^h_0, \qquad \text{whence also} \qquad F^h_{0,s_1} \restriction w^h_1 \not\subseteq F^h_{0,s_3} \restriction w^h_1,$$

and hence by Positive Change (1.2)(a), $E^2_{0,s_1} \restriction u^2_1 \neq E^2_{0,s_3} \restriction u^2_1$. On the other hand, since we made no enumeration into A_0 at stage s_1, $A_{0,s_1} \restriction v_1 = A_{0,s_2} \restriction v_1$ so $E^2_{0,s_1} \restriction u^2_1 = E^2_{0,s_2} \restriction u^2_1$.

By analogy with Theorem 1.2, we hope to use this information to compute s_2 from s_1 by looking for changes in $E^2_0 \restriction u^2_1$. However, E^2_0 is 2-REA rather than r.e. and it is consistent with the above conditions that there are changes between s_1 and s_2 which are undone at

s_2. There can be no such change in the r.e. part $E_0^1 \mid u_1^1$ and any such change in $F_0^2 \mid w_1^2$ at a stage $s_1 < t < s_2$ must consist in the acquisition of some new elements with axioms satisfied by a value of $E_{0,t}^1 \mid u^1(t, x_1)$ which changes by stage s_2 to remove these elements. In particular, since no correct initial segment of an approximation to an r.e. set ever changes,

$$E_{0,t}^1 \mid u^1(t, x_1) \neq E_0^1 \mid u^1(t, x_1).$$

Thus, if we set

$$t_0 = \text{least } t > s_1[\phi(t, x_1) \wedge E_{0,t}^2 \mid u_1^2 \neq E_{0,s_1}^2 \mid u_1^2 \wedge E_{0,t}^1 \mid u^1(t, x_1) = E_0^1 \mid u^1(t, x_1)],$$

then if t_0 exists, it is greater than s_2 and we can compute s_2 from s_1 by

$$s_2 = \text{least } s < t_0[\phi(s, x_2) \wedge C_s \mid v(s, x_2) = C_{t_0} \mid v(s, x_2)].$$

This computation uses oracle $B \oplus E_0^1$; extending this idea gives $C \leq_T B \oplus E_0^1 \oplus \cdots \oplus E_3^1$.

It remains to argue that t_0 does exist and that $B \oplus E_0^1$ (and analogously $B \oplus E_0^1 \oplus \cdots \oplus E_3^1$) is recursive in C. Both of these arguments depend on the device of constructing four (in general $n + 2$) rather than only two sets A_k. Of course, $B \leq_T C$ by assumption. The key claim is that $E_{0,s_3}^1 \mid u_3^1 = E_0^1 \mid u_3^1$. On the one hand, this guarantees that $t_0 \leq s_3$ and hence exists. On the other, since the function $l \mapsto s_l$ is C-recursive, this equation describes (the beginning of) an algorithm for computing E_0^1 from C

The claim follows from the fact that for all $t \geq s_3$, $A_{0.s_3} \mid v_3 = A_{0,t} \mid v_3$. This is true because the next witness after x_0 which is used for A_0 is x_4, which is chosen $> v_3$, so no number $\leq v_3$ is ever again added to A_0. Of course, the restraint imposed ensures that no other requirement does this either, at least on the true path. The claim now follows since for all $l \geq 3$, by $\phi(s_l, x_l)$,

$$E_{0,s_l}^1 \mid u_3^1 = \Theta_{0,1}^{A_{0,s_l} \oplus C_{s_l}} \mid u_3^1 = \Theta_{0,1}^{A_{0,s_3} \oplus C_{s_3}} \mid u_3^1 = E_{0,s_3}^1 \mid u_3^1,$$

and the fact that the approximations $E_{0,s}^1$ converge to E_0^1.

The remainder of the notational bookkeeping is very similar to that of Theorem 1.2. We maintain a function ρ such that if $\rho(s) = l$, then there are defined witnesses $x_j(s)$ for $j \leq l$ and uses $u_j^i(s)$, $v_j(s)$, and $w_j^i(s)$ for $j < l$. The proof that (1) is satisfied follows from

(1.3) if (1) fails, then $\liminf_s \rho(s) = \infty$;

(1.4) if $\liminf_s \rho(s) = \infty$, then $C \equiv_T B \oplus E_0^1 \oplus \cdots \oplus E_{n-1}^1$; hence, C is of r.e. degree, contrary to hypothesis.

We set

$$s_l = \text{least } s\, (\forall t > s)[\rho(t) \geq l + 1].$$

By the construction, if s_l exists, then $C_{s_l} \mid v_l = C \mid v_l$, so if all s_l exist, then C is recursive in the function $l \mapsto s_l$.

Set $\rho(0) = 0$ and $x_0(0) = 0$. At stage $s + 1$ we have one of three cases as follows. Any parameter not mentioned is assigned the same value at $s + 1$ as at s.

Case 1. If $\rho(s) > 0$ and for some (least) $h < \rho(s)$, $c_{s+1} < v_h(s)$, then $\rho(s+1) := h$; $u^i_j(s+1)$, $v_j(s+1)$, and $w^i_j(s+1)$ are undefined for all $j \geq h$; and $x_j(s+1)$ is undefined for all $j > h$;

Case 2. otherwise, if $\rho(s) = l$ and $\phi(s, x_l(s))$ holds, with associated uses $u^i(s, x_l(s))$, $w^i(s, x_l(s))$, and $v(s, x_l(s)) \leq c_{s+1}$, then

(a) $\rho(s+1) := l+1$, $u^i_l(s+1) = u^i(s, x_l(s))$, $w^i_l(s+1) = w^i(s, x_l(s))$, and $v_l(s+1) = v(s, x_l(s))$;

(b) for $j < \min\{l, n-1\}$, x_{l-j} is enumerated into $A_{\pi(l-j),j,s+1}$ with use $v_l(s+1)$;

(c) if $l \geq n$, then x_{l-n} is enumerated into $A_{\pi(l-n),n-1,s+1}$ with use $v_l(s+1)$;

(d) $x_{l+1}(s+1)$ is chosen greater than any number used so far;

Case 3. otherwise, $\rho(s+1) := \rho(s)$.

The proof now breaks into a series of lemmas as before. In many cases the proofs are straightforward adaptations of the corresponding parts of Theorem 1.2 and we omit them.

(2) For all s, h, $1 \leq i \leq n$, and j,

(a) $x_j(s)$ is defined for (exactly) $j \leq \rho(s)$; $u^i_j(s)$, $v_j(s)$, and $w^i_j(s)$ are defined for (exactly) $j \leq \rho(s) - 1$;

(b) $h < j \leq \rho(s) \implies v_h(s) \leq x_j(s)$;

(c) $j < \rho(s+1) \implies v_j(s) \leq c_{s+1}$;

(d) for $j < (n-1)$ and $h < \rho(s) - j$, $x_h(s) \in A_{\pi(h),j,s}$ with use $\leq c_s$, but for $\rho(s) - j \leq h \leq \rho(s)$, $x_h \notin A_{\pi(h),j,s}$;

(e) for $h < \rho(s) - n$, $x_h(s) \in A_{\pi(h),n-1,s}$ with use $\leq c_s$, but for $\rho(s) - n \leq h \leq \rho(s)$, $x_h(s) \notin A_{\pi(h),n-1,s}$;

(f) for all $t < s$, if $x_h(t)$ is defined but $x_h(s)$ is either undefined or $\neq x_h(t)$, then for all $s' \geq s$ and $j < n$, $x_h(t) \notin A_{\pi(h),j,s'}$. In particular, this holds when $\rho(s) < h \leq \rho(t)$.

Proof. For (f), suppose that $x_h(t)$ is enumerated into $A_{\pi(h),j,t}$ with use $v_{h+j}(t)$ (for $j < n-1$) or $v_{h+j+1}(t)$ (for $j = n-1$). In either of the cases of the hypothesis, for some t' with $t \leq t' < s$, x_h becomes undefined at stage $t'+1$ because $c_{t'+1} < v_h(t')$. If $v_{h+j(+1)}(t') = v_{h+j(+1)}(t)$, then also $c_{t'+1} < v_{h+j(+1)}(t)$, so $x_h(t)$ is removed from $A_{\pi(h),j,t'+1}$; otherwise it was already removed at an earlier stage. Now the number $x_h(t)$ will never again be used as a witness so it is never again enumerated into $A_{\pi(h),j}$. □

(3) For each $l \geq 0$, if s_l exists, then

(a) $\rho(s_l) = l$, $\phi(s_l, x_l)$, and $\rho(s_l + 1) = l + 1$;

(b) for all $s > s_l$, $\rho(s) \geq l + 1$;

(c) for all $j \leq l$, $x_{j+1}(t)$, $u^i_j(t)$, $w^i_j(t)$, and $v_j(t)$ have the same values for all $t > s_l$ (which we denote by x_{j+1}, u^i_j, w^i_j, and v_j);

(d) $C_{s_l} \mid v_l = C \mid v_l$;

(e) for all $s \geq s_l$,

(i) for all $j < (n-1)$ and $h \leq (l-j-1)$, $\quad x_h \in A_{\pi(h),j,s}$;

(ii) for all $h \leq (l-n-1)$, $\quad x_h \in A_{\pi(h),n-1,s}$;

(f) for all $s > s_l$,

(i) for all $j < (n-1)$, $\quad x_{l-j} \notin A_{\pi(l-j),j,s_l}$, $\quad$ but $\quad x_{l-j} \in A_{\pi(l-j),j,s}$;

(ii) $x_{l-n} \notin A_{\pi(l-n),n-1,s_l}$, $\quad$ but $\quad x_{l-n} \in A_{\pi(l-n),n-1,s}$.

(4) For any l and s, if $\rho(s) \geq l+1$ and $C_s \mid v_l(s) = C \mid v_l(s)$, then s_l exists and $s > s_l$.

(5) For each $l \geq 0$, if s_{l+n+1} exists, then

(a) for all $j < (n-2)$, $\quad A_{\pi(l),s_{l+j}} \mid v_{l+j} = A_{\pi(l),s_{l+j+2}} \mid v_{l+j}$;

(b) $A_{\pi(l),s_{l+n-2}} \mid v_{l+n-2} = A_{\pi(l),s_{l+n+1}} \mid v_{l+n-2}$;

(c) $A_{\pi(l),s_{l+n-1}} \mid v_{l+n-1} = A_{\pi(l),s_{l+n}} \mid v_{l+n-1}$.

Proof. Consider first (a) and fix $j < (n-2)$. Much as in the corresponding part of the proof of Theorem 1.2 we conclude that for each h and $t \leq s_{l+j+2}$, with the possible exception of $x_h := x_h(s_l)$ for $h \leq l+j$ and $h \equiv l \pmod{n+2}$, we have for all $g < n$,

$$A_{\pi(l),g,s_{l+j}}(x_h(t)) = A_{\pi(l),g,s_{l+j+2}}(x_h(t)).$$

Then by (3)(e) and (f), for such h,

$$\begin{gathered} h \leq l-(n+2) \quad \Longrightarrow \quad (\forall g < n)(\forall s \geq s_{l+j}) x_h \in A_{\pi(l),g,s}; \\ x_l \in A_{\pi(l),g,s_{l+j}} \quad \text{for exactly } g = 0,1,\ldots,j-1; \\ x_l \in A_{\pi(l),g,s_{l+j+2}} \quad \text{for exactly } g = 0,1,\ldots,j+1. \end{gathered}$$

Thus, the only permanent change in $A_{\pi(l)}$ between stage s_{l+j} and stage s_{l+j+2} is the enumeration of x_l into both $A_{\pi(l),j}$ and $A_{\pi(l),j+1}$, which has no net effect. The proof of (b) is similar; between s_{l+n-2} and s_{l+n+1}, x_l is enumerated into $A_{\pi(l),n-2}$ and $A_{\pi(l),n-1}$. For (c), we may show as above that there is no permanent change in any $A_{\pi(l),g}$ between s_{l+n-1} and s_{l+n}. $\square$

(6) For each $l > 0$,

(a) if s_{l+n+1} exists, then for all $j \leq (n-2)$,

$$E^{n-j+1}_{\pi(l),s_{l+j}} \mid u^{n-j+1}_{l+j} \neq E^{n-j+1}_{\pi(l),s_{l+j+1}} \mid u^{n-j+1}_{l+j};$$

(b) if s_{l+n} exists, then $E^2_{\pi(l),s_{l+n-1}} \mid u^2_{l+n-1} = E^2_{\pi(l),s_{l+n}} \mid u^2_{l+n-1}$;

(c) if s_{l+j} exists for all j, then $E^1_{\pi(l),s_{l+n+1}} \mid u^1_{l+n+1} = E^1_{\pi(l)} \mid u^1_{l+n+1}$;

(d) if s_{l+j} exists for all j, then

$$\begin{aligned} \exists t > s_{l+n-1}[\phi(t, x_{l+n-1}) \wedge & E^2_{\pi(l),t} \mid u^2_{l+n-1} \neq E^2_{\pi(l),s_{l+n-1}} \mid u^2_{l+n-1} \\ & \wedge E^1_{\pi(l),t} \mid u^1(t, x_{l+n-1}) = E^1_{\pi(l)} \mid u^1(t, x_{l+n-1})], \end{aligned}$$

and if t_l denotes the least such t, then $s_{l+n} < t_l$.

Proof. The proof of part (a) is similar to that of the corresponding part in the proof of Theorem 2.1. Using (4) and (5)(a) show first that for $i \leq n$ and $j < (n-2)$,

$$E^{i+1}_{\pi(l),s_{l+j}} \mid u^{i+1}_{l+j} \neq E^{i+1}_{\pi(l),s_{l+j+1}} \mid u^{i+1}_{l+j} \implies E^{i}_{\pi(l),s_{l+j+1}} \mid u^{i}_{l+j+1} \neq E^{i}_{\pi(l),s_{l+j+2}} \mid u^{i}_{l+j+1}.$$

Then (a) follows by induction on j. Part (b) follows from (5)(c). Part (c) is proved as in the sketch: for all $j \geq (n+1)$, $A_{\pi(l),s_{l+n+1}} \mid v_{l+n+1} = A_{\pi(l),s_{l+j}} \mid v_{l+n+1}$, so

$$\begin{aligned} E^{1}_{\pi(l),s_{l+j}} \mid u^{1}_{l+n+1} &= \Theta^{A_{\pi(l),s_{l+j}} \oplus C_{s_{l+j}}}_{\pi(l),1,s_{l+j}} \mid u^{1}_{l+n+1} \\ &= \Theta^{A_{\pi(l),s_{l+n+1}} \oplus C_{s_{l+n+1}}}_{\pi(l),1,s_{l+n+1}} \mid u^{1}_{l+n+1} = E^{1}_{\pi(l),s_{l+n+1}} \mid u^{1}_{l+n+1}. \end{aligned}$$

For (d), we have first by (a) for $j = (n-2)$,

$$E^{3}_{\pi(l),s_{l+n-2}} \mid u^{3}_{l+n-2} \neq E^{3}_{\pi(l),s_{l+n-1}} \mid u^{3}_{l+n-2},$$

so by (1.2)(b), for some $1 \leq h \leq 3$,

$$F^{h}_{\pi(l),s_{l+n-2}} \mid w^{3}_{l+n-2} \subset F^{h}_{\pi(l),s_{l+n-1}} \mid w^{3}_{l+n-2}.$$

On the other hand, by (5)(b),

$$F^{h}_{\pi(l),s_{l+n-2}} \mid w^{3}_{l+n-2} = F^{h}_{\pi(l),s_{l+n+1}} \mid w^{3}_{l+n-2},$$

whence as before

$$E^{2}_{\pi(l),s_{l+n-1}} \mid u^{2}_{l+n-1} \neq E^{2}_{\pi(l),s_{l+n+1}} \mid u^{2}_{l+n-1}.$$

Thus by (c), $t = s_{l+n+1}$ satisfies the condition in square brackets and t_l exists.

Towards a contradiction, suppose that $t_l \leq s_{l+n}$. If

$$E^{1}_{\pi(l),t_l} \mid u^{1}_{l+n-1} \neq E^{1}_{\pi(l),s_{l+n-1}} \mid u^{1}_{l+n-1},$$

then also $E^{1}_{\pi(l),s_{l+n}} \mid u^{1}_{l+n-1} \neq E^{1}_{\pi(l),s_{l+n-1}} \mid u^{1}_{l+n-1}$, contrary to (b). Hence the $E^{1}_{\pi(l)}$ parts agree and by (1.2)(a),

$$F^{2}_{\pi(l),s_{l+n-1}} \mid w^{2}_{l+n-1} \subset F^{2}_{\pi(l),t_l} \mid w^{2}_{l+n-1}.$$

But by the third condition of the definition of t_l,

$$E^{1}_{\pi(l),t_l} \mid u^{1}_{l+n-1} = E^{1}_{\pi(l),s_{l+n}} \mid u^{1}_{l+n-1},$$

so also $F^{2}_{\pi(l),s_{l+n-1}} \mid w^{2}_{l+n-1} \subset F^{2}_{\pi(l),s_{l+n}} \mid w^{2}_{l+n-1}$, contrary to (b). $\square$

Now (1.3) is proved exactly as for Theorem 1.2 and it remains to show

(1.4) Only finitely many s_l exist; in other words, $\liminf_s \rho(s) < \infty$.

Proof. Suppose to the contrary that all s_l exist. Then clearly the function $l \mapsto s_l$ is recursive in C, so by 6(c) we have $B \oplus E^1_0 \oplus \cdots \oplus E^1_{n+1} \leq_T C$. By (4),

$$s_{l+n} = \text{least } s < t_l \left[\phi(s, x_{l+n}) \wedge C_s \mid v(s, x_{l+n}) = C_{t_l} \mid v(s, x_{l+n})\right],$$

so s_{l+n} can be calculated from t_l and hence by 6(d) from s_{l+n-1} using $E^1_{\pi(l)}$ as an oracle. It follows that the function $l \mapsto s_l$ and hence C is recursive in $B \oplus E^1_0 \oplus \cdots \oplus E^1_{n+1}$. Thus $C \equiv_T B \oplus E^1_0 \oplus \cdots \oplus E^1_{n+1}$ so is of r.e. degree, contrary to hypothesis. □

In combining the strategies for all the requirements (1), there are several new problems, and we must modify the basic module. We use a priority tree $T = \omega^{<\omega}$ and assign to each $\alpha \in T$ of length e a strategy σ_α for satisfying the e-th requirement $(1)_e$ in some fixed listing. The nodes are ordered in the usual way:

$$\alpha \leq \beta \quad \Longleftrightarrow \quad \alpha \subseteq \beta \vee \exists e\, [\alpha \mid e = \beta \mid e \wedge \alpha(e) < \beta(e)].$$

For each α which is ***active*** (to be defined in the construction) at stage s we will define numbers $\rho_\alpha(s)$, $x_{\alpha\frown\langle j\rangle}(s)$ for $j \leq \rho_\alpha(s)$, and $u^i_{\alpha\frown\langle j\rangle}(s)$, $v_{\alpha\frown\langle j\rangle}(s)$, and $w^i_{\alpha\frown\langle j\rangle}(s)$ for $j < \rho_\alpha(s)$ and $1 \leq i \leq n$, which will play the same role in the action of σ_α as their counterparts do above. Strategy σ_α acts under the assumption that for all $e < |\alpha|$, $\sigma_{\alpha|e}$ has ***outcome*** $\alpha(e)$ — that is, $\liminf_s \rho_{\alpha|e}(s) = \alpha(e)$ — and only at those stages, called α-stages, when this assumption is predicted by the evidence gathered to that point in the construction. At each stage s we compute a sequence $\tau(s) \in T$ with $|\tau(s)| = s$ which represents our current prediction of the eventual outcome of the first s-many requirements. At stage $s+1$, if $\alpha = \tau(s+1) \mid e$, then $\tau(s+1)(e)$ will be the value of $\rho_\alpha(s+1)$ computed essentially as above: either $\rho_\alpha(s+1) = h < \rho_\alpha(s)$ because $c_{s+1} < v_{\alpha\frown\langle h\rangle}(s)$ for some (least) $h < \rho_\alpha(s)$, or $\rho_\alpha(s+1) = \rho_\alpha(s) + 1$ because a new agreement is verified at s, or neither of these holds and $\rho_\alpha(s+1) = \rho_\alpha(s)$. A stage s is an α-***stage*** iff $\alpha \subseteq \tau(s)$. Then we shall show that these values determine a ***true path*** f defined by

$$(*) \qquad f(e) = \liminf_s \{\, \tau(s)(e) : \tau(s) \mid e = f \mid e \,\},$$

and that $\sigma_{f|e}$ satisfies the e-th requirement. More precisely, we will prove the following two assertions.

(1.5) If C is not of r.e. degree, then there exists a path f through T such that for all e, there exists $\bar{s}$ such that if α denotes $f \mid e$, then for all $s \geq \bar{s}$,

(a) $\bar{s}$ is an α-stage;

(b) $\alpha \leq \tau(s)$ and α is active at s;

(c) if s is an α-stage, then for all $\beta \leq \alpha$, $x_\beta(s)$ is defined iff $x_\beta(\bar{s})$ is defined, in which case $x_\beta(s) = x_\beta(\bar{s})$ (which we denote by x_β), and for $k < (n+2)$ and all t, $A_{k,s}(x_\beta(t)) = A_{k,\bar{s}}(x_\beta(t))$;

(d) if s is C-true, then s is an α-stage.

Note that (b) and (d) together strengthen $(*)$.

(1.6) For all e, if $(1)_e$ fails, then $\liminf_s \rho_{f|e}(s) = \infty$.

As is usual in tree arguments, we need to take care that any actions taken at nodes $\beta < \tau(s+1)$ are preserved at stage $s+1$. Threats to such actions are of two sorts: (i) new enumerations of elements into some A_k below the use $v_\beta(s)$ of the agreement established at some earlier stage at β, and (ii) changes in C which cause elements to be removed from some A_k. Problem (i) will be handled simply by choosing witnesses large enough to ensure that

$$x_\alpha(s) \geq \max \left\{ v_\beta(s) : \beta <_L \alpha \wedge v_\beta(s) \text{ is defined} \right\}.$$

We note that $v_\beta(s+1)$ is undefined for $\beta \subseteq \tau(s+1)$.

Problem (ii) will be handled by ensuring that the uses of elements enumerated into A_k at node α and stage $s+1$, which we shall denote by $p_\alpha(s+1)$, are chosen (possibly larger than $v_\alpha(s)$) in such a way that if $c_{s+1} < p_\alpha(s)$, then $\tau(s+1) \leq \alpha$. These two facts are stated formally as (2)(b) and (c) below.

The full construction now goes as follows. At stage 0 only $\emptyset$ is active, $\rho_\emptyset(0) = 0$, and $x_{\langle 0 \rangle}(0) = 0$. At stage $s+1$ we proceed by induction on $e \leq s$. Any parameter not mentioned is assigned the same value at $s+1$ as at s. Let $\alpha = \tau(s+1) \mid e$; there are four cases. We write ϕ_e for the version of ϕ corresponding to $(1)_e$, but to prevent further degradation of readability, we will not attach this subscript to any of the other parameters of $(1)_e$.

Case 1. If α was not active at stage s, then set $\rho_\alpha(s+1) := 0$ and choose $x_{\alpha^\frown\langle 0 \rangle}(s+1)$ greater than any number used so far;

Case 2. If $\rho_\alpha(s) > 0$ and for some (least) $j < \rho_\alpha(s)$, $c_{s+1} < p_{\alpha^\frown\langle j \rangle}(s)$, then $\rho_\alpha(s+1) := j$; $p_\beta(s+1)$, $u^i_\beta(s+1)$, $v_\beta(s+1)$ and $w^i_\beta(s+1)$ $(1 \leq i \leq n)$ are undefined for $\beta = \alpha^\frown\langle j \rangle$ and all $\beta \geq \alpha^\frown\langle j+1 \rangle$; $x_\beta(s+1)$ is undefined for all $\beta \geq \alpha^\frown\langle j+1 \rangle$;

Case 3. otherwise, if $\rho_\alpha(s) = l$, and $\phi_e(s, x_{\alpha^\frown\langle l \rangle}(s))$ holds with associated uses $u^i(s, x_{\alpha^\frown\langle l \rangle}(s))$, $w^i(s, x_{\alpha^\frown\langle l \rangle}(s))$ and $v(s, x_{\alpha^\frown\langle l \rangle}(s)) \leq c_{s+1}$, then

(a) $\rho_\alpha(s+1) := l+1$, $u^i_{\alpha^\frown\langle l \rangle}(s+1) = u^i(s, x_{\alpha^\frown\langle l \rangle}(s))$, $v_{\alpha^\frown\langle l \rangle}(s+1) = v(s, x_{\alpha^\frown\langle l \rangle}(s))$, $w^i_{\alpha^\frown\langle l \rangle}(s+1) = w^i(s, x_{\alpha^\frown\langle l \rangle}(s))$, and

$$p_{\alpha^\frown\langle l \rangle}(s+1) := \max \left\{ v_{\alpha^\frown\langle l \rangle}(s+1) \right\} \cup \left\{ p_\beta(s) : \alpha^\frown\langle l \rangle \subseteq \beta \text{ and } p_\beta(s) \text{ is defined} \right\}$$

(b) for $j < \min\{l, n-1\}$, $x_{\alpha^\frown\langle l-j \rangle}(s)$ is enumerated into $A_{\pi(l-j),j,s+1}$ with use $p_{\alpha^\frown\langle l \rangle}(s+1)$;

(c) if $l \geq n$, then $x_{a^\frown\langle l-n \rangle}(s+1)$ is enumerated into $A_{\pi(l-n),n-1,s+1}$ with use $p_{\alpha^\frown\langle l \rangle}(s+1)$;

(d) $x_{\alpha^\frown\langle l+1 \rangle}$ is chosen greater than any number used so far;

Case 4. otherwise $\rho_\alpha(s+1) = \rho_\alpha(s)$.

Set $\tau(s+1)(e) = \rho_\alpha(s+1)$. A node α is ***active*** at stage $s+1$ iff either $\alpha \subseteq \tau(s+1)$ or $\alpha <_L \tau(s+1)$ and α was active at stage s. In particular, if $\tau(s+1) <_L \alpha$, then α is inactive at stage $s+1$. If $\beta = \gamma^\frown\langle j \rangle$, we write $\pi(\beta)$ for $\pi(j)$ and β^+ for $\gamma^\frown\langle j+1 \rangle$. We have first:

(2) All parameters corresponding to inactive nodes are undefined. For all s, all α active at s, and all β, $1 \le i \le n$, and j,

(a) $x_{\alpha^\frown\langle j\rangle}(s)$ is defined for (exactly) $j \le \rho_\alpha(s)$; $p_{\alpha^\frown\langle j\rangle}(s)$, $u^i_{\alpha^\frown\langle j\rangle}(s)$, $v_{\alpha^\frown\langle j\rangle}(s)$, and $w^i_{\alpha^\frown\langle j\rangle}(s)$ are defined for (exactly) all $j \le \rho_\alpha(s) - 1$;

(b) $\beta <_L \alpha \implies p_\beta(s) \le x_\alpha(s)$ (when both are defined);

(c) $\beta <_L \tau(s+1) \implies p_\beta(s) \le c_{s+1}$;

(d) for $j < (n-1)$ and $h < \rho_\alpha(s) - j$, $x_{\alpha^\frown\langle h\rangle}(s) \in A_{\pi(h),j,s}$ with use $\le c_s$, but for $\rho_\alpha(s) - j \le h \le \rho_\alpha(s)$, $x_{\alpha^\frown\langle h\rangle} \notin A_{\pi(h),j,s}$;

(e) for $h < \rho_\alpha(s) - n$, $x_{\alpha^\frown\langle h\rangle}(s) \in A_{\pi(h),n-1,s}$ with use $\le c_s$, but for $\rho_\alpha(s) - n \le h \le \rho_\alpha(s)$, $x_{\alpha^\frown\langle h\rangle}(s) \notin A_{\pi(h),n-1,s}$;

(f) for all $t < s$, if $x_\beta(t)$ is defined but $x_\beta(s)$ is either undefined or $\ne x_\beta(t)$, then for all $s' \ge s$ and $j < n$, $x_\beta(t) \notin A_{\pi(\beta),j,s'}$. In particular, this holds when $x_\beta(t)$ is defined and $\tau(s) <_L \beta$.

Proof. For (a), by induction, note that for $\alpha <_L \tau(s+1)$, no changes are made in any of the α-parameters at $s+1$, while for $\alpha \subseteq \tau(s+1)$, the construction ensures exactly these definitions. For (b), if $\beta <_L \alpha$, then $p_\beta(s) = p_\beta(t)$ for some β-stage $t < s$ for which $x_\alpha(t)$ is undefined. Hence when $x_\alpha(s)$ is defined, it is chosen larger than $p_\beta(s)$.

Next note that when both are defined we have

$$(*) \qquad \alpha \subseteq \beta \quad \implies \quad p_\beta(s) \le p_\alpha(s).$$

This inequality is true at the stage s at which $p_\alpha(s)$ is defined by Case 2. Any change in the relevant p_β occurs only at an α-stage t for which $p_\alpha(t)$ is undefined, so the change will be incorporated into the next definition of p_α, if any. Now, for (c), suppose that $\beta <_L \tau(s+1)$, so for some γ and $i \le k$, $\gamma^\frown\langle i\rangle \subseteq \beta$ and $\gamma^\frown\langle k+1\rangle \subseteq \tau(s+1)$. Then by the construction and $(*)$,

$$p_\beta(s) \le p_{\gamma^\frown\langle i\rangle}(s) \le p_{\gamma^\frown\langle k\rangle}(s) \le c_{s+1}.$$

Parts (d) and (e) hold as in the single requirement case using (c). For (f), suppose that $x_\beta(t)$ is defined and is enumerated into to $A_{\pi(\beta),j,t}$ with use $p_\beta(t)$ (or $p_{\beta^+}(t)$) but becomes undefined at some $t'+1$ with $t \le t' < s$. Then by the construction, there exists some γ and $h' < h$ such that $\gamma^\frown\langle h'\rangle \subseteq \tau(t')$, $\gamma^\frown\langle h\rangle \subseteq \beta$ and

$$c_{t'+1} < p_{\gamma^\frown\langle h'\rangle}(t') \le p_{\gamma^\frown\langle h-1\rangle}(t') = p_{\gamma^\frown\langle h-1\rangle}(t) \le x_\beta(t) \le p_\beta(t) \le p_{\beta^+}(t),$$

so $x_\beta(t)$ is removed from $A_{\pi(\beta),j}$ at stage $t'+1$, unless this already happened at an earlier stage. □

We begin now the proof of (1.5). Suppose that $\alpha = f \mid e$ has been defined to satisfy (1.5); we aim to calculate $f(e)$ so that $f \mid (e+1)$ also satisfies (1.5). By (1.5)(b), for all $s \ge \bar{s}$, $\rho_\alpha(s)$ is defined. For each $l \ge -1$ for which it exists, let

$$s_{\alpha^\frown\langle l\rangle} = \text{least } s \ge \bar{s}\ (\forall t > s)[\rho_\alpha(t) \ge l+1].$$

Clearly when it exists, $s_{\alpha^\frown\langle l\rangle}$ is an α-stage. The proof follows closely the pattern of the proof of (1.4) above and we use the same numbering for the lemmas.

(3) For each $l \geq 0$, if $s_{\alpha^\frown\langle l\rangle}$ exists, then,

(a) $\rho_\alpha(s_{\alpha^\frown\langle l\rangle}) = l, \quad \phi_e(s_{\alpha^\frown\langle l\rangle}, x_{\alpha^\frown\langle l\rangle}), \quad$ and $\quad \rho_\alpha(s_{\alpha^\frown\langle l\rangle}+1) = l+1$;

(b) for all $s > s_{\alpha^\frown\langle l\rangle}, \quad \rho_\alpha(s) \geq l+1$, $\alpha^\frown\langle l+1\rangle \leq \tau(s)$ and $\alpha^\frown\langle l+1\rangle$ is active at stage s;

(c) for all $j \leq l$, $u^i_{\alpha^\frown\langle j\rangle}(t)$, $v_{\alpha^\frown\langle j\rangle}(t)$, $w^i_{\alpha^\frown\langle j\rangle}(t)$, $p_{\alpha^\frown\langle j\rangle}(t)$, and $x_{\alpha^\frown\langle j+1\rangle}(t)$ have the same values for all $t > s_{\alpha^\frown\langle k\rangle}$ (which we denote by $u^i_{\alpha^\frown\langle j\rangle}$, $v_{\alpha^\frown\langle j\rangle}$, $w^i_{\alpha^\frown\langle j\rangle}$, $p_{\alpha^\frown\langle j\rangle}$ and $x_{\alpha^\frown\langle j+1\rangle}$);

(d) $C_{s_{\alpha^\frown\langle l\rangle}} \mid p_{\alpha^\frown\langle l\rangle} = C \mid p_{\alpha^\frown\langle l\rangle}$;

(e) for all $s \geq s_{\alpha^\frown\langle l\rangle}$,

(i) for all $j < n-1$ and all $h \leq (l-j-1), \quad x_{\alpha^\frown\langle h\rangle} \in A_{\pi(h),j,s}$;

(ii) for all $h \leq (l-n-1), \quad x_{\alpha^\frown\langle h\rangle} \in A_{\pi(h),n-1,s}$;

(f) for all $s > s_{\alpha^\frown\langle l\rangle}$,

(i) for all $j < (n-1), \quad x_{\alpha^\frown\langle l-j\rangle} \notin A_{\pi(l-j),j,s_{\alpha^\frown\langle l\rangle}}, \quad$ but $\quad x_{\alpha^\frown\langle l-j\rangle} \in A_{\pi(l-j),j,s}$;

(ii) $x_{\alpha^\frown\langle l-n\rangle} \notin A_{\pi(l-n),n-1,s_{\alpha^\frown\langle l\rangle}}, \quad$ but $\quad x_{\alpha^\frown\langle l-n\rangle} \in A_{\pi(l-n),n-1,s}$;

(g) for all $j \leq l$, $\beta \supset \alpha^\frown\langle j\rangle$, and $s \geq s_{\alpha^\frown\langle l\rangle}$, either both $x_\beta(s)$ and $x_\beta(s_{\alpha^\frown\langle l\rangle})$ are undefined or they are equal (denoted by x_β) and for all $g < n$,

$$A_{\pi(\beta),g,s}(x_\beta) = A_{\pi(\beta),g,s_{\alpha^\frown\langle l\rangle}}(x_\beta).$$

Proof. Part (a) follows as before and (b) follows from (1.5)(b) and the definition of $s_{\alpha^\frown\langle l\rangle}$. Using (2)(c) we can repeat the earlier argument for (c) and (d), since by (b), $\alpha^\frown\langle l\rangle <_L \tau(t)$. Parts (e) and (f) are proved as before. For (g), fix $j \leq l$ and $\beta \supset \alpha^\frown\langle j\rangle$. Then for all $s > s_{\alpha^\frown\langle l\rangle}$, $\beta <_L \tau(s)$, so s is not a β-stage, and x_β is neither defined nor undefined at stage s so maintains its status and value. Likewise, $x_\beta := x_\beta(s)$ will not be enumerated into any $A_{\pi(\beta),g}$ at stage s. Suppose that for some $g < n$, $x_\beta \in A_{\pi(\beta),g,s_{\alpha^\frown\langle l\rangle}}$ with use $p_\beta(s_{\alpha^\frown\langle l\rangle})$ (or $p_{\beta^+}(s_{\alpha^\frown\langle l\rangle})$). Then we can prove by induction on $s \geq s_{\alpha^\frown\langle l\rangle}$ that also $p_\beta(s) = p_\beta(s_{\alpha^\frown\langle l\rangle})$ and $x_\beta \in A_{\pi(\beta),g,s}$ — the induction step uses (2)(c) (and the fact that also $\beta^+ <_L \tau(s)$). □

We have also the converse to (3)(d)

(4) For any l and $s > \bar{s}$, if $\rho_\alpha(s) \geq l+1$ and $C_s \mid p_{\alpha^\frown\langle l\rangle}(s) = C \mid p_{\alpha^\frown\langle l\rangle}(s)$, then $s_{\alpha^\frown\langle l\rangle}$ exists and $s > s_{\alpha^\frown\langle l\rangle}$.

Proof. As for a single requirement. □

(5) For each $l \geq 0$, if $s_{\alpha^\frown\langle l+n+1\rangle}$ exists, then

(a) for all $j < (n-2), \quad A_{\pi(l),s_{\alpha^\frown\langle l+j\rangle}} \mid v_{\alpha^\frown\langle l+j\rangle} = A_{\pi(l),s_{\alpha^\frown\langle l+j+2\rangle}} \mid v_{\alpha^\frown\langle l+j\rangle}$;

(b) $A_{\pi(l),s_{\alpha^\frown\langle l+n-2\rangle}} \mid v_{\alpha^\frown\langle l+n-2\rangle} = A_{\pi(l),s_{\alpha^\frown\langle l+n+1\rangle}} \mid v_{\alpha^\frown\langle l+n-2\rangle}$;

(c) $A_{\pi(l),s_{\alpha^\frown\langle l+n-1\rangle}} \mid v_{\alpha^\frown\langle l+n-1\rangle} = A_{\pi(l),s_{\alpha^\frown\langle l+n\rangle}} \mid v_{\alpha^\frown\langle l+n-1\rangle}$.

Proof. Consider first (a) and fix $j < (n-2)$. It suffices to verify that for all β and all $t \leq s_{\alpha^\frown\langle l+j+2\rangle}$ such that $x_\beta(t) < p_{\alpha^\frown\langle l+j\rangle}$,

$$A_{\pi(l),s_{\alpha^\frown\langle l+j\rangle}}(x_\beta(t)) = A_{\pi(l),s_{\alpha^\frown\langle l+j+2\rangle}}(x_\beta(t)).$$

For β such that $\alpha^\frown\langle l+j+2\rangle <_L \beta$ this follows from (2)(f): for all $g < n$ and $s = s_{\alpha^\frown\langle l+j\rangle}$ or $s = s_{\alpha^\frown\langle l+j+2\rangle}$, $x_\beta(t) \notin A_{\pi(l),g,s}$. If $\alpha^\frown\langle l+j+1\rangle \subseteq \beta$ or $\alpha^\frown\langle l+j+2\rangle \subseteq \beta$, the same argument applies for $t \leq s_{\alpha^\frown\langle l+j\rangle}$, while if $s_{\alpha^\frown\langle l+j\rangle} < t \leq s_{\alpha^\frown\langle l+j+2\rangle}$, we have $p_{\alpha^\frown\langle l+j\rangle} \leq x_\beta(t)$ by (2)(b). If $\alpha^\frown\langle h\rangle \subset \beta$ for some $h \leq l+j$, we have by (2)(f) (for $t \leq s_{\alpha^\frown\langle l+j\rangle}$) and (3)(g) that for all $g < n$ and $s \geq s_{\alpha^\frown\langle l+j\rangle}$,

$$A_{\pi(l),g,s}(x_\beta(t)) = A_{\pi(l),g,s_{\alpha^\frown\langle l+j\rangle}}(x_\beta(t)).$$

If $\beta \leq \alpha$, the result follows from (1.5)(c) of the induction hypothesis. If $\beta = \alpha^\frown\langle h\rangle$ for some $h \leq l+j$ and $x_\beta(t) \neq x_\beta(s_{\alpha^\frown\langle l\rangle})$, then we may again apply (2)(f). We are left with the numbers $x_{\alpha^\frown\langle h\rangle}$ for $h \leq l+j$ with $\pi(h) = \pi(l)$. By (3)(e) and (f), for such h,

$$h \leq l-(n+2) \quad \Longrightarrow \quad (\forall g < n)(\forall s \geq s_{\alpha^\frown\langle l+j\rangle})\, [x_{\alpha^\frown\langle h\rangle} \in A_{\pi(l),g,s}];$$
$$x_{\alpha^\frown\langle l\rangle} \in A_{\pi(l),g,s_{\alpha^\frown\langle l+j\rangle}} \quad \text{for exactly } g = 0,1,\ldots,j-1;$$
$$x_{\alpha^\frown\langle l\rangle} \in A_{\pi(l),g,s_{\alpha^\frown\langle l+j+2\rangle}} \quad \text{for exactly } g = 0,1,\ldots,j+1.$$

Thus the result follows as above. We leave the similar proofs of (b) and (c) to the devoted reader. □

(6) For each $l > 0$,

(a) if $s_{\alpha^\frown\langle l+n+1\rangle}$ exists, then for all $j \leq (n-2)$,

$$E^{n-j+1}_{\pi(l),s_{\alpha^\frown\langle l+j\rangle}} \mid u^{n-j+1}_{\alpha^\frown\langle l+j\rangle} \neq E^{n-j+1}_{\pi(l),s_{\alpha^\frown\langle l+j+1\rangle}} \mid u^{n-j+1}_{\alpha^\frown\langle l+j\rangle};$$

(b) if $s_{\alpha^\frown\langle l+n\rangle}$ exists, then $E^2_{\pi(l),s_{\alpha^\frown\langle l+n-1\rangle}} \mid u^2_{\alpha^\frown\langle l+n-1\rangle} = E^2_{\pi(l),s_{\alpha^\frown\langle l+n\rangle}} \mid u^2_{\alpha^\frown\langle l+n-1\rangle}$;

(c) if $s_{\alpha^\frown\langle l+j\rangle}$ exists for all j, then $E^1_{\pi(l),s_{\alpha^\frown\langle l+n+1\rangle}} \mid u^1_{\alpha^\frown\langle l+n+1\rangle} = E^1_{\pi(l)} \mid u^1_{\alpha^\frown\langle l+n+1\rangle}$;

(d) if $s_{\alpha^\frown\langle l+j\rangle}$ exists for all j, then

$$\exists t > s_{\alpha^\frown\langle l+n-1\rangle}[\phi(t, x_{\alpha^\frown\langle l+n-1\rangle}) \wedge E^2_{\pi(l),t} \mid u^2_{\alpha^\frown\langle l+n-1\rangle} \neq E^2_{\pi(l),s_{\alpha^\frown\langle l+n-1\rangle}} \mid u^2_{\alpha^\frown\langle l+n-1\rangle}$$
$$\wedge\, E^1_{\pi(l),t} \mid u^1(t, x_{\alpha^\frown\langle l+n-1\rangle}) = E^1_{\pi(l)} \mid u^1(t, x_{\alpha^\frown\langle l+n-1\rangle})],$$

and if t_l denotes the least such t, then $s_{\alpha^\frown\langle l+n\rangle} < t_l$.

Proof. The computations here exactly mirror those of the corresponding part of the single-requirement case, with each subscript $l+\cdots$ replaced by $\alpha^\frown\langle l+\cdots\rangle$ and using the α-versions of (4) and (5). □

(1.4) Only finitely many $s_{\alpha^\frown\langle l\rangle}$ exist; in other words, $\liminf_s \rho_\alpha(s) < \infty$.

Proof. Exactly as before, under the assumption that for all l, $s_{\alpha^\frown\langle l\rangle}$ exists, we may derive from (4) and (6) that C is of r.e. degree, contrary to hypothesis. □

Now we may verify (1.5) for $e+1$ as follows. Set $f(e) = l := \liminf_s \rho_\alpha(s)$, $\alpha^+ = f \mid (e+1) := \alpha^\frown\langle l\rangle$, and $s^+ = s_{\alpha^\frown\langle l-1\rangle} + 1$. By (3)(a), s^+ is an α^+-stage. For $s \geq s^+$ either $\alpha <_L \tau(s)$ or $\alpha \subseteq \tau(s)$, so $\alpha^\frown\langle l\rangle \leq \tau(s)$ by (3)(b). In either case, $\alpha^+ \leq \tau(s)$ as required by (1.5)(b). For (c), let $s \geq s^+$ be an α^+-stage and $\beta \leq \alpha^+$. The new cases to verify are $\alpha^\frown\langle j\rangle \subseteq \beta$ for $j < l$ and $\beta = \alpha^+$. The first clause (concerning $x_\beta(s)$) follows from (3)(g) when $\alpha^\frown\langle j\rangle \subset \beta$ and $j < l$ and from (3)(c) when $\alpha^\frown\langle j\rangle = \beta$ and $j \leq l$. The second clause (concerning $A_{k,s}(x_\beta(t))$) follows from (2)(f) for $t < s^+$ and all β, from (3)(g) when $t > s^+$, $\alpha^\frown\langle j\rangle \subset \beta$, and $j < l$, and from (2)(d)(e) when $t > s^+$, $\alpha^\frown\langle j\rangle = \beta$ and $j \leq l$. Finally, let $s \geq s^+$ be a C-true stage. Then $\alpha \subseteq \tau(s)$, so for some $l' \geq l$, $\alpha^\frown\langle l'\rangle \subseteq \tau(s)$. Suppose that $l' > l$ so $\alpha^\frown\langle l\rangle <_L \tau(s)$. Then by (2)(c), $p_{\alpha^\frown\langle l\rangle}(s-1) \leq c_s$, so since s is C-true, $C_s \restriction p_{\alpha^\frown\langle l\rangle}(s-1) = C \restriction p_{\alpha^\frown\langle l\rangle}(s-1)$. But then by (4), $s_{\alpha^\frown\langle l\rangle}$ exists and $\liminf_s \rho_\alpha(s) \geq l+1$, contrary to the choice of l. Hence $\alpha^+ \subseteq \tau(s)$ and s is an α^+-stage.

Finally, (1.6) follows as in each of the other cases.

In conclusion, we want to explain the difficulty in extending this proof to cover cases of the Conjecture for $m \geq 2$. We return to the informal description preceding the proof and attempt to adapt it to the case (2,1) — that is, given C which is 3-REA but not of 2-REA degree, we want to find an r.e.[C] set A such that $A \oplus C$ is not of 3-REA degree. Since $n = 1$, the construction will resemble closely that of Theorem 1.2 except that we construct three sets A_0, A_1, and A_2. Assuming that $C_{s_l} \restriction v_l = C \restriction v_l$, we enumerate x_0 into A_{0,s_1+1} with use v_1, x_1 into A_{1,s_2+1} with use v_2, x_2 into A_{2,s_3+1} with use v_3, etc. Since

$$A_{0,s_0}(x_0) = A_{0,s_1}(x_0) \neq A_{0,s_2}(x_0),$$

we have

$$E^3_{0,s_0} \restriction u^3_0 = E^3_{0,s_1} \restriction u^3_0 \neq E^3_{0,s_2} \restriction u^3_0,$$

and we may define

$$t_0 = \text{least } t > s_0 \left[\phi(t,x_0) \wedge E^3_{0,t} \restriction u^3_0 \neq E^3_{0,s_0} \restriction u^3_0 \wedge E^2_{0,t} \restriction u^2(t,x_0) = E^2_0 \restriction u^2(t,x_0)\right].$$

If necessarily $s_1 \leq t_0$, then we could compute s_1 (and by extension the entire function $l \mapsto s_l$) recursively in $B \oplus E^2_0 \oplus E^2_1 \oplus E^2_2$ and conclude that C is Turing equivalent to this set and thus of 2-REA degree, contrary to hypothesis. Unfortunately, there seems to be no reason to expect that $s_1 \leq t_0$. In the earlier case we had

$$E^2_{0,s_0} \restriction u^2_0 = E^2_{0,s_1} \restriction u^2_0 \quad \text{and} \quad E^2_{0,s_0} \restriction u^2_0 \subset E^2_{0,t_0} \restriction u^2_0.$$

Elements of $E^2_{0,t_0} \restriction u^2_0$ are witnessed by E^1_0-correct axioms which never change, since E^1_0 is an r.e. set, and thus if $t_0 < s_1$, the new elements could not be removed by stage s_1. Here, however, $E^3_{0,t_0} \restriction u^3_0$ may differ from $E^3_{0,s_0} \restriction u^3_0$ by both gaining and losing elements. If $t_0 < s_1$, lost elements could be restored by stage s_1 and new elements could be removed; even though their axioms are eventually E^2_0-correct, since E^2_0 is only a 2-REA set, they may be temporarily unsatisfied at stage s_1. We see no way around this problem and expect that if the full conjecture is to be proved a quite different method will be needed.

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