Computably Enumerable Sets that are Automorphic to Low Sets

Peter Cholak and Rachel Epstein

Abstract

We work with the structure consisting of all computably enumerable (c.e.) sets ordered by set inclusion. The question we will partially address is which c.e. sets are autormorphic to low (or low_2) sets. Using work of R. Miller [9], we can see that every set with semilow complement is Δ^0_3 automorphic to a low set. While it remains open whether every set with semilow complement is effectively automorphic to a low set, we show that there are sets without semilow complement that are effectively automorphic to low sets. We also consider other lowness notions such as having a $\text{semilow}_{1.5}$ complement, having the outer splitting property, and having a semilow_2 complement. We show that in every nonlow c.e. degree, there are sets with $\text{semilow}_{1.5}$ complements without semilow complements as well as sets with semilow_2 complements and the outer splitting property that do not have $\text{semilow}_{1.5}$ complements. We also address the question of which sets are automorphic to low_2 sets.

1 Introduction

Our domain of discourse is the collection of all c.e. sets under inclusion. This structure is called $\mathcal{E}$. By adding intersection, union, the empty set and ω, this structure is a lattice. We should also mention that these binary operations in addition to $\emptyset$, ω, and being computable and finite are definable in this structure. Thus, if we have an automorphism of $\mathcal{E}$ then computable sets must go to computable sets and infinite sets to infinite sets.

If we take the quotient of $\mathcal{E}$ modulo the ideal of finite sets, we get the lattice $\mathcal{E}^*$ of c.e. sets up to finite difference. Soare [13, XV 2.5] showed that if there is an automorphism of $\mathcal{E}^*$ taking A to B, then there is one of $\mathcal{E}$ as well, so we can work in $\mathcal{E}^*$ to show that automorphisms of $\mathcal{E}$ exist.

A c.e. set W is *low* if and only if its jump, W', is Turing equivialent to $\emptyset'$. As we will discuss below, the low sets are special within $\mathcal{E}$. Our goal is to understand as best possible which sets look like and behave like low sets in this structure. That is, when does a set have a low set in its orbit?

The first result in this vein is a result of Soare [12] following [11]. Soare showed that if a c.e. set A is low then its collection of c.e. supersets of A under inclusion is isomorphic to $\mathcal{E}$. Formally, $\mathcal{L}(A) = \{W_e \cup A \mid e \in \omega\}$ is isomorphic to $\mathcal{E}$. This suggests that low sets are similar to computable sets. Again, we can work modulo the finite sets, showing that the quotient of $\mathcal{L}(A)$ modulo the finite sets, called $\mathcal{L}^*(A)$, is isomorphic to $\mathcal{E}^*$.

Actually Soare proved a stronger result. But for that we need a few definitions. First a set X is *semilow* if and only if $\{e \mid W_e \cap X \neq \emptyset\} \leq_T \mathbf{0}'$. This index set is Σ^X_1. If $\overline{X}$ is c.e. and low then this index set is Δ^0_2. So if A is low then A has a semilow complement. Now consider an isomorphism Φ between $\mathcal{L}^*(A)$ (or $\mathcal{E}^*$) and $\mathcal{E}^*$. There are functions g and h such that $\Phi(W_e \cup A) = W_{g(e)}$ and $\Phi^{-1}(W_e) = W_{h(e)} \cup A$ (or $\Phi(W_e) = W_{g(e)}$ and $\Phi^{-1}(W_e) = W_{h(e)}$). If computable ($\Delta^0_3$) g and h can be found for Φ, then we call Φ *effective* (Δ^0_3). It is standard to use the terms "effective" or "Δ^0_3" to describe isomorphisms between $\mathcal{L}(A)$ or $\mathcal{E}$ and $\mathcal{E}$, even though the isomorphisms produced may only be effective or Δ^0_3 on the quotient spaces $\mathcal{L}^*(A)$ and $\mathcal{E}^*$. This is the way we will use these terms in this paper.

Soare [12] showed that if A is c.e. set with semilow complement then $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$. There have been several improvements on this result. We observe, using work of R. Miller [9], that this can be improved to if A is a c.e. set with semilow complement then A is Δ^0_3 automorphic to a low set (see Section 2). This means there is a Δ^0_3 automorphism Φ such that $\Phi(A) = \widehat{A}$ is low. One recent open question raised by Soare (personal communication) is whether the above Δ^0_3 can be replaced by effective. Another question raised by Soare is to characterize the sets which are effectively automorphic to low sets. It was thought that perhaps a characterization would be all sets which have semilow complements. But, in Section 3, we show that this is not the case, as there are sets without semilow complements that are effectively automorphic to low sets.

Soare's result about sets with semilow complements was first improved by Maass [8]. Maass showed that for any c.e. set A with semilow$_{1.5}$ complements that $\mathcal{L}(A)$ is Δ^0_3 isomorphic to $\mathcal{E}$. A set X is *semilow*$_{1.5}$ if and only if the index set $\{e \mid W_e \cap X \text{ is finite}\} \leq_1 \emptyset''$. By an index set argument, we know that if A is effectively automorphic to a low set $\widehat{A}$ it must have semilow$_{1.5}$ complement ($W_e \cap A \neq^* \emptyset$ if and only if $W_{g(e)} \cap \widehat{A} \neq^* \emptyset$ and if $\widehat{A}$ is low then the latter is Π^0_2).

Now Harrington and Soare [5, Section 5] show that there is a property $NL(A)$ definable in the structure $\mathcal{E}$ such that if $NL(A)$ holds then A does not have semilow complement. Morever they showed that there is a set A with semilow$_{1.5}$ complement and $NL(A)$. Thus we know that not all sets A with semilow$_{1.5}$ complement can be automorphic to a low set. But can they be automorphic to low$_2$ sets?

We thought we had a positive answer but $NL(A)$ for nonlow$_2$ A seems to be a barrier. $NL(A)$ does provide a barrier for the question about whether two promptly simple sets with semilow complements are automorphic. This barrier is discussed in Section 5.4 of [5]. (Our situation is similar. The issue seems to be that we cannot get $\widehat{A}$ to cover A in real time. Even though we know a state ν is emptied into A, it is emptied slowly through a series of moves into RED and BLUE sets. On the $\widehat{A}$ side we have to match this series of BLUE and

RED moves and hence we cannot quickly cover A. Another problem here is that depending on e the series of RED and BLUE moves can change. The series of moves does not just depend on ν.)

There was one more extension of Soare's and Maass's work. Cholak [1] showed that if A has the outer splitting property and has semilow$_2$ complement then $\mathcal{L}(A)$ is isomorphic to $\mathcal{E}$. A has the *outer splitting property* if and only if there are computable functions g and h such that for all e, $W_e = W_{g(e)} \sqcup W_{h(e)}$, $W_{g(e)} \cap \overline{A} =^* \emptyset$ and if $W_e \cap \overline{A}$ is infinite then $W_{g(e)} \cap \overline{A}$ is nonempty. Maass [8, Lemma 2.3] shows in a very clever argument that if $\overline{A}$ is semilow$_{1.5}$ then A has the outer splitting property. X is *semilow$_2$* if and only if the index set $\{e \mid W_e \cap X \text{ is infinite}\} \leq_T \emptyset''$. At one time we thought we could show that if A has the outer splitting property and has semilow$_2$ complement then A is automorphic to a low$_2$ set. Note that if A is automorphic to a low$_2$ then it has semilow$_2$ complement.

There is also some related recent work of Epstein, [4]. Epstein shows that there is a properly low$_2$ degree such that every c.e. set in that degree is automorphic to a low set. We were wondering if that could be shown more modularly. We wondered if there is some property P such that every set with property P is automorphic to a low set and there is some properly low$_2$ degree such that every set in that degree has property P. One reasonable candidate for P would be having semilow complement. But Soare [13, IV 4.10] shows that every nonlow degree contains a c.e. set whose complement is not semilow (via a nice index set argument). Other later results rule out other possible P's.

Downey, Jockusch, and Schupp [3, Theorems 4.6 and 4.7] showed that every nonlow degree contains a c.e. A without the outer splitting property (so A's complement is not semilow$_{1.5}$). In related results, we show that every nonlow degree contains a c.e. set A whose complement is semilow$_{1.5}$ but not semilow (see Section 4) and a c.e. set A whose complement is semilow$_2$ but not semilow$_{1.5}$ and has the outer splitting property (see Section 5). We also provide a nice index set argument that every nonlow$_2$ degree contains a c.e. set A whose complement is not semilow$_2$.

We should mention that it has been long known that if a degree is nonlow$_2$ it must contain a c.e. set which is not automorphic to a low$_2$ set. Lachlan [7] showed, using a true stages construction, that every low$_2$ set has a maximal superset while Shoenfield [10] showed that every nonlow$_2$ degree contains a c.e. set with no maximal superset. These two results have been generalized by Cholak and Harrington [2]. One corollary of the work by Cholak and Harrington is that if $\mathbf{a}$ and $\mathbf{b}$ are two c.e. degrees and $\mathbf{a}'' \not\leq_T \mathbf{b}''$, then $\mathbf{a}$ contains a c.e. set not automorphic to anything computable from $\mathbf{b}$. It is open if this can be improved to show that $\mathbf{a}$ contains a c.e. set not automorphic to anything whose double jump is computable from $\mathbf{b}''$.

We mention one more open related question: if A is low$_2$ then is $\mathcal{L}(A)$ isomorphic to $\mathcal{E}$? We now know that there is a properly low$_2$ set without the outer splitting property. Thus, a positive result here may not use the outer splitting property and is very likely to use the a *true stage* construction in the style of Lachlan [7].

2 C.e. sets with semilow complements are automorphic to low

Soare [12] showed that if A is a c.e. set with semilow complement then $\mathcal{L}(A)$ is effectively isomorphic to $\mathcal{E}$. It has been conjectured that in fact any c.e. set A with semilow complement is effectively automorphic to a low set. Here we show that it is possible to modify a proof of R. Miller [9] to show that every c.e. set with semilow complement can be taken by a Δ^0_3 automorphism to a low set.

Theorem 2.1. *[R. Miller, [9], Theorem 1.1.1] For every c.e. set A with semilow complement and every noncomputable c.e. set C, there exists a Δ^0_3 automorphism of $\mathcal{E}$ mapping A to a set B such that $C \not\leq_{\mathrm{T}} B$.*

R. Miller states this theorem differently, saying that A is a low set instead of a set with semilow complement. However, he mentions that the construction only requires that A have semilow complement.

We modify R. Miller's proof to get the following theorem.

Theorem 2.2. *For every c.e. set A with semilow complement, there exists a Δ^0_3 automorphism of $\mathcal{E}$ mapping A to a low set B.*

Proof. Here we discuss the minor modifications to the proof of Theorem 2.1 that result in a proof of Theorem 2.2. Because our modifications are minor and the original proof using the complex Harrington-Soare automorphism construction of [6], we will not reproduce R. Miller's proof here. Instead we briefly sketch the proof and refer the reader of this section to Theorem 2.1 in [9] for more details.

To prove Theorem 2.1, R. Miller builds an automorphism on a tree, as in Harrington-Soare [6], which takes a given set A to a constructed set B with the desired property that $C \not\leq_{\mathrm{T}} B$. The primary challenge of the theorem is to allow enough flow of elements into B to match the flow of elements into A while simultaneously restraining elements from B so that B cannot compute C. There are two key components of the construction. The first is a list $\mathcal{L}^{\mathcal{G}}$ that keeps track of the states of elements flowing into A so that we can ensure that if infinitely many elements flow into A in a given state, then infinitely many will flow into B in the matching state. The second key component is Step $\hat{0}$, which enumerates elements into B that are in the appropriate states. Step $\hat{0}$ only allows elements to enter if they are large enough to preserve a given restraint. In our construction, we modify $\mathcal{L}^{\mathcal{G}}$ and Step $\hat{0}$ to reflect our new restraint, but little else is changed.

In R. Miller's construction, Step $\hat{0}$ was the only step that involved putting elements into B. He needed to guarantee the proper flow of elements into B, while preserving restraint that would ensure that B would not be able to compute C. In our modified construction, we just need to preserve a different restraint. In fact, this part of the construction can be done as in Step B of

Epstein's proof in [4] that there is a nonlow degree such that every set in that degree is Δ^0_3 automorphic to a low set.

In order to make this modification, we first must alter the list $\mathcal{L}^{\mathcal{G}}$. In R. Miller's construction, pairs $\langle \alpha, \widehat{\nu} \rangle$ are added to the list whenever an element x enters A from the α-state ν. (Note that $\widehat{\nu}$ is the corresponding state on the $\overline{B}/B$ side of the construction.) In our modification, we will instead add the triple $\langle \alpha, \widehat{\nu}, x \rangle$ to the list $\mathcal{L}^{\mathcal{G}}$ whenever x enters A from the α-state ν. Note that we can identify each triple with a number in ω.

Next, we replace R. Miller's Step $\hat{0}$ with the following new Step $\hat{0}$.

Step $\hat{0}$: (Moving elements into B.)

Find the first triple $\langle \alpha, \widehat{\nu}_0, x \rangle$ in $\mathcal{L}^{\mathcal{G}}$ such that there is a $\widehat{y} \in \widehat{\omega}$ that has never before caused action on this step satisfying all of the following:

($\hat{0}$.1) α is consistent;

($\hat{0}$.2) $\widehat{y} \in R_{\alpha,s}$;

($\hat{0}$.3) $\widehat{\nu}(\alpha, \widehat{y}, s) = \widehat{\nu}_0$; and

($\hat{0}$.4) for all $i < \langle \alpha, \widehat{\nu}_0, x \rangle$, $\varphi^B_i(i)[s]\!\downarrow \implies \varphi^B_i(i)[s] < \widehat{y}$.

Action. If $\langle \alpha, \widehat{\nu}_0, x \rangle$ is not checked, check $\langle \alpha, x, e \rangle$, and do not enumerate $\widehat{y}$ into B. If $\langle \alpha, \widehat{\nu}_0, x \rangle$ has been checked already, enumerate $\widehat{y}$ into B and remove $\langle \alpha, \widehat{\nu}_0, x \rangle$ from the list $\mathcal{L}^{\mathcal{G}}$. This will leave infinitely many elements in $\overline{B}$, while still matching the flow into A.

The purpose of this step is essentially the same as in the original construction. It creates a flow of elements into B matching the flow into A, while also respecting the restraint of the negative requirements and ensuring that infinitely many elements remain outside of B.

Most of the lemmas in the proof of Theorem 2.1 in [9] could be kept exactly the same. Lemmas 3.1.3 and 3.3.4 would need only very minor and straightforward changes, reflecting how the new construction still guarantees that $\overline{B}$ is infinite and that Step $\hat{0}$ is able to enumerate an element for every $\langle \alpha, \widehat{\nu}_0, x \rangle$ on $\mathcal{L}^{\mathcal{G}}$, with α on the true path, which works essentially the same as before, but with the old restraint replaced by the new one.

The only significant difference in the verification would be to replace Lemmas 3.3.1 and 3.3.2 with the following lemma, which is the same as Lemma 8.17 in [4].

Lemma 2.3. *The set B is low.*

Proof. Suppose there exist infinitely many s such that $\Phi^B_i(i)[s]\!\downarrow$.

Let s_0 be the least s such that for all $\langle \alpha, \widehat{\nu}, x \rangle \leq i$, $\langle \alpha, \widehat{\nu}, x \rangle$ has either been removed from the list $\mathcal{L}^{\mathcal{G}}$ already or will never be removed from the list (this can happen if it is never added to the list, or if it is added but never matched). Since each $\langle \alpha, \widehat{\nu}, x \rangle$ can enter $\mathcal{L}^{\mathcal{G}}$ only once, then after stage s_0, no $\widehat{y}$ will enter B in order to match $\langle \alpha, \widehat{\nu}, x \rangle$. Let $s > s_0$ be some stage with $\Phi^B_i(i)[s]\!\downarrow$. Then by ($\hat{0}$.4), nothing can enter B below the use of this computation. So $(\forall t > s)$ $[\Phi^B_i(i)[t]\!\downarrow]$. So either $(\forall^\infty s)[\Phi^B_i(i)[s]\!\downarrow]$ or $(\forall^\infty s)[\Phi^B_i(i)[s]\!\uparrow]$. Thus, B is low. $\square$

This completes the modification of R. Miller's proof to show that every c.e. set with semilow complement is Δ_3^0 automorphic to a low set.

□

It remains open whether every c.e. set with semilow complement is effectively automorphic to a low set. In the next section, we show that there are sets without semilow complement that are effectively automorphic to low sets.

3 Effectively automorphic to low but not semilow

In Theorem 3.2, we build an effective automorphism of $\mathcal{E}^*$ that takes a set A without semilow complement to a low set $\overline{A}$. To build an effective automorphism, we use Soare's Effective Extension Theorem.

The main tool in constructing an automorphism of $\mathcal{E}^*$ is matching infinite e-states. Suppose we are given listings of all the c.e. sets modulo finite difference, $\{U_e\}_{e\in\omega}$ and $\{V_e\}_{e\in\omega}$, and an invertible map Θ of $\mathcal{E}^*$ that takes U_e to $\widehat{U}_e$ and Θ^{-1} takes V_e to $\widehat{V}_e$. The *e-state* of an element x tells us which c.e. sets U_i and $\widehat{V}_i$, or $\widehat{U}_i$ and V_i contain the element x, for all $i \leq e$. The map Θ is an automorphism of $\mathcal{E}^*$ if there are infinitely many elements in an e-state ν with respect to U_i and $\widehat{V}_i$ if and only if there are infinitely many elements in the corresponding e-state $\widehat{\nu}$ with respect to $\widehat{U}_i$ and V_i.

More formally, we consider two copies of ω, which we will refer to as ω and $\widehat{\omega}$. We imagine that our automorphism is given by a permutation of ω, which can be represented as a function from ω to $\widehat{\omega}$. The e-state of an element $x \in \omega$ at stage s is given by the triple $\nu(e, x, s) = \langle e, \sigma(e, x, s), \tau(e, x, s)\rangle$, where $\sigma(e, x, s) = \{i \leq e \mid x \in U_{i,s}\}$ and $\tau(e, x, s) = \{i \leq e \mid x \in \widehat{V}_{i,s}\}$. The e-state of an element $\widehat{x} \in \widehat{\omega}$ is determined the same way, except with $\widehat{U}_{i,s}$ replacing $U_{i,s}$ and $V_{i,s}$ replacing $\widehat{V}_{i,s}$. The final e-state of an element is $\nu(e, x) = \langle e, \sigma(e, x), \tau(e, x)\rangle$, where $\sigma(e, x) = \{i \leq e \mid x \in U_i\}$ and $\tau(e, x) = \{i \leq e \mid x \in \widehat{V}_i\}$, and similarly for $\widehat{x} \in \widehat{\omega}$.

To see that Θ is an automorphism, it suffices to show that

$$(1)\qquad (\forall\nu)(\exists^\infty x \in \omega)[\nu(e, x) = \nu \textit{ w.r.t. } \{U_n\}_{n\in\omega} \text{ and } \{\widehat{V}_n\}_{n\in\omega}]$$
$$\iff (\exists^\infty \hat{y} \in \hat{w})[\nu(e, \hat{y}) = \nu \textit{ w.r.t. } \{\widehat{U}_n\}_{n\in\omega} \text{ and } \{V_n\}_{n\in\omega}].$$

The theorem stated below is actually a special case of Soare's Extension Theorem [1974]. The full version is stronger than is needed for this paper. Recall that for an given enumeration of two c.e. sets U and V, $U\backslash V = \{x|\exists s[x \in (U_{s+1} - V_s)]\}$, U *before* V and $U \searrow V = (U\backslash V) \cap V$, U *before* U and *then* V.

Theorem 3.1 (The Extension Theorem)**.** *(Soare[1974]) Let A and $\widehat{A}$ be infinite c.e. sets, and let $\{U_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$, $\{\widehat{U}_n\}_{n\in\omega}$, and $\{\widehat{V}_n\}_{n\in\omega}$ be computable arrays of c.e. sets. Suppose there is a simultaneous computable enumeration of all of the above such that $\widehat{A} \searrow \widehat{U}_n = \emptyset = A \searrow \widehat{V}_n$ for all n. For each full e-state ν, define*

$D_\nu^A = \{x \ : \ x \in A_{at\ s+1}$ *and* $\nu = \nu(e,x,s)$ *w.r.t.* $\{U_{n,s}\}_{n,s\in\omega}$ *and* $\{\widehat{V}_{n,s}\}_{n,s\in\omega}\}$

and

$D_{\widehat{\nu}}^{\widehat{A}} = \{\widehat{x} \ : \ \widehat{x} \in \widehat{A}_{at\ s+1}$ *and* $\widehat{\nu} = \nu(e,\widehat{x},s)$ *w.r.t.* $\{\widehat{U}_{n,s}\}_{n,s\in\omega}$ *and* $\{V_{n,s}\}_{n,s\in\omega}\}$.

D_ν^A *is the set of all elements that enter* A *from* e*-state* ν*, and similarly for* $D_{\widehat{\nu}}^{\widehat{T}}$.

Suppose that the simultaneous enumeration satisfies:

$$(\forall \nu)[D_\nu^A \text{ infinite} \iff D_{\widehat{\nu}}^{\widehat{A}} \text{ infinite}]. \tag{2}$$

Then there is a computable enumeration of c.e. sets $\{\widetilde{U}_n\}_{n\in\omega}$ *and* $\{\widetilde{V}_n\}_{n\in\omega}$*, where* $\widetilde{U}_n$ *extends* $\widehat{U}_n$ *and* $\widetilde{V}_n$ *extends* $\widehat{V}_n$ *such that* $\widehat{U}_n = \widetilde{U}_n \backslash \widehat{A}$*,* $\widehat{V}_n = \widetilde{V}_n \backslash A$*, and*

$$\begin{array}{c} \exists^\infty x \in A \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{U_{n,s}\}_{n,s\in\omega} \text{ and } \{\widetilde{V}_{n,s}\}_{n,s\in\omega} \\ \iff \\ \exists^\infty \widehat{x} \in \widehat{A} \text{ with final } e\text{-state } \widehat{\nu} \text{ w.r.t. } \{\widetilde{U}_{n,s}\}_{n,s\in\omega} \text{ and } \{V_{n,s}\}_{n,s\in\omega}. \end{array} \tag{3}$$

A *skeleton* of the c.e. sets is a simultaneous computable enumeration of c.e. sets such that every c.e. set is finitely different from some set on the list. Let $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ be skeletons of the c.e. sets. Thus, if we begin with a partial automorphism of $\mathcal{E}^*$ on the complements of A and $\widehat{A}$ that takes U_n to $\widehat{U}_n$ and $\widehat{V}_n$ to V_n, then we can extend it to an automorphism of $\mathcal{E}^*$ that takes A to $\widehat{A}$, U_n to $\widetilde{U}_n$, and $\widetilde{V}_n$ to V_n.

The way we build a partial automorphism on the complements of A and $\widehat{A}$ is to match infinite e-states for elements in $\overline{A}$ and $\overline{\widehat{A}}$. That is, if there are infinitely many elements $x \in \overline{A}$ such that $\nu(e,x) = \nu$, then there are infinitely many elements in $\widehat{y} \in \overline{\widehat{A}}$ such that $\widehat{\nu}(e,\widehat{y}) = \widehat{\nu}$, and vice versa.

We call the state ν ($\widehat{\nu}$) a *gateway state* if D_ν^T ($D_{\widehat{\nu}}^{\widehat{T}}$) is infinite. In the full version of the Extension Theorem, we do not need an exact matching of gateway states ν and $\widehat{\nu}$, but only a *covering* of the states, as described in Soare [1974]. Our construction gives an exact matching of gateway states, so we have stated only the special case of the theorem here.

Theorem 3.2 (Cholak/Epstein)**.** *There is a set* A *that does not have semilow complement, but is effectively automorphic to a low set.*

Proof. We will construct a c.e. set $\overline{A}$ that is not semilow and a c.e. set $\widehat{A}$ that is low, such that A and $\widehat{A}$ are effectively automorphic to each other. We use Soare's Effective Extension Theorem.

3.1 Requirements

Our first requirement is the automorphism requirement, which constructs an automorphism taking A to $\widehat{A}$. To do this, we build a simultaneous computable enumeration of c.e. sets $\{U_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$, $\{\widehat{U}_n\}_{n\in\omega}$, and $\{\widehat{V}_n\}_{n\in\omega}$ as in the extension theorem, such that $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ are skeletons. We will ensure that the map taking U_n to $\widehat{U}_n$ and $\widehat{V}_n$ to V_n gives a partial automorphism of $\mathcal{E}^*$ on the complements of A and $\widehat{A}$, and that we have equality of gateway states to meet the hypotheses of the Extension Theorem.

Let φ_i be a listing of all partial computable functions of two variables. We will use it to list all Δ_2 functions.

To achieve that A does not have semilow complement, we meet the following requirement for all $i \in \omega$:

P_i: $\widehat{\varphi}_i(e) := \lim_s \varphi_i(e,s)$ is not the characteristic function of $\{e : W_e \cap \overline{A}$ nonempty$\}$.

To this end, for each $i \in \omega$ we will build sets S^i_k for each $k \in \{0, 1, \dots 4^{i+1}\}$, such that for some k, S^i_k will be the c.e. set W_e and $\widehat{\varphi}_i(e) = \lim_s \varphi_i(e,s)$ will be wrong about whether W_e and $\overline{A}$ have nonempty intersection. The index e for S^i_k may be found computably in k, i, and the number of times n we have started over in building S^i_k. We call this computable function $g(i,k,n)$. By the Recursion Theorem with Parameters (see [13, II 3.5]), we may assume that we know the function g in advance. We can rewrite P_i as:

P_i: $(\exists k)(\exists n)[0 \le k \le 4^{i+1}$ and $\widehat{\varphi}_i(g(i,k,n)) = 1 \implies S^i_k \cap \overline{A} = \emptyset$, and $\widehat{\varphi}_i(g(i,k,n)) = 0 \implies S^i_k \cap \overline{A} \neq \emptyset$, for S^i_k having been reset n times$]$.

The number 4^{i+1} appears as it is the number of i-states. Since each i-state is determined by a subset of $\{U_0, \dots, U_i\}$ as well as a subset of $\{\widehat{V}_0, \dots, \widehat{V}_i\}$, there are $2^{i+1} \cdot 2^{i+1} = 4^{i+1}$. This will be important for matching entry states, as we will see in the next section.

To achieve that $\widehat{A}$ is low, we meet the usual requirement for all $j \in \omega$:

N_j: $(\exists^\infty s)\Phi^{\widehat{A}}_j(j)[s]\downarrow \implies \Phi^{\widehat{A}}_j(j)\downarrow$.

This guarantees that $\widehat{A}$ is low because it makes the jump of $\widehat{A}$ limit computable, and thus computable in $\mathbf{0}'$.

3.2 Basic strategy

Our construction will take place on two identical pinball machines, M and $\widehat{M}$, see Cholak [1], Soare [13] or Harrington and Cholak [6]. Each element of ω will flow through M, and each element of a copy of ω called $\widehat{\omega}$ will flow through $\widehat{M}$. The construction of the pinball machines is extremely simple. They each have a single corridor along which all elements flow. Along the corridor are gates G_e for each $e \in \omega$. Elements may be held at gates or pass through, according to the construction. In our construction, a *closed* gate will let nothing through, while an *open* gate will let all elements through except for those currently designated as witness, as explained later. Throughout the construction, we will match elements in ω to elements in $\widehat{\omega}$ by a matching function $m(x)$ with domain ω, where we begin with $m(x) = \widehat{x}$, the copy of x in $\widehat{\omega}$. During the construction, elements may be rematched.

Let $\{W_e\}_{e\in\omega}$ be a standard enumeration of the c.e. sets. We build two skeletons $\{U_e\}_{e\in\omega}$ and $\{V_e\}_{e\in\omega}$ as follows: If $x \in W_e$ and x is either at gate $G_{e'}$ for $e' \geq e$ or x has been removed from M, enumerate x into U_e. Similarly, if $x \in W_e$ and $\widehat{x}$ is either at gate $G_{e'}$ for $e' \geq e$ or $\widehat{x}$ has been removed from $\widehat{M}$, enumerate $\widehat{x}$ into V_e. For each e, only finitely many elements never reach gate G_e or leave the machines, so $U_e =^* W_e$ and $V_e =^* W_e$.

Elements move through M by flowing through the machine, starting at gate G_0, then moving to G_1, and so on. Every element on the machine is at a gate. As x moves, $m(x)$ copies its move on $\widehat{M}$. To meet the automorphism requirement, if x is in U_e while at gate $G_{e'}$ for $e' \geq e$, enumerate $m(x)$ into $\widehat{U}_e$. Similarly, if $m(x)$ is in V_e, enumerate x into $\widehat{V}_e$.

We define for each stage s a restraint function

$$r(j, s) = \max\{\varphi_{j'}^{\widehat{A}}(j')[s] \mid j' \leq j\}.$$

That is, $r(j, s)$ is the maximum use of any jump computation $\Phi_{j'}^{\widehat{A}}(j')[s]$ for $j' \leq j$.

To meet P_i, we would like to build a c.e. set S such that if φ_i guesses that $S \cap \overline{A}$ is empty, we put an element not currently in A into S, and if it later guesses that $S \cap \overline{A}$ is nonempty, we put that element into A. This is the standard method of constructing a set that does not have semilow complement. It is also frequently used to build a nonlow set, as any set without semilow complement is not low. In order to meet the automorphism requirement, if we put infinitely many elements into A, we must put infinitely elements into $\widehat{A}$ from the same i-state. We cannot simply enumerate $\widehat{x}$ into $\widehat{A}$ whenever we enumerate x into A because we need to ensure that $\widehat{A}$ is low. Instead, when we put an element into A, we would like to put a large enough element into $\widehat{A}$ that is in the same i-state. The way we will accomplish this is to build several c.e. sets S_k^i for each i and we will guarantee that one will act as the desired set S. When we want to enumerate an element x from some $S_{k_0}^i$ into A to satisfy a positive requirement, we will simultaneously enumerate a large enough element $\widehat{z}$ into $\widehat{A}$, where $\widehat{z}$ is in the same i-state as x. Since we have removed x and $\widehat{z}$ from the machines, we have left $m(x)$ and $m^{-1}(\widehat{z})$ without reasonable partners, and so we partner them with each other. By this process, whenever we enumerate an element into A from some i-state, we also enumerate an element into $\widehat{A}$ from the same i-state, so that we achieve exact matching of entry states, as desired by the Effective Extension Theorem.

For every i, we build S_k^i for each $k \in \{0, 1, \ldots 4^{i+1}\}$. We will start with all sets empty. If φ_i guesses that they all have empty intersection with $\overline{A}$, then we will close gate G_i and begin to fill each S_k^i with a single element, in increasing order of k. We fill S_k^i with the least element x at gate G_i such that $m(x) > r(j, s)$ for $j = i + k$ and $m(x)$ is greater than any element in any $S_{k'}^i$ for $k' < k$.

When a set S_k^i contains an element not in A, we call that element x and its partner $m(x)$ *witnesses*. Once each S^i_k contains an element, we reopen the gate G_i to all non-witnesses.

We then wait until φ_i guesses that each S^i_k has nonempty intersection with $\overline{A}$. Now, there are $4^{i+1}+1$ witnesses and only 4^{i+1} different i-states.Thus, by the pigeonhole principle, there are two witnesses in the same i-state. Say they are x_0 in $S^i_{k_o}$ and x_1 in $S^i_{k_1}$, with $k_0 < k_1$. We will enumerate x_0 into A and $m(x_1)$ into $\widehat{A}$. These elements are entering in the same state. Once elements enter A or $\widehat{A}$, they are removed from the pinball machine. Their previous matches, $m(x_0) = \widehat{y}$ and x_1, do not get removed from the pinball machine. Instead, set $m(x_1)$ to be $\widehat{y}$, the element previously known as $m(x_0)$. We reset all $S^i_{k'}$ for $k' > k_0$, removing the name "witness" from any elements in sets that are reset, and starting new empty $S^i_{k'}$. Note that $S^i_{k_0}$ has not been reset, but its only element has entered A.

If ever φ_i again guesses correctly which sets S^i_k have nonempty intersection with $\overline{A}$, then we must repeat the process, with some minor changes. We again close the gate G_i until there is a single element in each S^i_k that is not also in A. For $k = 0$, we fill S^i_0 as before, with the least element x such that $m(x) > r(j, s)$ for $j = i + 0 = i$. For $k > 0$, we wait until S^i_{k-1} has been filled and enumerate into it the least x such that $m(x) > r(j, s)$ for $j = i + k + n^i_k$, where n^i_k is the number of times S^i_k has been reset by the action of P_i. The purpose of n^i_k is so that if P_i acts infinitely often, it will still only injure each N_j finitely often. Once each S^i_k contains a witness, we reopen G_i to all non-witnesses. We then continue as in the previous paragraph.

In order that S^i_k respect N_j whenever $j \leq i + k + n^i_k$, if $\Phi^{\widehat{A}}_j(j)[s+1]\downarrow$ by a new computation, we reset S^i_k. This causes any witnesses to no longer be witnesses and the set to be built again from an empty set. Note that it does not cause n^i_k to increase because the resetting was caused by N_j and not by the action of P_i.

The primary reason for using multiple S^i_k sets instead of a single S^i is that we want to respect more and more computations when we enumerate elements into $\widehat{A}$, which we could not do with only a single S^i that we can only reset finitely often. We will show in the verification that there will be some S^i_k that is only reset finitely often that we will use to satisfy P_i.

3.3 Construction

Stage $s = 0$. Let each $n^i_k = 0$. All sets begin empty and all gates begin open.

Stage $s + 1$.

Step 1: Place s on machine M at gate G_0 and $\widehat{s}$ on machine $\widehat{M}$ at gate G_0. Let $m(s) = \widehat{s}$.

Step 2: For each x and e, if $x \in W_{e,s}$, and x is either at gate $G_{e'}$ for $e' \geq e$ or x has been removed from the machine, enumerate x into $U_{e,s+1}$. Similarly, if $\widehat{x} \in W_{e,s}$, and either $\widehat{x}$ is at gate $G_{e'}$ for $e' \geq e$ or $\widehat{x}$ has been removed from the machine, enumerate $\widehat{x}$ into $V_{e,s+1}$.

In addition, if $x \in U_{e,s+1}$ and is still on the machine, enumerate $m(x)$ into

$\widehat{U}_{e,s+1}$. Similarly, if $\widehat{x} \in V_{e,s+1}$ and is still on the machine, enumerate $m^{-1}(\widehat{x})$ into $\widehat{V}_{e,s+1}$.

Step 3: If $\Phi_j^{\widehat{A}}(j)[s]\downarrow$ via a new computation, reset all S_k^i with $k+i+n_k^i \geq j$ by creating a new S_k^i, increasing n_k^i by one, and cancelling all witnesses from S_k^i.

Step 4: For each i, in increasing order, check if either of the following cases apply.

Case 4A (filling S_k^i): At least one S_k^i has empty intersection with $\overline{A}$, and either the gate is closed or $\varphi_i(g(i,k,n_k^i),s)$ is equal to the characteristic function of $S_k^i \cap \overline{A}$ for each k. Close the gate if not already closed. For the least k such that $S_k^i \cap \overline{A}$ has empty intersection, check if there is an x at gate G_i such that both x and $m(x)$ are larger than any current or previous witness at the gate and $m(x) > r(j,s)$ for $j = i+k+n_k^i$. If so, enumerate x into S_k^i and call it and $m(x)$ witnesses. If we enumerated an element into S_k^i for $k = 4^{i+1}$, open the gate. Continue to Step 5.

Case 4B (enumerating into A and $\widehat{A}$): All S_k^i have nonempty intersection with $\overline{A}$ and $\varphi_i(g(i,k,n_k^i),s) = 1$ for all k. Find the least witness x_0 such that there is a witness x_1 in the same i-state as x_0. Such a pair is guaranteed by the pigeonhole principle. Say the element x_0 is in $S_{k_0}^i$ and x_1 is in $S_{k_1}^i$. Note that $k_0 < k_1$. Enumerate x_0 into A and $m(x_1)$ into $\widehat{A}$ and remove both x_0 and $m(x_1)$ from the machines. Now x_1 and $\widehat{y} = m(x_0)$ do not have matches on the machine, so set $m(x_1) = \widehat{y}$. Reset each S_k^i for $k > k_0$ by increasing n_k^i and starting S_k^i over with no witnesses. Note that $S_{k_0}^i$ does not get reset and so is not empty, but now has empty intersection with $\overline{A}$ and thus no witnesses.

Step 5: For each x at an open gate G_e such that x is not a witness and $x,\ m(x) > e$, move x and $m(x)$ to gate G_{e+1} on their respective machines.

3.4 Verification

Lemma 3.3. *Each N_j is injured finitely often and is satisfied, so $\widehat{A}$ is low.*

Proof. Induct on j. Suppose true for all $j' < j$. Then since $\mathrm{N}_{j'}$ is injured finitely often, $\lim_s r(j',s)$ exists.

If $\Phi_j^{\widehat{A}}(j)[s]\downarrow$, then the only S_k^i that can injure the computation by enumerating $m(x)$ into $\widehat{A}$ for $x \in S_k^i$ must satisfy $i+k < j$. Each of these S_k^i gets reset finitely often by $\mathrm{N}_{j'}$, for $j' < j$, by the induction hypothesis. By the construction, after an element $m(x)$ enters $\widehat{A}$, S_k^i gets reset and n_k^i increases by one, as S_k^i is playing the role of $S_{k_1}^i$ in the construction, where $k_1 > k_0$. Thus, after finitely many elements of the form $m(x)$ with $x \in S_k^i$ enter $\widehat{A}$, any future elements $x \in S_k^i$ must satisfy $m(x) > r(j,s)$ since $i+k+n_k^i > j$ for large enough n_k^i. Since N_j can only be injured finitely often by finitely many S_k^i, N_j will eventually be satisfied. Thus, the jump of $\widehat{A}$ is limit computable, so $\widehat{A}$ is low. □

Lemma 3.4. *For each gate G_i, infinitely many elements pass through gate G_i to gate G_{i+1}.*

Proof. Induct on i. Suppose true for all $i' < i$. Then infinitely many elements reach gate G_i.

Note that if gate G_i is ever closed, then it will reopen when an element enters S_k^i for $k = 4^{i+1}$. First we show that it will eventually reopen. As G_i is closed, we know that $S_{4^{i+1}}^i$ has empty intersection with $\overline{A}$, so Case 4A applies. While the gate is closed, Case 4B will not act for this i, so the only way for any S_k^i to be reset is for $\Phi_j^{\widehat{A}}(j)[s]$ to converge via a new computation for $j \leq k + i + n_k^i$. However, no n_k^i will change for any k while G_i is closed, and Lemma 3.3 tells us that $\Phi_j^{\widehat{A}}(j)[s]$ can only converge via a new computation finitely often, so each S_k^i will only be reset finitely often while G_i remains closed. Furthermore, for $j = i + k + n_k^i$, $r(j, s)$ will eventually stop increasing while the gate remains closed. As infinitely many elements arrive at gate G_i, each S_k^i will eventually receive an element, including $S_{4^{i+1}}^i$, at which point the gate is opened. Therefore, whenever gate G_i is closed, it is eventually reopened.

If the gate G_i is closed only finitely often, then almost all elements that enter G_i also enter G_{i+1}. Consider the case that G_i is closed infinitely often. Then it is opened infinitely often, meaning that infinitely often, all S_k^i have nonempty intersection with $\overline{A}$. In order for G_i to become closed again, some S_k^i must have empty intersection with $\overline{A}$. This can only happen by Step 4 Case 4B or Step 3 applies. In either case, there must be some S_k^i that gets reset and so its witness is no longer a witness, and on Step 5, it moves to gate G_{i+1} if large enough. Thus, infinitely often, an element moves from G_i to G_{i+1}. □

Lemma 3.5. *For each e, $W_e =^* U_e =^* V_e$, and thus $\{U_e\}_{e\in\omega}$ and $\{V_e\}_{e\in\omega}$ are skeletons.*

Proof. We will prove that $W_e =^* U_e$. The proof that $W_e =^* V_e$ is similar. Note that $U_e \subseteq W_e$, as we never enumerate an element into U_e until after it appears in W_e. Note also that the only elements in W_e that are not enumerated into U_e are those that are permanently held at a gate G_i for $i < e$. We know from Lemma 3.4 that all gates are open at infinitely many stages, thus no element is held forever at gate G_i unless it is forever labeled a witness or if either x or $m(x)$ are less than or equal to i. The latter situation happens for finitely many elements. When any S_k^i is reset, its witnesses are cancelled, meaning they are no longer considered to be witnesses, so any element that is forever a witness is in the final incarnation of S_k^i and never enters A. (For the V_e case, these witnesses are of the form $m(x)$ for $x \in S_k^i \cap \overline{A}$.) There is never more than one element of $S_k^i \cap \overline{A}$, for each $i < e$, so there are finitely many elements that are forever kept at G_i as witnesses. Thus, all other elements of W_e eventually either leave the machine or reach G_e and are enumerated into U_e. □

Lemma 3.6. *Requirement P_i is met, so $\overline{A}$ is not semilow.*

Proof. We must show that $\lim_s \varphi_i(e,s) \neq \{e \mid W_e \cap \overline{A} \neq \emptyset\}$. Suppose for a contradiction that it is false. Let i be the least such that $\lim_s \varphi_i(e,s)$ gives the characteristic function of the set of e such that W_e has nonempty intersection with $\overline{A}$.

Note that S^i_0 can only be reset by N_j for $j \leq i$, which will only reset S^i_0 finitely often. Let k^* be the greatest k such that S^i_k is only ever reset finitely often. If $k^* = 4^{i+1}$, then Step 4 only acts finitely often, meaning that Case 4A and Case 4B only apply finitely often, so for almost all s, φ_i guesses wrong about whether some S^i_k has nonempty intersection with $\overline{A}$. Thus, since we are assuming that φ_i is eventually correct, we cannot have that $k^* = 4^{i+1}$.

Now, let $S^i_{k^*}$ be reset n times in total. Then $\lim_s \varphi_i(g(i,k^*,n),s)$ is 1 if $S^i_{k^*}$ has nonempty intersection with $\overline{A}$ and 0 otherwise. Since $k^* \neq 4^{i+1}$, $S^i_{k^*+1}$ gets reset infinitely often, which means that Step 4 Case 4B acts infinitely often by enumerating an element $x \in S^i_{k^*}$ into A. Thus, infinitely often, $S^i_{k^*}$ gets a new witness in Case 4A and then has that witness enumerated into A in Case 4B. Therefore, there are infinitely many stages when $S^i_{k^*}$ has empty intersection with $\overline{A}$ and Case 4A applies because $\varphi_i(g(i,k^*,n),s) = 0$ and there are infinitely many stages when $S^i_{k^*}$ has nonempty intersection with $\overline{A}$ and Case 4B applies because $\varphi_i(g(i,k^*,n),s) = 1$. This contradicts the assumption that $\lim_s \varphi_i(g(i,k^*,n),s)$ exists. Therefore, requirement P_i is met, and since $\{e \mid W_e \cap \overline{A} \neq \emptyset\}$ is not limit computable, $\overline{A}$ is not semilow.

□

Lemma 3.7. *There is an effective automorphism of $\mathcal{E}^*$ taking A to $\widehat{A}$.*

Proof. Note that the construction provides a simultaneous computable enumeration of the sets $\{\widehat{U}_e\}_{e\in\omega}$ and $\{\widehat{V}_e\}_{e\in\omega}$. We guarantee that ν is infinite on the complement of A if and only if $\widehat{\nu}$ is infinite on the complement of $\widehat{A}$ by always enumerating elements simultaneously with their partners: x and $m(x)$ are in the same e-state, where G_e is the highest gate they reach. Note that there is no possibility for either x or $\widehat{x}$ to be rematched infinitely often, as they each only reach finitely many gates and can only be witnesses once at each gate. Also note that if x and $m(x)$ are permanently partners and remain forever at gate G_e, then they are in the same e'-state for all $e' > e$ as well, since neither are enumerated into U_i, V_i, $\widehat{U}_i$ or $\widehat{V}_i$.

Next, we apply the Effective Extension Theorem. It suffices to show that the gateway states for A equal the gateway states for $\widehat{A}$. We only ever enumerate into either A or $\widehat{A}$ during Step 4 Case 4B. When we do, we pick witnesses x_0 and x_1 at Gate G_i that are in the same i-state, which are guaranteed to exist by the pigeonhole principle. Then, we enumerate x_0 into A and $m(x_1)$ into $\widehat{A}$. If x_0 is in i-state ν, then $m(x_1)$ is in $\widehat{\nu}$. As above, note that x_0 and $m(x_1)$ are also in the same e-state for any e because if $e > i$, then neither element is in U_e, V_e, $\widehat{U}_e$, or $\widehat{V}_e$. Thus, we get that the gateway states D^A_ν equal the gateway states $D^{\widehat{A}}_{\widehat{\nu}}$.

By the Effective Extension Theorem, we can extend the partial automorphism on the complements of A and $\widehat{A}$ to a total automorphism of $\mathcal{E}^*$ taking A

to $\widehat{A}$.

□ □

4 Semilow$_{1.5}$ not semilow in all nonlow degrees

In Theorem 3.2, we showed that there is set A that does not have semilow complement and is effectively automorphic to a low set. Since A is effectively automorphic to a low set, A has semilow$_{1.5}$ complement (see the introduction).

In Theorem 4.1, we show that in every nonlow c.e. degree, there is a set that has semilow$_{1.5}$ complement but that does not have semilow complement. It remains open whether we can find such sets in every degree that are also effectively automorphic to low sets.

Theorem 4.1. *For every nonlow c.e. degree* $\mathbf{d}$*, there is a c.e. set* $A \in \mathbf{d}$ *such that* A *has semilow*$_{1.5}$ *complement, but does not have semilow complement.*

Corollary 4.2. *The nonlow c.e. degrees are precisely the degrees of c.e. sets that have semilow*$_{1.5}$ *complement but not semilow complement.*

The corollary follows immediately from the theorem, since every low c.e. set has semilow complement.

Proof. Given a nonlow c.e. set D, we will construct an $A \equiv_{\mathrm{T}} D$ such that $\overline{A}$ is semilow$_{1.5}$ but not semilow.

4.1 Requirements:

To ensure that $D \leq_{\mathrm{T}} A$, we will code D into A. To begin, we construct a computable list of disjoint finite sets $F_k \subset \omega$, with F_0 containing 1 element and each other F_k having $2k^2$ elements.

C_k: $k \in D$ if and only if some element of F_k is in A.

To make $\overline{A}$ not semilow, we proceed similarly as in Theorem 3.2 by meeting the following requirement for each total computable function h on two inputs:

P_h: $\lim\limits_s h(e,s)$ is not the characteristic function of $\{e \mid W_e \cap \overline{A} \neq \emptyset\}$.

Let $\{\varphi_i\}_{i\in\omega}$ be a listing of all partial computable functions on two inputs. Then we must meet P_h for all $h = \varphi_i$. We find it easier to refer to h instead of i in this situation. However, when we say $h < j$, please note that if we are using h to stand in for φ_i, then we mean that $i < j$.

We will try to meet P_h for each α of length i for $h = \varphi_i$ on our tree of strategies, and for each $e \in \omega$, so we will label our requirements $\mathrm{P}^{\alpha}_{h,e}$. To meet P^{α}_h, for each e, if $\mathrm{P}^{\alpha}_{h,e}$ has been reset n times, then we build $S^{\alpha}_{h,e} = W_{r(\alpha,e,n)}$, where r is known in advance by the Recursion Theorem with Parameters. If h guesses that $S^{\alpha}_{h,e} \cap \overline{A}$ is empty, we add an element, called a *witness*, to $S^{\alpha}_{h,e}$. If h changes its guess, then we want to enumerate the witness into A. When $\mathrm{P}^{\alpha}_{h,e}$ gets reset, its witnesses become inactive. We will meet the requirement for some e and α, so that $W_{r(\alpha,e,n)} \cap \overline{A}$ is nonempty if and only if $\lim_s h(r(\alpha,e,n),s) \neq 1$.

Define $Y_h = \{x \mid x \text{ is ever a witness for some } \mathrm{P}^{\alpha}_{h,e}, \text{ for any } e \text{ and } \alpha\}$ and $Z_h = \{x \mid x \text{ is a witness in } Y_h \text{ that becomes inactive but does not enter } A\}$.

Let Y^j be defined as $\bigcup_{h \leq j} Y_h$ and Z^j be defined as $\bigcup_{h \leq j} Z_h$. Thus Y^j is the set of witnesses for nodes of length at most j and Z^j is the subset of Y^j of witnesses that become inactive.

For A to have semilow$_{1.5}$ complement, we need to ensure:

L: $\{j \mid W_j \cap \overline{A} \text{ infinite}\} \leq_m \mathrm{Inf}$.

To do this, we will show that $W_j \cap \overline{A}$ is infinite if and only if either $W_j \cap Z^j$ is infinite or $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, where a stage is expansionary if $(W_j - (Y^j \cup A))[s]$ is larger than it has been at any previous stage. Since $W_j \cap Z^j$ and the set of expansionary stages of $W_j - (Y^j \cup A)$ are both c.e. sets, this will give us the desired m-reduction.

As mentioned, we will work on a binary tree of strategies. Let α be a length i string on the tree. For each $j < i$, $\alpha(j) = 0$ indicates a guess that $W_j - (Y^j \cup A)$ has infinitely many expansionary stages and $\alpha(j) = 1$ is a guess that it has finitely many. P^{α}_h will work to satisfy P_h while assuming that α is correctly guessing the true outcome.

The tree allows us to ensure that if $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, it is in fact an infinite set. The nodes guessing that there are infinitely many expansionary stages are not allowed to add witnesses when L_j is injured. Instead, they wait until a new expansionary stage is reached. This way, they do not create witnesses that may later enter $W_j - (Y^j \cup A)$, only to be forced into A. In addition, nodes guessing that $W_j - (Y^j \cup A)$ has finitely many expansionary stages are reset each time $W_j - (Y^j \cup A)$ reaches a new expansionary stage, meaning that any of their current witnesses will never enter A. The combination of these two actions guarantees that if $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, it has infinitely many elements that never enter A and so it is infinite.

In addition, we need that $A \leq_{\mathrm{T}} D$, which we achieve by a nonlow permitting argument using a technique of Downey, Jockusch, and Schupp [3]. An element only becomes a witness for $\mathrm{P}^{\alpha}_{h,e}$ when $\Phi^D_e(e)[s]\downarrow$. If this computation later converges, such a witness gets enumerated into A. For each α, we build a computable function on two inputs, $\ell_\alpha(e, s)$. If $\mathrm{P}^{\alpha}_{h,e}$ wishes to enumerate a witness into A at stage s, we set $\ell_\alpha(e, s) = 1$. Otherwise $\ell_\alpha(e, s) = 0$. If the computation $\Phi^D_e(e)[s]$ is injured, then D permits the witness to enter A and so the witness enters A as desired. We will only enumerate witnesses when permitted, so that we will achieve $A \leq_{\mathrm{T}} D$. Thus, we must ensure that D permits frequently enough. Since D is not low and ℓ_α is computable, we know that $K^D(e)$ is not the limit of $\ell_\alpha(e, s)$ as s approaches infinity, else K^D would be limit computable and thus D would be low. Therefore, there must be some e such that $K^D(e) \neq \lim_s \ell_\alpha(e, s)$. When $\mathrm{P}^{\alpha}_{h,e}$ wishes to enumerate an element, $\ell_\alpha(e, s) = 1$, so $\Phi^D_e(e)$ must diverge. When the current computation is injured by a new element entering D below the use, $\mathrm{P}^{\alpha}_{h,e}$ has permission to enumerate the element into A. Thus, for each α on the true path, there is some e such that D will give permission for $\mathrm{P}^{\alpha}_{h,e}$ to enumerate elements into A as often as needed.

4.2 Construction

For each k, we define a finite set F_k. Each of these sets will be contained within $\omega^{[0]}$, where $\omega^{[n]}$ is the nth "row" of ω, the set of all elements of the form $\langle x, n\rangle$ using the standard pairing function. We let F_0 be the singleton set $\{\langle 0,0\rangle\}$, and for $k > 0$, we let F_k be the least $2k^2$ elements in $\omega^{[0]}$ that have not appeared in any $F_{k'}$ for $k' < k$.

We build an approximation δ_s to the true path. We call s an α-stage if α is a substring of δ_s. Define $\delta_0 = \lambda$, the empty string. Let δ_s be the length s string defined by: $\delta_s(j) = 0$ if $W_j - (Y^j \cup A)$ has reached a new expansionary stage since the last $\delta_s \restriction j$-stage, and 1 otherwise.

Let $\ell_\alpha(e, 0) = 0$ and $\ell_\alpha(e, s+1) = \ell_\alpha(e, s)$ unless otherwise specified. We begin with all L_j allowing addition of new witnesses for every P^{α}_{h}. During the construction, in order to prevent further injury to itself, L_j may disallow some P^{α}_{h} from adding new witnesses. However, we will show that if α is on the true path, the construction ensures that P^{α}_{h} will be allowed to add witnesses infinitely often.

Nodes on the tree may be put in 1-1 correspondence with $\mathbb{N} - \{0\}$ via a function $f : 2^{<\omega} \to \mathbb{N} - \{0\}$. Using such a correspondence, let $\langle \alpha, e\rangle := \langle f(\alpha), e\rangle$. In Step 3, witnesses come from $\omega^{[\langle \alpha, e\rangle]}$.

Stage $s = 0$: $\ell_\alpha(e, 0) = 0$ and all sets $S^{\alpha}_{h,e}$ begin empty for all α, h, and e.

Stage $s + 1$:

Step 1 (new expansionary stage): For each $j \in \omega$, if $W_j - (Y^j \cup A)$ grows to a size we haven't seen before, i.e. it reaches a new expansionary stage, then do the following for each α of length greater than j: If α guesses that $W_j - (Y^j \cup A)$ has infinitely many expansionary stages (i.e. $\alpha(j) = 0$), then L_j now allows addition of new witnesses for P^{α}_{h}. If α guesses that $W_j - (Y^j \cup A)$ has finitely many expansionary stages ($\alpha(j) = 1$), we reset $\mathrm{P}^{\alpha}_{h,e}$ for all e, making $\ell_\alpha(e, s) = 0$ and declaring $\mathrm{P}^{\alpha}_{h,e}$ and any witnesses for $\mathrm{P}^{\alpha}_{h,e}$ inactive. When reset, we will start a new $S^{\alpha}_{h,e}$ that begins empty. This will ensure in Lemma 4.3 that nodes guessing incorrectly that $W_j - (Y^j \cup A)$ has finitely many expansionary stages will not prevent $W_j \cap \overline{A}$ from being infinite by putting too many elements into A.

Step 2 (allowing some disallowed witnesses): We say a node of length j *is allowed* new witnesses if no $j' < j$ disallows P^{α}_{h} from adding witnesses. For each j, N, and β, where β has length j, if L_j currently disallows for N (as defined in Step 5) and if β is allowed new witnesses, then set L_j to allow new witnesses for all α unless Step 2 has acted at some previous stage for these particular j, N, and β. Do this for all possible triples, of which there are only finitely many. This step guarantees that nodes on the true path will be allowed to act in Step 3 infinitely often.

Step 3 (adding witnesses): We act for each α of length at most s such that P^{α}_{h} is allowed to add witnesses by each L_j with $j < s$ such that $\alpha(j) = 0$, and such that either s is an α-stage or α is to the left of δ_s and if s_α was the last α-stage, P_h^α has not been allowed to act at any stage t, $s_\alpha \leq t < s$. In other words, we act on the approximation to the true path δ_s and for any nodes to the left of δ_s that have not been allowed to act since before they were last on the approximation to the true path. We act in length-lexicographic order. For each such α, we ask if there is any $e < s$ such that

- $\mathrm{P}_{h,e}^\alpha$ is inactive,
- $\Phi_e^D(e)[s]\downarrow$,
- $h(r(\alpha, e, n)) = 0$ for n the number of times $\mathrm{P}_{h,e}^\alpha$ has been reset, and
- for all $e' < e$, $\Phi_{e'}^D(e')[s]\downarrow \iff \ell_\alpha(e', s) = 1$ or $\Phi_{e'}^D(e')[s]\downarrow$ and has changed computations at least e times since the last time that $\mathrm{P}_{h,e}^\alpha$ added a witness.

If so, then for each such e, let $x > s$, $x \in \omega^{[\langle\alpha,e\rangle]}$, be a fresh witness, larger than any previously chosen witness. Enumerate $x \in S_{h,e}^\alpha$. Declare x to have use s, which we note is larger than the use of $\Phi_e^D(e)[s]$. We call both $\mathrm{P}_{h,e}^\alpha$ and x *active*.

Step 4 (changing ℓ_α when h changes its guess): If $\mathrm{P}_{h,e}^\alpha$ is active, $\ell_\alpha(e, s) = 0$, and $h(r(\alpha, e, n), s + 1) = 1$ where n is the number of times $\mathrm{P}_{h,e}^\alpha$ has been reset, then set $\ell_\alpha(e, s + 1) = 1$. Reset $\mathrm{P}_{h,e'}^\alpha$ for all $e' > e$ by making $\mathrm{P}_{h,e'}^\alpha$ and all elements in $S_{h,e'}^\alpha$ inactive and starting new empty sets $S_{h,e'}^\alpha$, as well as setting $\ell_\alpha(e', s + 1) = 0$. Perform this step for each α and for the least e such that it applies.

Step 5 (enumerating witnesses into A and disallowing new witnesses): If $x \in S_{h,e}^\alpha$ is active with use u and $D_{s+1}\upharpoonright u \neq D_s\upharpoonright u$, then put x into A_{s+1} and set $\ell_\alpha(e, s + 1) = 0$. Declare $\mathrm{P}_{h,e}^\alpha$ and x inactive.

If by such an enumeration, we cause an element of $(W_j - (Y^j \cup A))[s]$ to enter A, then we say L_j does not *allow* P_h^α to add witnesses for any α such that $\alpha(j) = 0$. In addition, if the size of $(W_j - (Y^j \cup A))$ has reached the number N, we say that L_j *disallows for* N.

Step 6 (coding D): If $k \in D_{s+1} - D_s$, enumerate one element from F_k into A_{s+1} such that for each $M > 0$, if $j + M = k$, then the element is not one of the least M elements in $(W_j - (Y^j \cup A))[s]$.

4.3 Verification

Lemma 4.3. *For each $j \in \omega$, if the set of expansionary stages of $W_j - (Y^j \cup A)$ is infinite, then $W_j \cap \overline{A}$ is infinite.*

Proof. Assume the set of expansionary stages of $W_j - (Y^j \cup A)$ is infinite. We will show that for each M, $W_j - (Y^j \cup A)$ has at least M elements.

No element in $(W_j - (Y^j \cup A))[s]$ ever enters Y^j since new witnesses are chosen to be larger than s, and elements of $(W_j - (Y^j \cup A))[s]$ are smaller than s.

Induct on M. Assume true for $M-1$.

For $k \leq j$, only finitely many elements are ever put into A by C_k. For $k > j$, C_k can only bring the size of $W_j - (Y^j \cup A)$ below M if $M + j > k$, which happens finitely often. Let s_0 be a stage by which the least $M-1$ elements in $(W_j - (Y^j \cup A))[s_0]$ never enter A and after which no C_k enumerates any of the least M elements of $(W_j - (Y^j \cup A))[s]$ into A. Let $s_1 > s_0$ be a stage after which Step 2 never acts for $(j, M-1, \beta)$ for any β of length j. We may assume that at every stage $s > s_1$, the Mth least element of $(W_j - (Y^j \cup A))[s]$ is a witness for some $\mathrm{P}^\alpha_{h,e}$, else it would never enter A and we would be guaranteed to have at least M elements in $W_j - (Y^j \cup A)$, as desired. Note that the length of α is greater than j since the witnesses are not in Y^j.

Let $s_2 > s_1$ be a stage such the Mth least element in $(W_j - (Y^j \cup A))[s_2]$, called x_M, is the least that will ever be in the Mth position of any $(W_j - (Y^j \cup A))[s]$ for $s \geq s_2$. Now, if x_M never enters A by Step 5, we are done, so assume x_M enters A. When x_M enters A, all larger witnesses also enter A, and since the next element in the Mth position is a witness, and it cannot be less than x_M due to minimality, then the next element that enters the Mth position has yet to become a witness. When x_M enters A, L_j does not allow any P^α_h to add witnesses for any α such that $\alpha(j) = 0$. Step 2 has finished acting for j and $M-1$, so it will not cause L_j to later allow. The next time L_j allows will be by Step 1, which means $(W_j - (Y^j \cup A))$ has already reached a new expansionary stage. Thus, at the first expansionary stage after x_M enters A, the only witness that could be in the Mth position must be a witness for an α with $\alpha(j) = 1$. At that new expansionary stage, Step 1 inactivates the witnesses for α. Thus the element in the Mth position can never enter A, so $(W_j - (Y^j \cup A))$ has at least M elements.

□

We say that α is on the *true path* if $\alpha(j) = 0$ if and only if $W_j - (Y^j \cap \overline{A})$ has infinitely many expansionary stages during the construction.

Lemma 4.4. *Assume α is on the true path. Let e be the least such that $\lim_s \ell_\alpha(e,s) \neq K^D(e)$. Then $\mathrm{P}^\alpha_{h,e}$ is reset finitely often (n times) and $\lim_s h(r(\alpha,e,n),s) = 0 \iff W_{r(\alpha,e,n)} \cap \overline{A}$ is nonempty. Thus, $\overline{A}$ is not semilow.*

Proof. $\mathrm{P}^\alpha_{h,e}$ can only be reset in two ways: In Step 1 by $W_j - (Y^j \cup A)$ growing while α is guessing it has finitely many expansionary stages, which can only happen finitely often for the correct α, and by Step 4 acting for $\mathrm{P}^\alpha_{h,e'}$, for $e' < e$. Such a $\mathrm{P}^\alpha_{h,e'}$ only resets $\mathrm{P}^\alpha_{h,e}$ when $\ell_\alpha(e',s)$ changes to 1 from 0. This can only happen finitely often since $\lim_s \ell_\alpha(e',s)$ must exist for $e' < e$. Thus, $\mathrm{P}^\alpha_{h,e}$ is reset n times for some finite n.

Case 1: Suppose $S^\alpha_{h,e} \cap \overline{A}$ is nonempty, for $S^\alpha_{h,e} = W_{r(\alpha,e,n)}$.

Then there is some $x \in S^\alpha_{h,e} \cap \overline{A}$ that was enumerated into $S^\alpha_{h,e}$ by Step 3 at stage $s_0 + 1$ when $\Phi^D_e(e)[s_0]\downarrow$ and $\ell_\alpha(e,s_0) = 0$. We want to show that $\lim_s h(r(\alpha,e,n),s) = 0$.

Assume $h(r(\alpha,e,n),t)$ equals 1 at some stage $t > s_0$, so $\ell_\alpha(e,t) = 1$. Since $x \in \overline{A}$, $\Phi_e^D(e)[s_0] = \Phi_e^D(e)\downarrow$ and both computations are the same. Since e was chosen so that $\lim_s \ell_\alpha(e,s) \neq K^D(e)$, we know that $\ell_\alpha(e,s)$ must change back to 0, which can only happen in Step 5. However, this can't happen since $\Phi_e^D(e)[s_0]\downarrow$ and never changes below the use. This gives a contradiction. Thus, if $S^\alpha_{h,e} \cap \overline{A}$ is nonempty, then $\lim_s h(r(\alpha,e,n),s) = 0$.

Case 2: Suppose $S^\alpha_{h,e} \cap \overline{A}$ is empty.

First, note that if $S^\alpha_{h,e}$ has infinitely many witnesses, all of which enter A, then Step 5 acts infinitely often, meaning that $\Phi_e^D(e)[s]$ changes infinitely often, so $\Phi_e^D(e)$ diverges, and thus $\lim_s \ell_\alpha(e,s) \neq 0$. Since $\ell_\alpha(e,s) = 1$ infinitely often, we must have that $h(r(\alpha,e,n),s)$ switches from 0 to 1 infinitely often, as it must be 0 when each witness is appointed and 1 when $\ell_\alpha(e,s)$ changes to 1. Thus $\lim_s h(r(\alpha,e,n,s)$ doesn't exist.

We may thus assume that after a finite stage, $S^\alpha_{h,e} \cap \overline{A}$ is empty and never gets more elements. We may also assume that $\lim_s h(r(\alpha,e,n),s) = 0$, else we have achieved our goal. We now show that $\mathrm{P}^\alpha_{h,e}$ will eventually be able to act by adding a new witness in Step 3, contradicting that $S^\alpha_{h,e}$ never gets a new element. We check that all the conditions of Step 3 will eventually be met.

For the first bullet point, we know that $\mathrm{P}^\alpha_{h,e}$ will be inactive for almost all s since it is only active when $S^\alpha_{h,e} \cap \overline{A}$ is nonempty. Since $\mathrm{P}^\alpha_{h,e}$ is inactive, we know that $\ell_\alpha(e,s) = 0$ at almost all stages, as it only can be 1 when $\mathrm{P}^\alpha_{h,e}$ is active. Now, we also know that $\lim_s \ell_\alpha(e,s) \neq K^D(e)$, so $\Phi_e^D(e)\downarrow$. This gives us the second bullet point for almost all s. By the assumption in the previous paragraph, for almost all s, $h(r(\alpha,e,n),s) = 0$, so we meet the third bullet point.

To check that the fourth bullet point is met, note that for all $e' < e$, either $K^D(e') = 1 = \lim_s \ell_\alpha(e',s)$, in which case for almost all s, e' does not prevent e from acting, or $K^D(e') = 0 = \lim_s \ell_\alpha(e',s)$. In the latter case, there are two possibilities. If for almost all s, $\Phi_{e'}^D(e')[s]\uparrow$, then for almost all s, e' does not prevent e from acting. Otherwise, there are infinitely many stages such that $\Phi_{e'}^D(e')[s]$ converges, but since $\Phi_{e'}^D(e')\uparrow$, the computation changes infinitely often, so eventually the computation will have changed at least e times. Thus, e' only prevents e from adding a new witness for finitely many stages.

We have shown that for almost all s, the conditions in all the bullet points are met. We still must show that α is *allowed* to add a witness at a stage where it is able to act.

Claim: For every α on the true path, α is allowed to add witnesses infinitely often.

Proof of Claim: Induct on the length of α. Assume true for the predecessor of α, α^-. Let α^- have length j. If $\alpha(j) = 1$, then α is allowed to act at every stage that α^- is allowed to act. If $\alpha(j) = 0$, we need to show that infinitely often, when α^- is allowed to act, L_j allows action as well. Suppose not. Then there is some stage s_0 such that α is not allowed to act for any $s \geq s_0$. Each time α^- is allowed to act after stage s_0, L_j disallows α from adding new witnesses. This means that Step 2 does not apply for j at any of these stages. We know that by the proof of Lemma 4.3, L_j disallows for each N at most finitely often. Since α is on the true path, $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, so L_j eventually disallows only for N's that Step 2 never acted for before stage s_0. Thus eventually Step 2 will apply to j, N, and α^-, so α will be allowed to add new elements, which gives a contradiction, proving the claim.

Now wait until a stage s_0 such that the approximation to the true path never goes to the left of α and the bullet points do not prevent $\mathrm{P}^\alpha_{h,e}$ from acting. Let $s_1 > s_0$ be the next α-stage, and $s_2 \geq s_1$ the next stage that α is allowed to add witnesses. Then s_2 is either an α-stage or δ_{s_2} is to the right of α, so we may perform Step 3 for P^α_h.

At infinitely many stages, Step 3 will be able to act by adding a witness. Thus, it will act, contradicting that no new witnesses are added. Therefore, if no new witnesses are ever added, then we must have that $\lim_s h(r(\alpha, e, n), s)$ does not equal 0, satisfying the lemma.

□

Lemma 4.5. *For each partial computable $\varphi_i = h$, $Y_h \cap \overline{A} \subseteq^* Z_h$.*

Proof. Proving this lemma amounts to showing that there are only finitely many elements in $Y_h \cap \overline{A}$ that remain active. For each e and each α, there can only be one active element at a time. Let e be such that $\mathrm{P}^\alpha_{h,e}$ is met, as in Lemma 4.4. Either e resets all $e' > e$ infinitely often or $\lim_s \ell_\alpha(e, s)$ exists.

If $\lim_s \ell_\alpha(e, s) = 0$, then $\Phi^D_e(e)\downarrow$. Then $\Phi^D_e(e)[s]$ changes computations some finite number of times, so by the final bullet of Step 3, only finitely many $e' > e$ will ever be able to add witnesses, and will only be able to do so finitely often. If $\lim_s \ell_\alpha(e, s) = 1$, then $\Phi^D_e(e)\uparrow$, but this is not possible since $\ell_\alpha(e, s) = 0$ if $\Phi^D_e(e)[s]\uparrow$.

Thus, for a fixed α and h, either almost all $\mathrm{P}^\alpha_{h,e}$ are reset infinitely often or only finitely many ever become active. Since there are only finitely many α of length i, $h = \varphi_i$, we conclude that $Y_h \cap \overline{A} \subseteq^* Z_h$. □

Lemma 4.6. *For each $j \in \omega$, $W_j \cap \overline{A}$ is infinite if and only if either the set of expansionary stages of $W_j - (Y^j \cup A)$ is infinite or $W_j \cap Z^j$ is infinite. Thus, $\overline{A}$ is semilow$_{1.5}$.*

Proof. This Lemma gives us that $\overline{A}$ is semilow$_{1.5}$ because to tell if $W_j \cap \overline{A}$ is infinte, we can ask Inf if the c.e. set that is the union of $W_j \cap Z^j$ and the set of expansionary stages of $W_j - (Y^j \cup A)$ is infinite. This gives us an m-reduction from $W_j \cap \overline{A}$ to Inf.

First, note that $Z^j \subseteq \overline{A}$, so if $W_j \cap Z^j$ is infinite, so is $W_j \cap \overline{A}$. By Lemma 4.3, we know that if the set of expansionary stages of $W_j - (Y^j \cup A)$ is infinite, then $W_j \cap \overline{A}$ is infinite.

Assume $W_j \cap \overline{A}$ is infinite. Suppose $W_j - (Y^j \cup A)$ has finitely many expansionary stages. Then $W_j - (Y^j \cup A)$ is finite. Thus, $W_j \cap \overline{A} \subseteq^* W_j \cap Y^j$. By Lemma 4.5, $Y_h \cap \overline{A} \subseteq^* Z_h$, so $W_j \cap \overline{A} \subseteq^* W_j \cap Z^j$. Thus $W_j \cap Z^j$ is infinite. □

Lemma 4.7. $A \leq_{\mathrm{T}} D$.

Proof. To determine if $k \in D$, ask if any elements of the set F_k are in A. The only way any of the elements can enter A is if $k \in D$. It is always possible to enumerate one of the elements into A because we must keep out at most $\sum_{M=1}^{k} M = k(k+1)/2$ elements, and $\frac{(k+1)k}{2} = \frac{k^2}{2} + \frac{k}{2} < 2k^2$, which is the size of F_k for $k > 0$. For $k = 0$, we do not need to keep out any elements, and F_0 is nonempty. □

Lemma 4.8. *$D \leq_{\mathrm{T}} A$.*

Proof. $\omega^{[0]} \cap A \leq_{\mathrm{T}} D$ since no element from $\omega^{[0]}$ enters A unless it is in F_k and k enters D.

For $\omega - \omega^{[0]}$, $x \in A$ if and only if x becomes a witness and then enters A. The set of witnesses is computable because if x is not a witness by stage x, it will never become a witness. If x is a witness, we ask what its use u is as a witness. Let s_x be such that $D \upharpoonright u = D_{s_x} \upharpoonright u$. Then $x \in A$ if and only if $x \in A_{s_x}$. □

□

5 Semilow$_2$, O.S.P, not semilow$_{1.5}$ in all nonlow degrees

As mentioned previously, a set B is *semilow*$_2$ if $\{e \mid W_e \cap B \text{ is infinite}\} \leq_{\mathrm{T}} \mathbf{0}''$. It follows immediately that if B is semilow$_{1.5}$, then B is semilow$_2$.

A set A has the *outer splitting property* if there exist total computable functions f and g such that, for each $i \in \omega$,

(a) $W_i = W_{f(i)} \sqcup W_{g(i)}$,

(b) $W_{f(i)} \cap \overline{A}$ is finite, and

(c) if $W_i \cap \overline{A}$ is infinite, then $W_{f(i)} \cap \overline{A}$ is nonempty.

Maass [8] showed that if a set A has semilow$_{1.5}$ complement, then A has the outer splitting property. Thus, the class of c.e. sets with semilow$_{1.5}$ complement is contained in the intersection of the sets with semilow$_2$ complement and the sets with the outer splitting property. We show that in every nonlow c.e. degree, this containment is strict.

Theorem 5.1. *For every nonlow c.e. degree* $\mathbf{d}$*, there is a c.e. set* $A \in \mathbf{d}$ *such that* A *has the outer splitting property and semilow*$_2$ *complement, but does not have semilow*$_{1.5}$ *complement.*

Corollary 5.2. *The nonlow c.e. degrees are precisely the degrees of c.e. sets that have the outer splitting property and semilow*$_2$ *complement but not semilow*$_{1.5}$ *complement.*

The corollary follows immediately from the theorem, since every low c.e. set has semilow$_{1.5}$ complement.

Proof. Given a nonlow c.e. set D, we will construct an $A \equiv_{\mathrm{T}} D$ such that $\overline{A}$ is semilow$_2$ but not semilow$_{1.5}$ and such that A satisfies the outer splitting property.

5.1 Requirements:

To ensure that $D \leq_{\mathrm{T}} A$, we will code D into A. To begin, we construct a computable list of disjoint finite sets $F_k \subset \omega^{[0]}$, such that F_0 contains two elements and each other F_k has $6k^2$ elements.

C_k: $k \in D$ if and only if some element of F_k is in A.

To make $\overline{A}$ not semilow$_{1.5}$, we meet the following requirement for each total computable function h:

P_h: $\{e \mid W_e \cap \overline{A} \text{ infinite}\} \neq \{e \mid W_{h(e)} \text{ infinite}\}$.

Let $\{\varphi_i\}_{i\in\omega}$ be a listing of all partial computable functions on two inputs. Then we must meet P_h for all $h = \varphi_i$. As in Theorem 4.1, we find it easier to refer to h instead of i in this situation and when we say $h < j$, we mean that $i < j$.

We will try to meet P_h for each α of length i for $h = \varphi_i$ on our tree of strategies and each $e \in \omega$, so we will label our requirements $\mathrm{P}^{\alpha}_{h,e}$. To meet P^{α}_{h}, for each e, if $\mathrm{P}^{\alpha}_{h,e}$ has been reset n times, then we build $S^{\alpha}_{h,e} = W_{r(\alpha,e,n)}$, where $r$is known in advance by the Recursion Theorem with Parameters. We add elements, called witnesses, to $S^{\alpha}_{h,e}$. For some e, we will guarantee that if $W_{h(r(\alpha,e,n))}$ is infinite, then we dump the witnesses into A, so $S^{\alpha}_{h,e} \cap \overline{A}$ is finite, and if $W_{h(r(\alpha,e,n))}$ is finite, $S^{\alpha}_{h,e}$ gets infinitely many witnesses that never enter A. Thus, $r(\alpha, e, n)$ will satisfy that $W_{r(\alpha,e,n)} \cap \overline{A}$ is infinite if and only if $W_{h(r(\alpha,e,n))}$ is finite.

Define

$Y^{\alpha}_{h,e} = \{x \mid x \text{ is ever a witness for } \mathrm{P}^{\alpha}_{h,e}\}$,

$Y_h = \{x \mid x \text{ is in } Y^{\alpha}_{h,e} \text{ for some } e \text{ and } \alpha\}$, and

$Y^j = \bigcup_{h\leq j} Y_h$.

Thus, Y^j is the set of all x that are ever witnesses for any $P^{\alpha}_{h,e}$ for any $h \leq j$.

To meet the outer splitting property, we build functions f and g so that for each i, we satisfy requirement O_i:

(a) $W_i = W_{f(i)} \sqcup W_{g(i)}$,

(b) $W_{f(i)} \cap \overline{A}$ is finite, and

(c) if $W_i \cap \overline{A}$ is infinite, then $W_{f(i)} \cap \overline{A}$ is nonempty.

For A to have semilow$_2$ complement, we need to ensure $\{j \mid W_j \cap \overline{A} \text{ infinite}\} \leq_{\mathrm{T}} \mathbf{0}''$.

To achieve this, we will meet for each $j \in \omega$:

L_j: $W_j \cap \overline{A}$ is infinite if and only if either $W_j - (Y^j \cup A)$ has infinitely many expansionary stages or $W_j \cap Y^j \cap \overline{A}$ is infinite.

We will also show that $\mathbf{0}''$ is able to determine if $W_j \cap Y^j \cap \overline{A}$ is infinite.

We use a binary tree of strategies. Let α be a length i string on the tree. For each $j < i$, $\alpha(j) = 0$ indicates a guess that $W_j - (Y^j \cup A)$ has infinitely many expansionary stages and $\alpha(j) = 1$ is a guess that it has finitely many. P_h^α will work to satisfy P_h while assuming that α is correctly guessing the true outcome.

The tree serves two purposes. As in Theorem 4.1, it allows us to ensure that if $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, it is in fact an infinite set. The nodes guessing that there are infinitely many expansionary stages are not allowed to add witnesses when L_j is injured. Instead, they wait until a new expansionary stage is reached. This way, they do not create witnesses that may later enter $W_j - (Y^j \cup A)$, only to be forced into A. In addition, nodes guessing that $W_j - (Y^j \cup A)$ has finitely many expansionary stages are reset each time $W_j - (Y^j \cup A)$ reaches a new expansionary stage, meaning that any of their current witnesses will never enter A. The combination of these two actions guarantees that if $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, it has infinitely many elements that never enter A and so it is infinite.

The other purpose of the tree is to allow $\mathbf{0}''$ to compute whether $W_j \cap Y^j \cap \overline{A}$ is infinite. In Lemma 5.5, we analyze which elements of Y^j stay in $\overline{A}$ forever by examining what happens to witnesses for P_h^α when α has various relationships to the true path. Since $\mathbf{0}''$ can determine the true path, we can use $\mathbf{0}''$ to find the index for a c.e. set that is finitely different from $Y^j \cap \overline{A}$. Thus $\mathbf{0}''$ can determine whether $W_j \cap Y^j \cap \overline{A}$ is infinite.

In addition, we need that $A \leq_{\mathrm{T}} D$, which we achieve by a nonlow permitting argument, as in Theorem 4.1.

5.2 Construction

We build an approximation δ_s to the true path. We call s an α-stage if α is a substring of δ_s. Define $\delta_0 = \lambda$, the empty string. Let δ_s be the length s string defined by: $\delta_s(j) = 0$ if $W_j - (Y^j \cup A)$ has reached a new expansionary stage since the last $\delta_s \restriction j$-stage, and 1 otherwise.

Let $\ell_\alpha(e, 0) = 0$ and $\ell_\alpha(e, s+1) = \ell_\alpha(e, s)$ unless otherwise specified.

We begin with all L_j allowing addition of new witnesses for every P_h^α. In Step 5, if L_j is injured, then it will disallow new witnesses for P_h^α, where α guesses the infinite outcome for L_j. L_j will again allow new witnesses if $W_j - (Y^j \cup A)$ reaches a new expansionary stage (Step 1). Step 2 also has the ability to make L_j allow again, and serves the purpose of ensuring that if $W_j - (Y^j \cup A)$ has infinitely many expansionary stages, it allows each P_h^α to add witnesses infinitely often.

Nodes on the tree may be put in 1-1 correspondence with $\mathbb{N} - \{0\}$ via a function $f : 2^{\in \omega} \to \mathbb{N} - \{0\}$. Using such a correspondence, let $\langle \alpha, e \rangle := \langle f(\alpha), e \rangle$. In Step 3, witnesses come from $\omega^{[\langle \alpha, e \rangle]}$.

Note that in the following, Steps 1 and 2 are identical to those in Theorem 4.1.

Stage $s+1$:

Step 1 (new expansionary stage): For each $j \in \omega$, if $W_j - (Y^j \cup A)$ grows to a size we haven't seen before, i.e. it reaches a new expansionary stage, then do the following for each α of length greater than j: If α guesses that $W_j - (Y^j \cup A)$ has infinitely many expansionary stages (i.e. $\alpha(j) = 0$), then L_j now allows addition of new witnesses for P^α_h. If α guesses that $W_j - (Y^j \cup A)$ has finitely many expansionary stages ($\alpha(j) = 1$), we reset $\mathrm{P}^\alpha_{h,e}$ for all e, making $\ell_\alpha(e,s) = 0$ and declaring $\mathrm{P}^\alpha_{h,e}$ and any witnesses for $\mathrm{P}^\alpha_{h,e}$ inactive. When reset, we will start a new $S^\alpha_{h,e}$ that begins empty.

Step 2 (allowing some disallowed witnesses): We say a node of length j *is allowed* new witnesses if no $j' < j$ disallows P^α_h from adding witnesses. For each j, N, and β, where β has length j, if L_j currently disallows for N (as defined in Step 5) and if β is allowed new witnesses, then set L_j to allow new witnesses for all α unless Step 2 has acted at some previous stage for these particular j, N, and β. Do this for all possible triples, of which there are only finitely many. This step guarantees that nodes on the true path will be allowed to act in Step 3 infinitely often.

Step 3 (adding witnesses): We act for each α of length at most s such that P^α_h is allowed to add witnesses by each L_j with $j < s$ such that $\alpha(j) = 0$, and such that either s is an α-stage or α is to the left of δ_s and if s_α was the last α-stage, P^α_h has not been allowed to act at any stage t, $s_\alpha \leq t < s$. In other words, we act on the approximation to the true path δ_s and for any nodes to the left of δ_s that have not been allowed to act since before they were last on the approximation to the true path. We act in length-lexicographic order. For each such α, we ask if there is any $e < s$ such that

- $\ell_\alpha(e,s) = 0$,
- $\Phi^D_e(e)[s]\downarrow$, and
- for all $e' < e$, $\Phi^D_{e'}(e')[s]\downarrow \iff \ell_\alpha(e',s) = 1$ or $\Phi^D_{e'}(e')[s]\downarrow$ and has changed computations at least e times since the last time that $\mathrm{P}^\alpha_{h,e}$ added a witness.

If so, then for each such e, let $x > s$, $x \in \omega^{\langle \alpha, e\rangle}$, be a fresh witness, larger than any previously chosen witness. Enumerate $x \in S^\alpha_{h,e}$. Declare x to have use $\varphi^D_e(e)[s]$, which is defined to be the use of $\Phi^D_e(e)[s]$. We call the witness x *active*.

Step 4 (changing ℓ_α when $W_{h(r(\alpha,e,n))}$ grows): Suppose $\ell_\alpha(e,s) = 0$, $S^\alpha_{h,e} \cap \overline{A}$ contains at least one witness, and $|W_{h(r(\alpha,e,n)),s+1}| > |W_{h(r(\alpha,e,n)),s}|$ where n is the number of times $\mathrm{P}^\alpha_{h,e}$ has been reset. Then set $\ell_\alpha(e,s+1) = 1$ and reset $\mathrm{P}^\alpha_{h,e'}$ for all $e' > e$ by starting new empty sets $S^\alpha_{h,e'}$, as well as setting $\ell_\alpha(e',s+1) = 0$ and declaring any witnesses for $\mathrm{P}^\alpha_{h,e'}$ inactive. Perform this step for each α and for the least e such that it applies.

Step 5 (enumerating witnesses into A and disallowing new witnesses): If $x \in S^\alpha_{h,e}$ is active with use u and $D_{s+1}\restriction u \neq D_s\restriction u$, then put x into A_{s+1} and set $\ell_\alpha(e, s+1) = 0$.

If by such an enumeration, we cause an element of $(W_j - (Y^j \cup A))[s]$ to enter A, then we say L_j does not *allow* P^α_h to add witnesses for any α such that $\alpha(j) = 0$. In addition, if the size of $(W_j - (Y^j \cup A))$ has become N, we say that L_j *disallows for* N.

Step 6 (outer splitting property): Let $X_i = \{x \mid x$ is ever a witness for any $\mathrm{P}^\alpha_{h,e}$ with $\langle h, e\rangle < i\}$. For each $x \in W_{i,s+1}$ such that x is not yet in $W_{f(i)}$ or $W_{g(i)}$, do the first of the following that applies:

(a) If $x \in W_i - (X_i \cup A)$ and $|W_{f(i)} - (X_i \cup A)| < i+1$, put x in $W_{f(i)}$. Reset all $\mathrm{P}^\alpha_{h,e}$ for $\langle h, e\rangle \geq i$.

(b) If there is an $\langle h, e\rangle < i$ and any α such that x is an inactive witness for $\mathrm{P}^\alpha_{h,e}$, and $W_{f(i)}$ does not already contain an inactive witness, then put x into $W_{f(i)}$.

(c) If $W_{f(i)}$ contains no inactive witness and there is an $\langle h, e\rangle < i$ such that $x \in W_i \cap S^\alpha_{h,e}$ and $W_{f(i)} \cap S^\alpha_{h,e}$ is empty, then put x into $W_{f(i)}$.

(d) If none of the above applies, put x into $W_{g(i)}$.

Step 7 (coding D): If $k \in D_{s+1} - D_s$, enumerate one element from F_k into A_{s+1} such that it obeys the following: for each $M > 0$ and $j + M = k$, the element is not one of the least M elements in $(W_j - (Y^j \cup A))[s]$, and for each $i \leq k$, the element is not one of the least $i+1$ elements in $W_{f(i)} - (X_i \cup A)$.

5.3 Verification

Lemma 5.3. *For each $j \in \omega$, $W_j \cap \overline{A}$ is infinite if and only if either $W_j \cap \overline{A} \cap Y^j$ is infinite or $W_j - (Y^j \cup A)$ has infinitely many expansionary stages.*

Proof. Note that the proof of this lemma is a very slight variation on the proof of Lemma 4.3.

If $W_j \cap \overline{A}$ is infinite then either $W_j \cap \overline{A} \cap Y^j$ is infinite or $W_j - (Y^j \cup A)$ is infinite, in which case it has infinitely many expansionary stages.

If $W_j \cap \overline{A} \cap Y^j$ is infinite, then $W_j \cap \overline{A}$ is infinite. Suppose $W_j - (Y^j \cup A)$ has infinitely many expansionary stages. We will show that for each M, $W_j - (Y^j \cup A)$ has at least M elements.

Induct on M. Assume true for $M - 1$.

No element in $(W_j - (Y^j \cup A))[s]$ ever enters Y^j since new witnesses are chosen to be larger than s, and elements of $(W_j - (Y^j \cup A))[s]$ are smaller than s.

For $k \leq j$, only finitely many elements are ever put into A by C_k. For $k > j$, C_k can only bring the size of $W_j - (Y^j \cup A)$ below M if $M + j > k$, which happens finitely often. Let s_0 be a stage by which the least $M - 1$ elements in $(W_j - (Y^j \cup A))[s_0]$ never enter A and after which no C_k enumerates any of the least M elements of $(W_j - (Y^j \cup A))[s]$ into A. Let $s_1 > s_0$ be a stage after which Step 2 never acts for $(j, M-1, \beta)$ for any β of length j. We may assume that at every stage $s > s_1$, the Mth least element of $(W_j - (Y^j \cup A))[s]$ is a witness for some $\mathrm{P}^{\alpha}_{h,e}$, else it would never enter A and we would be guaranteed to have at least M elements in $W_j - (Y^j \cup A)$, as desired. Note that the length of α is greater than j since the witnesses are not in Y^j.

Let $s_2 > s_1$ be a stage such the Mth least element in $(W_j - (Y^j \cup A))[s_2]$, called x_M, has the least use of any that will ever be in the Mth position of any $(W_j - (Y^j \cup A))[s]$ for $s \geq s_2$. Now, if x_M never enters A by Step 5, we are done, so assume x_M enters A. When x_M enters A, all witnesses with equal or larger uses also enter A, and since the next element in the Mth position is a witness, and it cannot have smaller use than x_M due to minimality, then the next element that enters the Mth position has yet to become a witness. When x_M enters A, L_j does not allow any P^{α}_{h} to add witnesses for any α such that $\alpha(j) = 0$. Step 2 has finished acting for j and $M-1$, so it will not cause L_j to later allow. The next time L_j allows will be by Step 1, which means $(W_j - (Y^j \cup A))$ has already reached a new expansionary stage. Thus, at the first expansionary stage after x_M enters A, the only witness that could be in the Mth position must be a witness for an α with $\alpha(j) = 1$. At that new expansionary stage, Step 1 inactivates the witnesses for α. Thus the element in the Mth position can never enter A, so $(W_j - (Y^j \cup A))$ has at least M elements. □

We say that α is on the *true path* if $\alpha(j) = 0$ if and only if $W_j - (Y^j \cap \overline{A})$ has infinitely many expansionary stages during the construction.

Lemma 5.4. *Let α be on the true path. Let e be the least such that either $\lim_s \ell_\alpha(e, s)$ does not exist or it exists and $\lim_s \ell_\alpha(e, s) \neq K^D(e)$. Then $P^{\alpha}_{h,e}$ is reset finitely often (n times) and $W_{h(r(\alpha,e,n))}$ is infinite if and only if $W_{r(\alpha,e,n)} \cap \overline{A}$ is finite. Thus $\overline{A}$ is not semilow$_{1.5}$.*

Proof. Since α is on the true path, $\mathrm{P}^{\alpha}_{h,e}$ is reset only finitely often by L_j, $j < h$. $\mathrm{P}^{\alpha}_{h,e}$ can only be reset by O_i in Step 6 when $\langle h, e\rangle \geq i$, and each of these O_i can only reset it finitely often, so it only gets reset finitely often by O_i.

$\mathrm{P}^{\alpha}_{h,e}$ can only be reset by $\mathrm{P}^{\alpha}_{h,e'}$ for $e' < e$ when $W_{h(r(\alpha,e',n'))}$ grows, where n' is the number of times $\mathrm{P}^{\alpha}_{h,e'}$ has been reset. When it grows, we set $\ell_\alpha(e', s+1) = 1$ from $\ell_\alpha(e', s) = 0$, so if $\mathrm{P}^{\alpha}_{h,e'}$ resets $\mathrm{P}^{\alpha}_{h,e}$ infinitely often, then $\lim_s \ell_\alpha(e', s)$ does not exist, contradicting the assumption that it exists and equals $K^D(e')$. Thus $\mathrm{P}^{\alpha}_{h,e}$ is reset finitely often.

Case 1: $W_{h(r(\alpha,e,n))}$ is infinite.

Case 1a: $\Phi^D_e(e)$ diverges. Thus, $\lim_s \ell_\alpha(e, s) \neq 0$. If $S^{\alpha}_{h,e}$ is finite, we're done, since $W_{r(\alpha,e,n)} = S^{\alpha}_{h,e}$. If $S^{\alpha}_{h,e}$ is infinite, then $\Phi^D_e(e)[s]$ converges infinitely often. Any elements that we add when $\Phi^D_e(e)[s]$ converges will be enumerated into A when $\Phi^D_e(e)[t]$ diverges for $t > s$ because the use of each witness equals the use of the computation $\Phi^D_e(e)[s]$. Thus $W_{r(\alpha,e,n)} \cap \overline{A} = S^{\alpha}_{h,e} \cap \overline{A}$ is finite.

Case 1b: $\Phi_e^D(e)$ converges. Thus, $\lim_s \ell_\alpha(e,s) \neq 1$. If $W_{r(\alpha,e,n)} \cap \overline{A}$ is infinite, then $\lim_s \ell_\alpha(e,s)$ cannot equal 0 because there will be infinitely many stages where Step 4 applies, setting $\ell_\alpha(e, s+1)$ to 1. Thus, if $\Phi_e^D(e)\downarrow$, then $\lim_s \ell_\alpha(e,s)$ does not exist, but this is also impossible since the only time that $\ell_\alpha(e,s)$ changes from 1 to 0 is when $\Phi_e^D(e)[s]$ diverges or changes its computation, meaning that $\Phi_e^D(e)$ must diverge.

Thus, if $W_{h(r(\alpha,e,n))}$ is infinite, then $W_{r(\alpha,e,n)} \cap \overline{A}$ is finite.

Case 2: $W_{h(r(\alpha,e,n))}$ is finite. Note that $\lim_s \ell_\alpha(e,s)$ must exist because it can change to 1 only finitely often since $W_{h(r(\alpha,e,n))}$ is finite.

Case 2a: $\Phi_e^D(e)$ diverges. Thus, $\lim_s \ell_\alpha(e,s) = 1$. Note that at the final stage s when $\ell_\alpha(e,s)$ changes from 0 to 1, $S^\alpha_{h,e}$ contains an active witness not in A. This means that $\Phi_e^D(e)[s]\downarrow$, else the active witness would have entered A. Since $\Phi_e^D(e)\uparrow$, then the computation $\Phi_e^D(e)[s]$ must eventually be injured, at which point $\ell_\alpha(e,t)$ will become 0, contradicting that the limit is 1.

Case 2b: $\Phi_e^D(e)$ converges. Thus, $\lim_s \ell_\alpha(e,s) = 0$. Because $\Phi_e^D(e)$ converges, if infinitely many elements enter $S^\alpha_{h,e}$, then infinitely many elements stay in $S^\alpha_{h,e} \cap \overline{A}$, since they only enter if $\Phi_e^D(e)[s]$ has its computation injured. Thus it suffices to show that infinitely elements enter $S^\alpha_{h,e}$. To do so, we will show that at infinitely many stages, Step 3 acts. We already know that for almost all s, $\ell_\alpha(e,s) = 0$ and $\Phi_e^D(e)[s]\downarrow$, so the first two bullet points of Step 3 are met for almost all s.

To check that the third bullet point is met, note that for each $e' < e$, either $K^D(e') = 1 = \lim_s \ell_\alpha(e',s)$, in which case for almost all s, e' does not prevent e from acting, or $K^D(e') = 0 = \lim_s \ell_\alpha(e',s)$. In the latter case, there are two possibilities. If for almost all s, $\Phi_{e'}^D(e')[s]\uparrow$, then for almost all s, e' does not prevent e from meeting the third bullet point. Otherwise, there are infinitely many stages such that $\Phi_{e'}^D(e')[s]\downarrow$, but since $\Phi_{e'}^D(e')\uparrow$, the computation changes infinitely often, so eventually the computation will have changed at least e times. Thus, e' only prevents e from adding a new witness for finitely many stages.

Now that we have seen that for almost all s, the three bullet points are met, we must still show that α is *allowed* to add a witness at a stage where it is able to act.

Claim: For every α on the true path, α is allowed to add witnesses infinitely often.

For the proof of the claim, see Lemma 4.4, replacing the reference to Lemma 4.3 with Lemma 5.3.

Now wait until a stage s_0 such that the approximation to the true path never goes to the left of α and the bullet points do not prevent $\mathrm{P}^\alpha_{h,e}$ from acting after stage s_0. Let $s_1 > s_0$ be any α-stage, and $s_2 \geq s_1$ the next stage that α is allowed to add witnesses. Then s_2 is either an α-stage or δ_{s_2} is to the right of α, so we may perform Step 3 for $\mathrm{P}^\alpha_{h,e}$. Thus, there are infinitely many stages at which we will add witnesses to $S^\alpha_{h,e}$, so it will be infinite and have infinite intersection with $\overline{A}$.

□

Lemma 5.5. $\mathbf{0}''$ *can compute* $\{j \mid W_j \cap \overline{A}$ *is infinite*$\}$.

Proof. By Lemma 5.3, it suffices to show that $\mathbf{0}''$ can compute $\{j \mid W_j \cap \overline{A} \cap Y^j$ is infinite$\}$, since $\mathbf{0}''$ can determine if the set of expansionary stages of $W_j - (Y^j \cup A)$ is infinite since the set of expansionary stages is a c.e. set.

We will show that $\overline{A} \cap Y^j$ is a c.e. set and that there is a $\mathbf{0}''$-computable function $m(j)$ such that $W_{m(j)} =^* \overline{A} \cap Y^j$, so $W_j \cap \overline{A} \cap Y^j$ is infinite if and only if $W_j \cap W_{m(j)}$ is infinite, which is $\mathbf{0}''$-computable.

Note that $\mathbf{0}''$ can determine the true path on the tree of strategies. Consider each $Y^\alpha_{h,e}$ that makes up Y^j. If α is to the right of the true path, then $\mathrm{P}^\alpha_{h,e}$ gets reset infinitely often, so the set of elements in $\mathrm{Y}^\alpha_{h,e} \cap \overline{A}$ is the c.e. set given by the set of all x in $\mathrm{Y}^\alpha_{h,e} \cap \overline{A}$ at a stage when $\mathrm{P}^\alpha_{h,e}$ is reset. Put these into $W_{m(j)}$.

If α is to the left of the true path, there are only finitely many α-stages. Step 3 is either allowed to add witnesses at the final α-stage or at one later stage. Thus $\mathrm{S}^\alpha_{h,e}$ only gets finitely many elements, so it may be ignored.

For each α on the true path of length $i \leq j$, use $\mathbf{0}''$ to find the least e such that $\lim_s \ell_\alpha(e,s) \neq K^D(e)$. For $e' < e$, note that if there are infinitely many witnesses for $\mathrm{P}^\alpha_{h,e'}$, then $\ell_\alpha(e',s)$ must be 0 infinitely often, so its limit must be 0. We also have that $\Phi^D_{e'}(e')[s]$ must converge infinitely often for us to add infinitely many witnesses, so it must also diverge infinitely often so that $K^D(e') = \lim_s \ell_\alpha(e',s) = 0$. Thus, once $\mathrm{P}^\alpha_{h,e'}$ stops being reset, which we know must happen by the proof of Lemma 5.4, all new witnesses eventually enter A. Thus, $\mathrm{P}^\alpha_{h,e'}$ only contributes finitely much to $Y^j \cap \overline{A}$ and can be ignored.

Now consider $e' > e$. If $\mathrm{P}^\alpha_{h,e}$ resets $\mathrm{P}^\alpha_{h,e'}$ infinitely often, then the contribution of $\mathrm{P}^\alpha_{h,e'}$ to $Y^j \cap \overline{A}$ is given by the set of elements that are witnesses at stages when it gets reset, and these can be added to $W_{m(j)}$. Suppose $\mathrm{P}^\alpha_{h,e}$ resets $\mathrm{P}^\alpha_{h,e'}$ only finitely often. First, consider the case where $\lim_s \ell_\alpha(e,s) = 1$. Then $\Phi^D_e(e)\uparrow$ by the choice of e. However, if $\lim_s \ell_\alpha(e,s) = 1$, then when $\ell_\alpha(e,s)$ gets defined as 1, it contains an active witness, and the witness cannot enter A else it would reset e', so $\Phi^D_e(e)\downarrow$. This is a contradiction, so $\lim_s \ell_\alpha(e,s) \neq 1$. Since $\mathrm{P}^\alpha_{h,e}$ resets $\mathrm{P}^\alpha_{h,e'}$ only finitely often, we must have that $\lim_s \ell_\alpha(e,s) = 0$. Thus, $\Phi^D_e(e)\downarrow$, so $\Phi^D_e(e)[s]$ changes computations some finite number of times. By the final bullet of Step 3, only finitely many $e' > e$ will ever be able to add witnesses, and will only be able to do so finitely often. Thus after a finite stage, no $\mathrm{P}^\alpha_{h,e'}$ can add any witnesses, for $e' > e$, so these collectively contribute only finitely many elements to $Y^j \cap \overline{A}$ and can be ignored.

For e itself, $S^\alpha_{h,e} \cap \overline{A}$ is infinite if and only if $W_{h(r(\alpha,e,n))}$ is finite, for n the number of times $\mathrm{P}^\alpha_{h,e}$ is reset, by Lemma 5.4. We can ask $\mathbf{0}''$ to determine this. If we know that $S^\alpha_{h,e} \cap \overline{A}$ is finite, we can ignore it. If we know it is infinite, then $W_{h(r(\alpha,e,n))}$ is finite, so $\lim_s \ell_\alpha(e,s) = 0$, and thus $\Phi^D_e(e)\downarrow$ and so almost all elements that we put into $S^\alpha_{h,e}$ remain in $\overline{A}$, so we can put them all into $W_{m(j)}$. □

Lemma 5.6. *The following hold for all $i \in \omega$:*

(a) $W_i = W_{f(i)} \sqcup W_{g(i)}$,

(b) $W_{f(i)} \cap \overline{A} =^* \emptyset$, *and*

(c) $W_i \cap \overline{A}$ *infinite* $\implies W_{f(i)} \cap \overline{A}$ *is nonempty.*

Proof. By construction, we put every element of W_i into either $W_{f(i)}$ or $W_{g(i)}$, but not both, so part (a) holds.

Proof of (b): We may add elements to $W_{f(i)}$ by Steps 6a, 6b, and 6c. The set $(W_{f(i)} \cap \overline{A}) - X_i = W_{f(i)} - (X_i \cup A)$ must be finite because only Step 6a can add elements to it and only when its size is less than $i+1$, so at most $i+1$ elements will stay in $W_{f(i)} - (X_i \cup A)$ forever.

To see that $W_{f(i)} \cap \overline{A} \cap X_i$ is finite, we look at Steps 6b and 6c. If Step 6b adds an inactive witness to $W_{f(i)}$, it will never be able to do so again, as inactive witnesses remain inactive witnesses forever for the remainder of the construction. Step 6c allows us to take an element from $W_i \cap S^{\alpha}_{h,e} \cap \overline{A}$ only when there is not already an element in $W_{f(i)} \cap S^{\alpha}_{h,e} \cap \overline{A}$, for $\langle h, e\rangle < i$. Thus, even if we put infinitely many elements in by Step 6c, only finitely many remain forever.

Proof of (c): Suppose $W_i \cap \overline{A}$ is infinite. Suppose there are eventually $i+1$ elements that enter $W_{f(i)}$ by Step 6a. When we add elements by Step 6a, they are taken from $W_i - (X_i \cup A)$, which means that they cannot be put into A by action of $\mathrm{P}^{\alpha}_{h,e}$ for any $\langle h, e\rangle < i$. When we put the elements into $W_{f(i)}$, we also reset $\mathrm{P}^{\alpha}_{h,e}$ for each $\langle h, e\rangle \geq i$, so they cannot enter A from action of these requirements, either. They can only be enumerated into A by C_k, for $k < i$, which each only act once. Thus, at least one element will remain in $W_{f(i)} \cap \overline{A}$. If there are not eventually $i+1$ elements that enter $W_{f(i)}$ by Step 6a, then $W_i - (X_i \cup A)$ must be finite, so $W_i \cap \overline{A} \subseteq^* X_i$, and so $W_i \cap \overline{A} \cap X_i$ is infinite. If Step 6b ever acts by enumerating into $W_{f(i)}$, then $W_{f(i)}$ contains an inactive witness, which will never enter A. Otherwise, we know there is some $\langle h, e\rangle < i$ such that there are infinitely many witnesses for $\mathrm{P}^{\alpha}_{h,e}$ contained in $W_i \cap \overline{A}$, by the pigeonhole principle. Since Step 6b never acts, all of the witnesses are active when they enter W_i. Suppose $W_{f(i)} \cap \overline{A}$ is empty. Then every time Step 6c acts by adding an active witness to $W_{f(i)}$, that element later enters A when the current computation $\Phi^D_e(e)[s]$ changes because D changes below the use. For each x that is an active witness for $\mathrm{P}^{\alpha}_{h,e}$ when it enters W_i, it either enters $W_{f(i)}$, in which case it later enters A, or there is already an active witness y for $\mathrm{P}^{\alpha}_{h,e}$ in $W_{f(i)} \cap \overline{A}$ when it enters W_i. In that case, we know that y goes into A, at which stage x must go into A as well because they have the same use. Thus, if $W_{f(i)} \cap \overline{A}$ is empty, then $W_i \cap \overline{A}$ is finite, proving part (c). □

Lemma 5.7. *$D \leq_{\mathrm{T}} A$.*

Proof. To determine if $k \in D$, ask if A contains any elements of F_k. $F_k \cap A$ is nonempty if and only if $k \in D$. This is because only C_k can enumerate elements of F_k into A, and C_k is always able to act because F_k has more elements in it than are prohibited. To see that F_k has more elements than are prohibited, recall that for each $M > 0$ and $j + M = k$, we prohibit the least M elements in $(W_j - (Y^j \cup A))[s]$, and for each $i \leq k$, we prohibit the least $i+1$ elements in $W_{f(i)} - (X_i \cup A)$. For $k = 0$, one element is prohibited and F_0 has two elements. For $k > 0$, the first clause prohibits less than $2k^2$ elements because $\sum_{M=1}^{k} M = \frac{(k+1)k}{2} = \frac{k^2}{2} + \frac{k}{2} < 2k^2$. The second clause prohibits less than $4k^2$ elements because $\sum_{i=0}^{k} i+1 = \frac{(k+1)(k+2)}{2} = \frac{k^2}{2} + \frac{3k}{2} + 1 < 4k^2$. Thus fewer than $6k^2$ elements are prohibited and F_k contains $6k^2$ elements.

□

Lemma 5.8. *$A \leq_{\mathrm{T}} D$.*

Proof. $\omega^{[0]} \cap A \leq_{\mathrm{T}} D$ since no element from $\omega^{[0]}$ enters A unless it is in F_k and k enters D.

For $\omega - \omega^{[0]}$, $x \in A$ if and only if x becomes a witness and then enters A. The set of witnesses is computable because if x is not a witness by stage x, it will never become a witness. If x is a witness, we ask what its use u is as a witness. Let s_x be such that $D\restriction u = D_{s_x}\restriction u$. Then $x \in A$ if and only if $x \in A_{s_x}$.
□

This concludes the proof of the theorem. □

6 Non-Low$_2$ degrees and sets whose complement is not semilow$_2$

In this section we will provide an index set argument that every nonlow$_2$ c.e degree contains a c.e. set whose complement is not semilow$_2$. Soare [13, IV 4.11] shows via an index set argument that every non-low c.e degree contains a c.e. set whose complement is not semilow.

Recall that Inf is the set of indices for infinite c.e. sets. Inf^B is Π^0_2 complete. $Inf^B = \{e \mid W_e^B \text{ is infinite}\}$ and $\overline{B''} \leq_1 Inf^B$. Let $Inf(X) = \{e \mid W_e \cap X$ is infinite$\}$. Recall $\overline{A}$ is semilow$_{1.5}$ if $Inf(\overline{A}) \leq_1 Inf$ and $\overline{A}$ is semilow$_2$ if $Inf(\overline{A}) \leq_T Inf$

Theorem 6.1. *For every c.e. set B, there is a c.e. $A \equiv_{\mathrm{T}} B$ such that $Inf^B \leq_1 Inf(\overline{A})$.*

Corollary 6.2. *If B is nonlow$_2$, then there exists a c.e. $A \equiv_{\mathrm{T}} B$ such that $\overline{A}$ is not semilow$_2$ (and not semilow$_{1.5}$).*

Proof. Let B be nonlow$_2$. Then there exists a c.e. $A \equiv_{\mathrm{T}} B$ such that $Inf <_T Inf^B \leq_1 Inf(\overline{A})$, so $\overline{A}$ is not semilow$_2$ (and not semilow$_{1.5}$). □

Proof of Theorem 6.1. Let $\{\Phi_i\}_{i\in\omega}$ be a listing of all Turing functionals. Let $\{B_s\}_{s\in\omega}$ be a computable enumeration of B. We may assume that if $\Phi_i^B(x)[s]\downarrow \neq \Phi_i^B(x)[t]\downarrow$, then there is a stage between s and t such that the computation diverges. We may also assume that at each stage s, there is at most one pair $\langle i, x\rangle$ such that $\Phi_i^B(x)[s]\downarrow$ and $\Phi_i^B(x)[s-1]\uparrow$. The use function of the computation

$\Phi_i^B(x)[s]$ is denoted $\varphi_i^B(x)[s]$ and is the maximal element of $B_{i,s}$ seen in the computation.

Construction:

Stage $s+1$:

Step 1: If $b \in B_{s+1} - B_s$, enumerate all marked elements into A_{s+1} whose markers have uses $u \geq b$. Whenever an element is enumerated into A, its marker is removed.

Step 2: If $\Phi_i^B(x)[s+1]\downarrow$, and there is no current marker $M_{\langle i,x\rangle}$, then choose the least element $y \in \overline{A}_s$, $y > s+1$, without a marker and place marker $M_{\langle i,x\rangle}$ on it. The use of this marker is $\varphi_i^B(x)[s+1]$.

Step 3: Let $b \notin B_{s+1}$ be the least such that there is no current marker Γ_b. Place Γ_b on the least $y \in \overline{A}_s$, $y > s+1$, without any marker, and let b be the use of this marker.

End construction.

Lemma 6.3. *$A \leq_{\mathrm{T}} B$.*

Proof. To determine if $y \in A$, first run the construction until stage y to see if y ever has a marker. If not, then $y \notin A$. If so, then if the marker was added at stage s and the use of the marker on y is u, ask if B ever changes at or below u after stage s. If so, then $y \in A$. Otherwise $y \notin A$. □

Lemma 6.4. *$B \leq_{\mathrm{T}} A$.*

Proof. Recall that the use of any marker Γ_b is b. Thus, once B_s has settled up through b, no current or future marker Γ_b will ever enter A. To determine if $b \in B$, run the construction until either Γ_b is placed on an element in $\overline{A}$, in which case $b \notin B$, or b enters B_s. If $b \notin B$, then eventually a marker Γ_b will appear in Step 3 on an element in $\overline{A}$, as desired. □

Lemma 6.5. *Let $W_{f(i)} = \{y \mid (\exists s)(\exists x)[M_{\langle i,x\rangle}$ is on y at stage $s]\}$. W_i^B is infinite if and only if $W_{f(i)} \cap \overline{A}$ is infinite.*

Proof. If W_i^B is infinite, then there exist infinitely many x such that $\Phi_i^B(x)\downarrow$. For such x, for almost all s, $\Phi_i^B(x)[s]\downarrow$ and B_s never later changes at or below the use $\varphi_i^B(x)[s]$. For each such x, $M_{\langle i,x\rangle}$ will be undefined before the final computation $\Phi_i^B(x)[s]$ converges, because any previous computation would have been injured below the use, causing the marker to be removed. When the final computation appears, the marker will be placed by Step 2 on an element in $\overline{A}_s$, and this element will never enter A since nothing enters B at or below the use. Thus for each of these infinitely many x values, the final resting place of $M_{\langle i,x\rangle}$ is in $\overline{A}$, and thus $W_{f(i)} \cap \overline{A}$ is infinite.

If W_i^B is finite, then for almost all x, $\Phi_i^B(x)$ diverges, which means that for almost all x, every computation $\Phi_i^B(x)[s]$ that converges is injured below the use. Thus, for almost all x, whenever $M_{\langle i,x\rangle}$ is placed on an element, that element later enters A. There are only finitely many x values where $M_{\langle i,x\rangle}$ is ever on an element that remains outside of A, and once a marker is placed on an element outside A, it remains there forever, and there is always only one current $M_{\langle i,x\rangle}$. Thus, only finitely many elements of $W_{f(i)}$ are in $\overline{A}$. $\square$

$\square$

References

[1] Peter Cholak. Automorphisms of the lattice of recursively enumerable sets. *Mem. Amer. Math. Soc.*, 113(541):viii+151, 1995.

[2] Peter Cholak and Leo A. Harrington. On the definability of the double jump in the computably enumerable sets. *J. Math. Log.*, 2(2):261–296, 2002.

[3] Rodney G. Downey, Carl G. Jockusch, Jr., and Paul E. Schupp. Asymptotic density and computably enumerable sets. *J. Math. Log.*, 13(2):1350005, 43, 2013.

[4] Rachel Epstein. The nonlow computably enumerable degrees are not invariant in $\mathcal{E}$. *Trans. Amer. Math. Soc.*, 365(3):1305–1345, 2013.

[5] Leo Harrington and Robert I. Soare. Definable properties of the computably enumerable sets. *Ann. Pure Appl. Logic*, 94(1-3):97–125, 1998. Conference on Computability Theory (Oberwolfach, 1996).

[6] Leo A. Harrington and Robert I. Soare. The Δ^0_3-automorphism method and noninvariant classes of degrees. *J. Amer. Math. Soc.*, 9(3):617–666, 1996.

[7] A. H. Lachlan. Degrees of recursively enumerable sets which have no maximal supersets. *J. Symbolic Logic*, 33:431–443, 1968.

[8] W. Maass. Characterization of recursively enumerable sets with supersets effectively isomorphic to all recursively enumerable sets. *Trans. Amer. Math. Soc.*, 279:311–336, 1983.

[9] Russell Miller. Orbits of computably enumerable sets: low sets can avoid an upper cone. *Ann. Pure Appl. Logic*, 118(1-2):61–85, 2002.

[10] Joseph R. Shoenfield. Degrees of classes of recursively enumerable sets. *J. Symbolic Logic*, 41:695–696, 1976.

[11] Robert I. Soare. Automorphisms of the lattice of recursively enumerable sets I: maximal sets. *Ann. of Math. (2)*, 100:80–120, 1974.

[12] Robert I. Soare. Automorphisms of the lattice of recursively enumerable sets II: low sets. *Ann. Math. Logic*, 22:69–107, 1982.

[13] Robert I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987.