Thin sets and the Preservation of Hyperimmunities

Peter Cholak

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Consider combinatorial principles $P$ as instances-solutions pairs, $(I_P, S_P)$.

**The Thin Set Theorem:** $RT^n_{<\infty, \ell}$

**Instance** Let $c$ be a coloring of all finite sets of size $n$ (all subsets of $\omega$) by finitely many colors, not necessarily computable.

**Solution** A set $T$ is $l$-thin iff $c$ uses at most $l$ colors to color all the sets of size $n$ from $T$ and $T$ is infinite. So $|c([T]^n)| \leq l$. 
Cone Avoidance

**Definition (Cone Avoidance)**

*Cone avoidance* for a principle $P$ says every set $Z$, every non-$Z$-computable set $X$ and every $Z$-computable instance $\mathcal{I}_P$, there is a solution $S_P$ such that $X \not\leq_T Z \oplus S_P$-computable.

**Theorem**

*Given an noncomputable set $X$ there is an (c.e.) set $W$ such that $X \not\leq_T W$.*

**Corollary**

*For every $Z$, for every non-$Z$-computable set $X$ there is an set $W$ such that $X \not\leq_T Z \oplus W$.*
The Catalan and Schröder numbers

The $n$th Catalan number is the number of paths from $(0, 0)$ to $(n, n)$ that take steps $(0, 1)$ and $(1, 0)$, and don’t go above main diagonal; the $n$th Schröder number is the same, except the paths are also allowed to take $(1, 1)$ steps.
Theorem (Wang)

For $l$ the $n$th Schröder number, $RT^n_{<\infty,l}$ satisfies cone avoidance. Let $C_0, C_1, \ldots$ be the sequence of Catalan numbers. In particular, $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, C_7 = 429.$

Theorem

$RT^n_{<\infty,C_n}$ satisfies cone avoidance. Moreover this bound is tight.
Preservation of Hyperimmunities

Definition
A function \( f : \mathbb{N} \to \mathbb{N} \) is \( Z \)-hyperimmune if it is not dominated by any function computable in \( Z \). An infinite set \( A = \{ x_0 < x_1 < \ldots \} \) is \( Z \)-hyperimmune if its principal function, \( p_A(n) = x_n \), is \( Z \)-hyperimmune.

Definition
A problem \( P \) admits the preservation of \( p \) hyperimmunities if for every set \( Z \) and every collection \( \{ f_s : s < p \} \) of \( Z \)-hyperimmune functions, every instance \( \mathcal{I}_P \leq_T Z \) has a solution \( S_P \) such that, for every \( s \leq m \), \( f_s \) is \( Z \oplus S_P \)-hyperimmune.
So \( S_P \) does not contain the information needed to dominate any of the \( f_s \)'s.
What does preservation of hyperimmunities get us?

Theorem (Patey)

$RT^2_{<\omega,k}$ admits preservation of $k$, but not $k + 1$, hyperimmunities.

Hence $RT^2_{<\omega,k+1}$ does not "follow" from $RT^2_{<\omega,k}$.

Definition

The thin set theorem for $n$ and $\omega$, $TS^n$, says for all $\omega$-colorings, $c : [\mathbb{N}]^n \to \omega$, there is an infinite set $T$ such that $|c([T])^n| \neq \omega$.

Theorem

$TS^n$ preserves $k$-hyperimmunities, for every $k \in \omega$, but not $\omega$-hyperimmunities.
Hyperimmune-free degrees

Definition
A Turing degree $d$ is hyperimmune-free iff, for all $f \leq_T d$, $f$ is not hyperimmune.

Lemma
Every instance of a problem $P$ has a solution of hyperimmune-free degree iff $P$ preserves all continuum many hyperimmune functions.

Corollary (Jockusch and Soare)
$WKL$ preserves all continuum many hyperimmune functions.
Patey also showed that problems where all instances have generic or random solutions admit the preservation of countable many hyperimmunities.
Cone Avoidance and preservation of 1-hyperimmunity

Theorem (Downey, Greenberg, Harrison-Trainor, Patey, and Turestsky)

A problem admits the preservation of 1-hyperimmunity iff the problem satisfies cone avoidance.

Corollary

$RT_{<\infty,C_n}^n$ admits the preservation of 1-hyperimmunity. Moreover this bound is tight.
Main new result

**Theorem**

$RT^n <_\infty, p^n C_n$ admits the preservation of $p$-hyperimmunities. Moreover this bound is tight.

So $RT^n <_\infty, (p+1)^n C_n$ does not follow from $RT^n <_\infty, p^n C_n$. 
Nonhyperarithmetic hyperimmune functions

Definition
A problem $P$ admits the preservation of $p$ nonhyperarithmetic hyperimmunities if for every set $Z$ and every collection \( \{f_s : s < p\} \) of $Z$-nonhyperarithmetic $Z$-hyperimmune functions, every instance $I_P \leq_T Z$ has a solution $S_P$ such that, for every $s \leq m$, $f_s$ is $Z \oplus S_P$-hyperimmune (and also likely $Z \oplus S_P$-nonhyperarithmetic).

Theorem
$RT^n_{< \infty, 2^n}$ preserves one nonhyperarithmetic hyperimmunity. Moreover this bound is tight.
Question
For which $\ell$ does $RT_{n,\ell}^n < \infty$ preserve $p$ hyperimmunities and $q$ nonhyperarithmetic hyperimmunities?

Question
Is there an $n$ and $\ell$ such that $RT_{n,\ell+1}^n$ follows from $RT_{n,\ell}^n$.  
