MAXIMAL CONTIGUOUS DEGREES

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Abstract. A computably enumerable (c.e.) degree is a *maximal contiguous degree* if it is contiguous and no c.e. degree strictly above it is contiguous. We show that there are infinitely many maximal contiguous degrees. Since the contiguous degrees are definable, the class of maximal contiguous degrees provides the first example of a definable infinite anti-chain in the c.e. degrees. In addition, we show that the class of maximal contiguous degrees forms an automorphism base for the c.e. degrees and therefore for the Turing degrees in general. Finally we note that the construction of a maximal contiguous degree can be modified to answer a question of Walk about the array computable degrees and a question of Li about isolated formulas.

§1. Introduction. We will work within the c.e. Turing degrees, $\mathcal{R}$, ordered by Turing reducibility, $\leq_T$. (We will suppress the $\leq_T$ for readability.) Our concern is that of definability and, specifically, the relationships between automorphisms of $\mathcal{R}$ and properties of various definable classes of degrees within $\mathcal{R}$.

1.1. Contiguous Degrees. Of particular interest is the class of *contiguous* degrees. Contiguity is an idea that relates Turing reducibility and weak truth-table reducibility, $\leq_{wtt}$.

DEFINITION 1. A c.e. degree **a** is contiguous iff for every c.e. $A, B \in$ **a**, $A \equiv_{wtt} B$. A c.e. degree **a** is strongly contiguous iff for *every* $A, B \in$ **a**, $A \equiv_{wtt} B$ (i.e. **a** is a single wtt-degree).

Contiguous degrees have been useful in transferring splitting results in the (c.e.) wtt-degrees to the (c.e.) Turing degrees. The first such use is by Ladner and Sasso [1975], who show that any contiguous degree **a** has the *anti-cupping property*: there is a $\mathbf{b} < \mathbf{a}$ such that **b** does not cup to **a** with any other Turing degree below **a**. Other such uses of the

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contiguous degrees can be found in Ambos-Spies [1984], Ambos-Spies and Fejer [To appear], Downey [1987], Downey and Stob [1986], and Stob [1983], and many other papers. Contiguous degrees have also been used to study degree-theoretical splitting properties of c.e. sets; for examples, see Downey and Stob [1993].

Ambos-Spies [1984] showed that the contiguous degrees are an automorphism base for $\mathcal{R}$. An *automorphism base* for $\mathcal{R}$ (respectively, $\mathcal{D}$) is a class of degrees $\mathcal{C}$ such that any nontrivial automorphism of $\mathcal{R}$ (resp. $\mathcal{D}$) moves some element of $\mathcal{C}$. Equivalently, any automorphism's behavior on the elements of $\mathcal{C}$ completely determines its behavior on all of $\mathcal{R}$ (resp. $\mathcal{D}$). Slaman and Woodin [To appear] have shown that the c.e. degrees are an automorphism base for $\mathcal{D}$. Hence the class of contiguous degrees is an automorphism base for $\mathcal{D}$.

More recently, local results about the degree structures (such as the results mentioned above) have been used to prove global results—such as the definability in $\mathcal{R}$ of any relation which is invariant under double jump and definable in second order arithmetic; see Nies, Shore, and Slaman [1998]. Lempp and Nies [1995] showed that the c.e. *wtt*-degrees have an undecidable Π_4-theory, and then used contiguous degrees to transfer this result to the c.e. Turing degrees. Lempp, Nies, and Slaman [1998] improved this, to show that the c.e. Turing degrees have an undecidable Π_3-theory, *without* using contiguous degrees. However, the contiguous degrees still have potential to allow us to transfer global results about the c.e. *wtt*-degrees to the c.e. Turing degrees.

Downey and Lempp [1997] showed that the contiguous degrees are definable in $\mathcal{R}$. Later Ambos-Spies and Fejer [To appear] slightly modified the work of Downey and Lempp to give a different definition:

THEOREM 2 (Downey and Lempp [1997]). *A c.e. degree* $\mathbf{a}$ *is contiguous iff it is strongly contiguous iff it is* locally distributive, *i.e. if* $\mathbf{d} < \mathbf{b} \cup \mathbf{c} = \mathbf{a}$ *then there are c.e. degrees* $\mathbf{b_0}$ *and* $\mathbf{c_0}$ *such that* $\mathbf{b_0} < \mathbf{b}$, $\mathbf{c_0} < \mathbf{c}$, *and* $\mathbf{b_0} \cup \mathbf{c_0} = \mathbf{d}$.

THEOREM 3 (Ambos-Spies and Fejer [To appear]). *A c.e. degree is contiguous iff it is not the top of an embedding of the pentagon into* $\mathcal{R}$.

Hence the contiguous degrees are a beautiful example of a definable class of c.e. degrees that not only is an automorphism base but also has properties which have proven useful in various different constructions and settings. We point out that the definability of the contiguous degrees has yet to be exploited in the recent theme of transferring local results about $\mathcal{R}$ to global results about $\mathcal{R}$.

The prompt degrees and the low_m and high_n degrees (for all $m \geq 2$, $n \geq 1$) are other similarly attractive classes. We direct the reader to Shore [2000] for details. One major difference between the contiguous degrees and the low_m and high_n degrees is that the formula defining the contiguous degrees is "natural," while this is not the case for the low_m and high_n degrees. The formulas defining the contiguous degrees in both Theorems 2 and 3 are reasonably understandable properties of partial orders. This is also the case for the prompt degrees, but it is not the case for the low_m and high_n degrees; again we direct the reader to Shore [2000] for details.

1.2. Maximal Contiguous Degrees. In this paper we focus on a subclass of the contiguous degrees, the maximal contiguous degrees:

DEFINITION 4. A c.e. degree $\mathbf{a}$ is called a *maximal contiguous* degree if $\mathbf{a}$ is contiguous and no c.e. degree $\mathbf{d} > \mathbf{a}$ is contiguous.

THEOREM 5. *A maximal contiguous degree exists.*

Actually we merely sketch (in Section 2) the proof of Theorem 5, since it is subsumed in our main result:

THEOREM 6. *Let* $\mathbf{d}$ *be any nonlow*$_2$ *c.e. degree, and* $\mathbf{c}$ *any c.e. degree such that* $\mathbf{d} \not\leq \mathbf{c}$. *Then there is a maximal contiguous degree* $\mathbf{a}$ *such that*
(i) $\mathbf{a} < \mathbf{d}$,
(ii) $\mathbf{a} \not\leq \mathbf{c}$, *and*
(iii) $\mathbf{a} \cup \mathbf{c}$ *is incomplete.*

Theorem 6 is proved in Sections 3 and 4. Messy as it is, Theorem 6 leads directly to the rather tidy Corollaries 7 and 8.

COROLLARY 7. *There are infinitely many maximal contiguous degrees.*

Clearly no two such degrees are comparable. Hence by Downey and Lempp [1997] the maximal contiguous degrees are an example of a definable infinite anti-chain. Furthermore,

COROLLARY 8. *The maximal contiguous degrees form an automorphism base for* $\mathcal{R}$.

This is the first known automorphism base to consist of an infinite anti-chain.

Proof of Corollaries. To prove Corollary 7, we show by induction that there are at least n maximal contiguous degrees whose n-fold join is incomplete. For $n = 1$, let $\mathbf{d} = \mathbf{0}'$ (and $\mathbf{c}$ any incomplete c.e. degree) and then apply Theorem 6. For the inductive step, given n maximal contiguous degrees with incomplete join, let $\mathbf{c}$ be the join, let $\mathbf{d}$ be $\mathbf{0}'$, and use Theorem 6 again to get an $(n+1)$st such degree.

To prove Corollary 8, we note that, as observed by Lerman in unpublished work, results of Sacks can be used to demonstrate that the high c.e. degrees are an automorphism base of $\mathcal{R}$ (in fact for any n, the low$_n$ and high$_n$ degrees are an automorphism base; see Odifreddi [1999]). Thus, let σ be any nontrivial automorphism and $\mathbf{d}$ any high degree moved by σ. We proceed as in Ambos-Spies's proof that the contiguous degrees form an automorphism base: Namely, suppose $\sigma(\mathbf{d}) \not> \mathbf{d}$. Then we let $\mathbf{c} = \sigma(\mathbf{d})$ in Theorem 6 to find a maximal contiguous degree $\mathbf{a}$ that is below $\mathbf{d}$ but not below $\sigma(\mathbf{d})$, so that $\sigma(\mathbf{a}) \neq \mathbf{a}$. On the other hand, suppose that $\sigma(\mathbf{d}) > \mathbf{d}$. Then $\sigma(\mathbf{d})$ is nonlow$_2$, so Theorem 6 yields a maximal contiguous degree $\mathbf{a}$ that is below $\sigma(\mathbf{d})$ but not below $\mathbf{d}$. Then $\sigma^{-1}(\mathbf{a}) \neq \mathbf{a}$, so $\sigma(\mathbf{a}) \neq \mathbf{a}$.

As noted, Sections 2, 3, and 4—most of the rest of the present paper—are devoted to proving Theorems 5 and 6.

1.3. The array noncomputable degrees. The array noncomputable (anc) degrees, defined by Downey, Jockusch, and Stob [1990], are another natural class of degrees. What is important here is that they capture the ability to do multiple permitting in a fashion intermediate between the permitting arguments that just use noncomputable degrees and those which use nonlow$_2$ degrees.

It is an open question whether the anc degrees are (naturally) definable. Walk [1999] had asked whether the array computable degrees were definable in $\mathcal{R}$ by the formula "x is bounded by a contiguous degree or is itself contiguous". This is a natural question since every contiguous degree is array computable (see Downey [1990] or Downey and Stob [1993]).

The basic construction of Section 2 can easily be altered so that $\mathbf{a}$ is array computable and noncontiguous, rather than contiguous, thus proving the following:

THEOREM 9. *There is an array computable c.e. degree* $\mathbf{a}$ *such that no c.e. degree* $\mathbf{d} \geq \mathbf{a}$ *is contiguous.*

Hence Walk's question has a negative answer. In section 5, we discuss how to modify the proof of Theorem 5 to get a proof of Theorem 9. We point out that Theorem 9 is not implied by Theorem 5 (or vice versa) and needs to be proved separately. We actually proved Theorem 9 first and then were able to modify the construction to prove Theorem 5.

1.4. Definability. We noted earlier that the definability of the contiguous degrees has yet to be exploited in the recent theme of transferring local results about $\mathcal{R}$ to global results about $\mathcal{R}$. Clearly the same can be said about the maximal contiguous degrees, though work in that direction has begun (see for instance Section 6 below).

For example, Steffen Lempp and Reed Solomon have attempted to use the maximal contiguous degrees to prove a version of the biinterpretability (with parameters) conjecture as suggested by Nies (see Nies [2000] for more details about this version of the biinterpretability conjecture).

They intended to show that for all n there is a c.e. degree $\mathbf{d}$ which is the join of exactly n maximal contiguous degrees. Instead, however, they showed that for any two incomparable contiguous c.e. degrees $\mathbf{a_0}$ and $\mathbf{a_1}$, there is a contiguous degree that is below their join but not below either of them individually. (Lempp notes that "the proof is the canonical one": As the set X is being built below the join $A_0 \oplus A_1$, if it appears that $X \leq_{wtt} A_i$ with a given reduction, then a reduction of $A_{(1-i)}$ from A_i is extended. For more details see Lempp and Solomon.)

We note that this result (of Lempp and Solomon) does not by itself prohibit the existence of a degree $\mathbf{d}$ with finitely many maximal contiguous degrees below it, since it is currently unknown whether every contiguous degree is bounded by a maximal contiguous degree.

With Andre Nies we have been able to answer to a question of Li's[1] (see Slaman [1993, 3.10(a)]). More or less Li's question asked if there is a formula, $\Theta(x)$, in the language of $\mathcal{R}$, an interval of degrees and only one degree, $\mathbf{a}$, in the interval such that $\mathcal{R}$ realizes $\Theta(\mathbf{a})$. In Section 6, we give two such answers: we show that the property of being maximal contiguity works in an upper cone and we come up with a property that works in a lower cone.

1.5. Notation. Our notation is generally standard as in Soare [1987] and/or Odifreddi [1989] and Odifreddi [1999]. We assume the hat trick, and we assume that the uses of all functionals are increasing in the argument and nondecreasing in the stage. Also, we think of our sets and functionals as being in a state of formation, and an expression such as "X_s" represents the set X at the very moment it is mentioned; we will append "$[s]$" to the name of any object when we wish to consider that object at the *end* of the stage s. Nonstandard notations are as follows:

1. The use of a computation $\Phi_{e,s}(X_s; x)$ at a stage s will be denoted $u(\Phi_e, X, x, s)$. If the oracle is the join of two or more sets, we

[1]We would like to thank Steffen Lempp for suggesting that we consider answering Li's question.

assume that the use is computed separately on each of these sets, and if we are interested only in, say, the A-use from a computation $\Phi_{e,s}(A_s \oplus C_s; x)$, then we will denote it $u(\Phi_e(A \oplus C), A, x, s)$.

2. Given any two sets X and Y (or functionals acting on sets), we will denote the *length of agreement* between X and Y at stage s as

$$l(X, Y, s) = \max\{x : (\forall y \leq x)[X_s(y) = Y_s(y)]\}.$$

§2. To ensure maximal contiguity. In Section 3, we construct a c.e. set A whose degree is maximal contiguous and has certain additional properties. In the current section, we present an intuitive discussion of the basic construction, that is, a construction that simply creates a maximal contiguous c.e. degree with no extra features. The requirements to ensure noncontiguity of all degrees above deg(A) are as follows:

$$\mathcal{Q}_{\mathbf{e}} : \quad (\exists Q_e)\,[\, Q_e \equiv_{\mathrm{T}} V_e \oplus A \;\;\&\;\; (\forall i)\, \mathcal{Q}_{\mathbf{e},\mathbf{i}}\,],$$
$$\text{where } \mathcal{Q}_{\mathbf{e},\mathbf{i}} \text{ is}$$
$$\mathcal{Q}_{\mathbf{e},\mathbf{i}} : \quad \widehat{\Phi}_i(Q_e) \neq A \;\vee\; V_e \leq_{\mathrm{T}} A.$$

(Here $\widehat{\Phi}_i$ is a *wtt*-reduction.) The requirements to make deg(A) contiguous are as follows:

$$\mathcal{N}_{\mathbf{e}} : \quad \Psi_e(A) = W_e \;\;\&\;\; \Phi_e(W_e) = A \implies A \equiv_{wtt} W_e.$$

In this section the reader should concentrate on absorbing only the gist of the procedure, deferring concern for details until Section 3.

2.1. The noncontiguous bounding strategies. First we consider the requirements $\mathcal{Q}_{\mathbf{e}}$ and their subrequirements $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$. The idea is that for every c.e. set V_e, we would like to build a set $Q_e \equiv_{\mathrm{T}} V_e \oplus A$ such that $A \not\leq_{wtt} Q_e$. This will guarantee that deg($V_e \oplus A$) is noncontiguous, since otherwise $A \leq_{wtt} V_e \oplus A \equiv_{wtt} Q_e$.

We will ensure that $Q_e \equiv_{\mathrm{T}} V_e \oplus A$ via a "kick set" procedure as in Downey [1993] or Downey and Stob [1993]. That is, at every stage s each number n in ω has a marker $\Lambda_s(n)$. Initially, each $\Lambda_0(n) = n$. For any stage s, let $f(s)$ be the least number (if any) enumerated into $V_e \oplus A$ during stage s. Then $\Lambda_s(f(s))$ is put into Q_e at stage s, and the markers $\Lambda_s(n)$ for all $n \geq f(s)$ will be moved to numbers at least as large as s. At all times we require the following:

- $\Lambda_s(m) < \Lambda_s(n)$ for $m < n$;
- $\Lambda_s(m) \leq \Lambda_{s+1}(m)$; and
- $\Lambda_s(m) \notin Q_{e,s}$.

These are the only enumerations we make into Q_e. This procedure will automatically make $Q_e \equiv_{\mathrm{T}} V_e \oplus A$.

While making $Q_e \equiv_{\mathrm{T}} V_e \oplus A$, we will also try to meet all requirements of the form

$$\widehat{\Phi}_i(Q_e) \neq A,$$

where $\widehat{\Phi}_i$ is the ith "*wtt*-reduction" in the indexing $\{\widehat{\Phi}_k\}_{k\in\omega}$. The reader should note that by $\{\widehat{\Phi}_k\}_{k\in\omega}$ we really mean an appropriate indexing of all pairs $\langle \Phi_i, \widehat{\varphi}_j \rangle_{\langle i,j\rangle \in \omega}$ consisting of one computable functional and one (possibly partial and thus unusable) computable function. We will always read the expression "$\widehat{\Phi}_{k,s}(X_s; n)\downarrow$," where $k = \langle i, j\rangle$, to mean "$\Phi_{i,s}(X_s; n)\downarrow$, $\widehat{\varphi}_{j,s}(n)\downarrow$, and $u(\Phi_i, X, n, s) \leq \widehat{\varphi}_j(n)$."

The natural strategy for meeting $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ proceeds as follows: First choose a witness x, and wait until $\widehat{\Phi}_{i,s}(Q_{e,s}; y)\downarrow$ for all $y \leq x$. There are two cases: If $\widehat{\Phi}_{i,s}(Q_{e,s}; y) \neq A_s(y)$ for some such y, then restrain A below this y. Also restrain from A any m such that $\Lambda_{e,s}(2m+1) \leq u(\widehat{\Phi}_i, Q_e, y)$—that is, any m whose enumeration into A would require us to put into Q_e a kick set marker that would injure the computation for y. These restraints preserve the disagreement, *as long as* no m such that $\Lambda_{e,s}(2m) \leq u(\widehat{\Phi}_i, Q_e, y)$ is enumerated into V_e; but of course we have no control over V_e, and if the disagreement is lost, we will simply be in the second case.

In the second case, there is no $y \leq x$ such that $\widehat{\Phi}_{i,s}(Q_{e,s}; y) \neq A_s(y)$, so we will try to manufacture a disagreement by putting y into A. However, this action requires us to enumerate $\Lambda_{e,s}(2y+1)$ into Q_e, possibly injuring the $\widehat{\Phi}_{i,s}(Q_{e,s}; y) = 0$ computation. This problem can seemingly be solved by picking *two* witnesses y and z: for the enumeration of y into A at stage s, and the corresponding enumeration of $\Lambda_{e,s}(2y+1)$ into Q_e, will cause $\Lambda_{e,s+1}(2z+1)$ to be reset higher than the *wtt*-use $u(\widehat{\Phi}_i, Q_e, z)$, so that the enumeration of z (if needed later) will not ruin the computation for z.

All would be well if not for the fact that, as noted earlier, V_e is entirely out of our control; and an enumeration into V_e may ruin a diagonalization the same way an enumeration into A would have. We can prepare for this by picking even more witnesses, say n of them. Let x_0 be the least of these witnesses, and suppose we attempt diagonalization with x_0 before using any of the other $x_1, \ldots, x_n$. Then the enumeration of x_0 into A will cause the $\Lambda_s(z)$ for $z \geq 2x_0 + 1$ to be lifted above the *wtt*-use of *any* of the x_i's (since we will have waited for convergence on all of them before bothering to enumerate x_0); this means that the enumeration of some number $z \leq x_0$ into V_e may ruin some computations, but the enumeration of $z > x_0$ cannot. Thus we can solve this newest problem by simply requiring that $n \geq x_0$—as V_e cannot ruin our diagonalizations more than x_0 times.

2.2. Mixing the basic strategy for noncontiguous bounding with the contiguity requirements. The problem now becomes one of mixing the requirements $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ with higher-priority contiguity requirements $\mathcal{N}_{\mathbf{k}}$, any of which may have infinite restraint. (Mixing them with lower-priority contiguity requirements will not be an issue because each $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ takes only finitely much positive action.) Part of the problem will be solved by the use of "thresholds": To say that "$\mathcal{Q}_{\mathbf{e},\mathbf{i}}$'s threshold with respect to $\mathcal{N}_{\mathbf{k}}$ is n" means intuitively that we may not enumerate anything into A, on behalf of $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$, that would injure the $\mathcal{N}_{\mathbf{k}}$-computations for n. That is, if $m \leq n$ and

$$m < l(\Phi_k(W_k), A, s) \quad \text{and } u = u(\Phi_k, W_k, m, s) < l(\Psi_k(A), W_k, s),$$

then $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ is not allowed to enumerate anything into A below $u(\Psi_k, A, u, s)$.

We shall attempt the basic strategy for $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ infinitely often if we need to; each attempt, or *module*, will receive its own threshold with respect to each higher-priority contiguity requirement. The first step for any module is to choose a large number v_j, known as a *challenge number*. Each v_j plays the role of x_0 in the discussion of the basic strategy; that is, we choose v_j large, then work on choosing $v_j + 1$ many witnesses for use in diagonalizations. These $v_j + 1$ numbers $x_{j,0}, x_{j,1}, \ldots, x_{j,v_j}$ will form a *witness set* for $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$.

It will be necessary for us to induce a "privilege ordering" on all modules for all positive requirements; this ordering will be based on when those modules were begun and not on the usual priority ordering $\mathcal{Q}_0$, $\mathcal{Q}_1, \ldots$. We will stipulate that privilege and threshold are linked; that is, for a given contiguity requirement $\mathcal{N}_{\mathbf{k}}$, modules with "stronger" privilege will have lower thresholds with respect to $\mathcal{N}_{\mathbf{k}}$ than will modules with "weaker" privilege.

We will implicitly employ a "confirmation and cancellation" strategy, as in Ambos-Spies [1984] and Stob [1983], to make deg(A) contiguous; details for this strategy will be discussed later, on page 13 of Section 3 (see the informal discussion on meeting the requirement $\mathcal{N}_e$). Such a strategy will succeed, provided that certain cancellation relationships among positive requirements are met; one of these is that witnesses must be chosen one at a time, and in particular, new witnesses should not be chosen until the computations for $\mathcal{N}_{\mathbf{k}}$ have converged at least up to the previous witness.

Another very important provision is that any enumeration of a number y into A cancels all witnesses $x > y$. To observe this provision, we will have to alter the basic strategy somewhat: rather than enumerating the witnesses $x_{j,0}, x_{j,1}, \ldots, x_{j,v_j}$ in ascending order, we will enumerate them in descending order. Of course, the enumeration of $x_{j,0}$ was supposed to lift the witnesses' kick-set markers and clear the way for us, so if we can't put $x_{j,0}$ in first, we will have to find another way to lift those markers. To this end, we will allow V_e to do our dirty work for us: Namely, once the witness set is chosen, we will not bother to put any of the witnesses into A until V_e changes below v_j. Such a change will force a change in Q_e, lifting the markers as desired. If we fail, for cofinitely many modules, to get such a change, then V_e is computable and need not concern us.

Thus, once the witness set is chosen, we wait for V_e to change below v_j, lifting the markers for the witness set so that these markers will not injure the Q_e-computations we might want to diagonalize later. Only then (actually when the Q_e-computations recover) do we start enumerating the witness set elements into V_e, and we do so in descending order. Every time V_e changes below v_j and spoils *one* diagonalization, we wait for convergence and then create *another*.

The alert reader will be wondering whether we will, indeed, be able to create another diagonalization *every* time one is lost; after all, perhaps a higher-priority contiguity requirement will have increased its restraint since we chose the witness set, thus prohibiting us from making the needed enumeration if V_e changes. However, this is where the thresholds and the privilege ordering—and the connection between them—really come in handy. For suppose that the contiguity requirement $\mathcal{N}_{\mathbf{k}}$ *has* increased its restraint on our threshold; that is, the use has increased for some n whose computations our witness set is not allowed to injure. Because the use has increased, we see that some number z has been put into A; but then z came from a witness set with stronger privilege than our witness set, as otherwise it would have had to respect the threshold as well. Since z had stronger privilege than our witness set, our witness set must not have been chosen before z was put into A; z's enumeration canceled the witness set at that time, and the current witness set was chosen in the meantime.

How does this help? Up until now, we have ignored, for simplicity, the fact that the construction was taking place on a tree. We need to remember that fact now, however, since the tree allows our $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$-strategy ($\mathcal{Q}_{\mathbf{e},\mathbf{i}}$-*node*, actually; say γ) to predict whether the contiguity strategy (again, a node; say δ) will have infinitely many expansionary stages. Thus, our $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$-node γ can always defer the choice of any of its witness sets *until* δ's computations have converged up to the appropriate threshold. This means that our current witness set was not chosen *until* we knew the use for n, which means the numbers in the witness set are large and are not restrained from entering A after all.

So if δ increases its restraint, thus forcing a new witness set to be chosen when the computations converge, and *then* V_e changes in the appropriate place (below v_j, say), γ will not be restrained from creating a diagonalization. The only other thing to worry about is that V_e might change for the last time *before* the witness set is chosen; then, since γ waits for a V_e-change before making an enumeration, γ would never enumerate anything into A from this witness set even though $\widehat{\Phi}_i(Q_e; x_k) = A(x_k)$ for all x_k in the witness set. This could happen for almost all of γ's witness sets, so that γ is never able to win the requirement. However, we will argue that if this occurs, then $V_e \leq_{\mathrm{T}} A$: For whenever a witness set $x_{j,0}, x_{j,1}, \ldots, x_{j,v_j}$ is chosen during the construction, we define (implicitly) a reduction of $V_e \upharpoonright v_j$ from $A \upharpoonright x_{j,v_j}$. The witness set may be canceled if a stronger-privilege witness set makes an enumeration into A, but then this enumeration records any change in $V_e \upharpoonright v_j$ that might happen before the next witness set is chosen. The fact that only *stronger-privilege* witness sets can make such enumerations will guarantee, by induction, that the reduction is well-defined; so $V_e \leq_{\mathrm{T}} A$, and $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ is met.

These are the ideas behind the *basic* construction, which allows us to prove the following:

THEOREM 5. *A maximal contiguous degree exists.*

§3. The Main Theorem.

The previous section sketched the construction of a maximal contiguous degree. We know give such a construction explicitly, adding the requirements necessary to prove the following:

THEOREM 6. *Let* $\mathbf{d}$ *be any nonlow*$_2$ *c.e. degree, and* $\mathbf{c}$ *any c.e. degree such that* $\mathbf{d} \not\leq \mathbf{c}$. *Then there is a maximal contiguous degree* $\mathbf{a}$ *such that*
(i) $\mathbf{a} < \mathbf{d}$,
(ii) $\mathbf{a} \not\leq \mathbf{c}$, *and*
(iii) $\mathbf{a} \cup \mathbf{c}$ *is incomplete.*

Let C be a c.e. set in the degree $\mathbf{c}$. In addition to meeting the requirements $\mathcal{N}_{\mathbf{e}}$ and $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ of the previous section, we must also meet requirements that will ensure parts (ii) and (iii) of Theorem 6. We do not need to include requirements to ensure part (i), since **d** is nonlow$_2$ and all contiguous degrees are low$_2$ (see for example Stob [1983]). Thus our list of requirements is the following:

$$\mathcal{Q}_{\mathbf{e}}: \quad (\exists Q_e)\,[\, Q_e \equiv_{\mathrm{T}} V_e \oplus A \ \ \& \ \ (\forall i)\, \mathcal{Q}_{\mathbf{e},\mathbf{i}}\,],$$
where $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ is
$$\mathcal{Q}_{\mathbf{e},\mathbf{i}}: \quad \widehat{\Phi}_i(Q_e) \neq A \ \vee \ V_e \leq_{\mathrm{T}} A.$$

$$\mathcal{N}_{\mathbf{e}}: \quad \Psi_e(A) = W_e \ \ \& \ \ \Phi_e(W_e) = A \implies A \equiv_{wtt} W_e.$$

$$\mathcal{P}_{\mathbf{e}}: \quad \Phi_e(C) \neq A, \ \text{and}$$

$$\mathcal{R}_{\mathbf{e}}: \quad \Phi_e(A \oplus C) \neq K.$$

The construction will take place on a subtree of $\omega^{<\omega}$. For ease of reading, the sets and functionals associated with the node β will often be referred to with subscript β instead of numerical subscripts, unless numerical subscripts are necessary to make a point. The requirements $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$, $\mathcal{N}_{\mathbf{e}}$, $\mathcal{P}_{\mathbf{e}}$, and $\mathcal{R}_{\mathbf{e}}$ will be worked on by nodes of lengths $4\langle e, i\rangle$, $4e+1$, $4e+2$, and $4e+3$, respectively.

Each requirement $\mathcal{R}_{\mathbf{e}}$ will be met via a preservation strategy; any node working on such a requirement will impose finite restraint on A. The working of the strategy is standard, as is the following notation: If δ is a node that is working on the incompleteness requirement $\mathcal{R}_{\mathbf{e}}$, then we say that a stage s is *δ-expansionary* if

$$l(\Phi_e(A \oplus C), K, s) > \max\{l(\Phi_e(A \oplus C), K, v) : v < s\}.$$

A node working on $\mathcal{R}_{\mathbf{e}}$ has only one possible outcome: 0.

The requirements $\mathcal{P}_{\mathbf{e}}$ are met via the usual Sacks coding strategy, complicated appreciably by the maximal contiguity details. Each $\mathcal{P}_{\mathbf{e}}$ node γ will have markers $\mu_{\gamma,s}(n)$ at stage s for some $n \leq l(\Phi_e(C), A, s)$. The idea is that, at the first stage s for which the length of agreement $l = l(\Phi_e(C), A, s)$ is nontrivial, we put the numbers $0, 1, \ldots, l$ into a set that we'll call the first *module*, $\mathsf{H}_{\gamma,1,s}$. The module $\mathsf{H}_{\gamma,1,s}$ gets a privilege number and thresholds, as any noncontiguity module would. Also, $\mathsf{H}_{\gamma,1,s}$ never gets more elements, though it will lose the elements $l'+1, l'+2, \ldots, l$ if the length of agreement $l(\Phi_e(C), A, s)$ ever dips to $l' < l$.

Like a noncontiguous bounding node, the Sacks coding node γ does not try to pick any witnesses for this module until the contiguity nodes above it have reached their appropriate lengths of agreement (up to the thresholds of γ, that is). From then on, at any stage t at which γ is active, it will work toward completing its *witness set* $\mathsf{G}_{\gamma,1,t}$; i.e., it will try to pick a marker $\mu_{\gamma,t}(n)$ for the least $n \in \mathsf{H}_{\gamma,1,t}$ that has no marker yet. We do not begin any work on the *second* module $\mathsf{H}_{\gamma,2,t}$ until the first module's witness set is full; that is, until every element of the module has a witness in $\mathsf{G}_{\gamma,1,t}$, or $|\mathsf{G}_{\gamma,1,t}| = |\mathsf{H}_{\gamma,1,t}|$. Only when a number has a marker in some complete witness set do we ever use such a marker to record the number's enumeration into D.

Recall the idea behind the Sacks coding strategy: γ is trying to build, for the sake of contradiction, a reduction of D from A. The argument for the contradiction will run more or less as follows: Wait until positive nodes above γ are finished acting (and we will have argued that this does happen). Let $x \in \omega$. If $\Phi_e(C) = A$, then $l(\Phi_e(C), A, s)$ will be greater than x at cofinitely many stages s, so eventually x will be put into some module $\mathsf{H}_{\gamma,j,t}$, with some privilege k, and will not be removed from it. The point of the argument will be to show that we can tell, uniformly A-computably in x, what the final witness set for x's module is, and thus that $D \leq_{\mathrm{T}} A \oplus C \equiv_{\mathrm{T}} A$. We note here that it is the privilege ordering that allows us to make this argument and show that such a reduction would be well defined.

A node working on a requirement $\mathscr{P}_{\mathbf{e}}$ can have as an outcome any element of ω.

Remark 10. For the sake of uniformity (and, we hope, readability), we will use the same witness set notation for $\mathscr{Q}_{\mathbf{e},\mathbf{i}}$-nodes that we just introduced for $\mathscr{P}_{\mathbf{e}}$-nodes. Namely, if the node α is working on requirement $\mathscr{Q}_{\mathbf{e},\mathbf{i}}$, and if α's nth module has filled or is working to fill its witness set at stage s, then we will denote this set as $\mathsf{G}_{\alpha,n,s}$.

Turning to the contiguity requirements $\mathscr{N}_{\mathbf{e}}$, we first need a bit of notation. First, if δ is a contiguity node, we define the following length of agreement to measure the equivalence between A and W_δ:

$$L(\delta, s) = \max\left\{x : (\forall y < x)\left[\begin{array}{l} y < l(\Phi_\delta(W_\delta), A, s), \text{ and} \\ u(\Phi_\delta, W_\delta, y, s) < l(\Psi_\delta(A), W_\delta, s) \end{array}\right]\right\}.$$

Then, for any $x < L(\delta, s)$, we define what might be called the "two-layer use" of x:

$$U(\delta, x, s) = u(\Psi_\delta, A, u, s), \text{ where } u = u(\Phi_\delta, W_\delta, x, s).$$

We are trying to create a *wtt*-reduction $\widehat{\Theta}$ at δ, and the reduction can be ruined at x by the following sequence of events:

1. After seeing convergence of a computation pertaining to x at stage s_1, we set the computable use function $\widehat{\theta}$ equal to the current use on x.

2. Then the computation is injured, and when it recovers at stage s_2, the new use u' is greater than $\widehat{\theta}(x)$. We define $\widehat{\Theta}_{s_2}(x) = 0$ since $x \notin A_{s_2}$ (or $x \notin W_{\delta,s_2}$, depending on which computation we are considering).
3. Later, x enters A (or x enters W_δ), allowed in by a change in the oracle that is below u' but *not* below $\widehat{\theta}(x)$.

To prevent this, as noted earlier, each contiguity node will employ a confirmation-and-cancellation strategy, but the strategy will be implemented implicitly by the movement of the tree. By this we mean that a contiguity node δ will not acknowledge an expansionary stage, and thus will not have outcome 0, unless the length of agreement associated with δ is at least as great as any witness for any positive node extending $\delta^\frown 0$. (The only possible outcomes for δ will be 0 and 1.) Those familiar with the standard confirmation-and-cancellation strategy will note that such witnesses are automatically "confirmed" at expansionary stages, and witnesses to the right of $\delta^\frown 0$ are *canceled* at expansionary stages.

More precisely, suppose that x is chosen, at stage s, as a witness for a positive node α; and that δ is a contiguity node such that $\delta^\frown 0 \subset \alpha$. Suppose also that α and δ are on the true path. The key is that *the only witnesses that can injure the δ-computations for x are witnesses less than or equal to x*. For at the next δ-expansionary stage after s, all witnesses greater than x are canceled, as they pertain to nodes to the right of $\delta^\frown 0$; and all witnesses chosen later are greater than the uses that are present at the expansionary stage. Further, if the computations *are* injured later, by some witness that is less than x, then the same comments apply at the *next* δ-expansionary stage. This prevents the "worst-case scenario" described earlier.

To implement the strategy we will assign δ, at the end of each stage s, a number $\nu(\delta, s)$ that represents the greatest witness of any node extending $\delta^\frown 0$, which is then the length of agreement δ must reach before acknowledging an expansionary stage. This detail, together with the threshold-privilege connection and judicious choice and enumeration of the $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ witnesses, will ensure that $\deg(A)$ is contiguous.

We turn now to the noncontiguous bounding requirements $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ and the explicit basic strategy for such a requirement. If σ is a string of length $4\langle e, i\rangle$, so that σ is working on requirement $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$, then the string $\tau \subset \sigma$ of length $4\langle e, 0\rangle$ is called the *parent* of σ and is denoted $\pi(\sigma)$. Conversely, σ is called a *child* of $\pi(\sigma)$. All children of this parent τ work to build one c.e. set Q_τ such that $Q_\tau \equiv_{\mathrm{T}} V_e \oplus A$. The set Q_τ will be built via a kick set procedure as indicated earlier, altered so that the only stages at which we reset any markers $\Lambda_{\tau,s}(n)$ are stages at which τ is active. Each child of length $4\langle e, i\rangle$ uses its own version of the basic strategy of Section 2 to try to guarantee that either $A \neq \widehat{\Phi}_i(Q_\tau)$ or $V_e \leq_{\mathrm{T}} A$. As noted earlier, each $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$-node σ will make (if necessary) infinitely many attempts to meet $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$; the $(n+1)$st attempt will be referred to as σ's *nth module*.

To create the privilege ordering, we simply assign each module a number from the current *privilege list*. The privilege list is a universal object in the construction; we assign privileges to modules of positive requirements in the order those modules begin their work. The list starts out as ω, and every time a privilege n is assigned to a module, that n is removed from the privilege number list and is never assigned to any other module.

A node working on a requirement $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ has 0 as its only possible outcome.

We now make precise the discussion from Section 2.

Strategy for $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ at stage $s+1$. Let β be a node working on the requirement $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$. If there is some y such that $\widehat{\Phi}_{i,s+1}(Q_{\pi(\beta),s+1}; y)\downarrow \neq A_{s+1}(y)$, then set β's restraint to be

$$r(\beta, s+1) = \max\left(\{m+1 : \Lambda_{\pi(\beta),s+1}(m) \leq \widehat{\varphi}_i(y)\} \cup \{y+1\}\right);$$

and for any positive node α that extends β, and any witness set $\mathsf{G}_{\alpha,n',s+1}$ such that $\min \mathsf{G}_{\alpha,n',s+1} \leq r(\beta, s+1)$, cancel $\mathsf{G}_{\alpha,n',s+1}$. If there is no such y, then proceed as follows:

Let $t < s+1$ be the most recent stage such that β was active at stage t, that is, $\beta \subseteq f_t$. If there is an n such that the nth module for β was in State 5 at stage t and $V_{e,s+1} \upharpoonright v_n + 1 \neq V_{e,t} \upharpoonright v_n + 1$, then let n be least such and do the following:

1. initialize β's mth modules for all $m > n$;
2. set

$$r(\beta, s+1) = \max\left\{z : \Lambda_{\pi(\beta),s+1}(z) \leq \widehat{\varphi}_i(y) \text{ for some } y \in \mathsf{G}_{\beta,n,s+1}\right\};$$

3. for any positive node α that extends β, and any witness set $\mathsf{G}_{\alpha,n',s+1}$ such that $\min \mathsf{G}_{\alpha,n',s+1} \leq r(\beta, s+1)$, cancel $\mathsf{G}_{\alpha,n',s+1}$;
4. and put the nth module into State 6 at the next β-stage.

Otherwise find the least n such that β's nth module is not in State 5. If this nth module has not been started, start it (that is, put it into State 1); in any case, proceed as described for the appropriate State:

State 1 First, choose a large number v_n to be the "challenge number" for β's nth module. Next, set $\texttt{priv}(\beta, n, s+1)$ equal to the lowest privilege number still on the privilege number list, and then remove $\texttt{priv}(\beta, n, s+1)$ from the privilege number list. Finally, for every contiguity node δ such that $\delta^\frown 0 \subset \beta$, set $\texttt{thr}(\beta, \delta, n, s+1)$ to be 1 plus the maximum of the following set:

$$\{L(\delta, s+1)\} \bigcup \left\{ \texttt{thr}(\alpha, \delta, m, v) : \begin{array}{l} \alpha \text{ is any positive} \\ \text{node}, m, v \leq s+1, \text{and} \\ \texttt{thr}(\alpha, \delta, m, v)\downarrow \end{array} \right\}.$$

Proceed to State 2 at the next β-stage.

State 2 If there is some contiguity node δ such that $\delta^\frown 0 \subset \beta$ and $L(\delta, s+1) < \texttt{thr}(\beta, \delta, n, s+1)$, do nothing, and remain in State 2 at the next β-stage. Otherwise, proceed to State 3 at the next β-stage.

State 3 If the witness set $\mathsf{G}_{\beta,n,s+1}$ is incomplete, that is, $\left|\mathsf{G}_{\beta,n,s+1}\right| < v_n + 1$, then *at the end of the present stage*, attempt to choose a witness x for the witness set $\mathsf{G}_{\beta,n,s+1}$. In this case, remain in State 3 at the next β-stage unless directed, at the end of the present stage, to proceed to State 4 at the next β-stage.

State 4 If it is not the case that $\widehat{\Phi}_{i,s+1}(Q_{\pi(\beta),s+1}; y)\downarrow = A_{s+1}(y)$ for all $y \in \mathsf{G}_{\beta,n,s+1}$ $(= \mathsf{G}_{\beta,n,t})$, then do nothing, and remain in State 4 at the next β-stage. Otherwise, proceed to State 5 at the next β-stage.

State 5 If $V_{e,s+1} \restriction v_n + 1 \neq V_{e,t} \restriction v_n + 1$, then set

$$r(\beta, s+1) = \max\left\{z : \Lambda_{\pi(\beta),s+1}(z) \leq \widehat{\varphi}_i(y) \text{ for some } y \in \mathsf{G}_{\beta,n,s+1}\right\}.$$

For any positive node α that extends β, and any witness set $\mathsf{G}_{\alpha,n',s+1}$ such that $\min \mathsf{G}_{\alpha,n',s+1} \leq r(\beta, s+1)$, cancel $\mathsf{G}_{\alpha,n',s+1}$. Proceed to State 6 at the next β-stage.

However, if $V_{e,s+1} \restriction v_n + 1 = V_{e,t} \restriction v_n + 1$, then keep the nth module in State 5 at the next β-stage and proceed (at the next β-stage) to State 1 for the $(n+1)$-st module.

State 6 If it is not the case that $\widehat{\Phi}_{i,s+1}(Q_{\pi(\beta),s+1}; y)\downarrow = A_{s+1}(y)$ for all $y \in \mathsf{G}_{\beta,n,s+1}$, then do nothing, and remain in State 6 at the next β-stage. Further, do not start the $(n+1)$-st module.

Otherwise (that is, if $\widehat{\Phi}_{i,s+1}(Q_{\pi(\beta),s+1}; y)\downarrow = A_{s+1}(y)$ for all $y \in \mathsf{G}_{\beta,n,s+1}$), let x be the greatest member of $\mathsf{G}_{\beta,n,s+1} \cap \overline{A}_{s+1}$ that is not already in the permitting bin. Put x into the permitting bin and give it "permitting number" $k+1$, where k

is the cardinality of the following set:

$$\left\{ m : \begin{array}{l} m < n; \text{ the } m\text{th module for } \beta \text{ has reached State 6 since} \\ \text{the last initialization of } \beta; \text{ and no } m'\text{-module, } m' < m, \\ \text{has reached its own State 6 since the } m\text{th module did} \end{array} \right\}.$$

For every $m > n$, initialize the mth module.

At the next β-stage, put the nth module back into State 5 and start the $(n + 1)$-st module (that is, proceed to State 1 for the $(n + 1)$-st module).

End of Strategy

Remark 11. Note that the witness set for the nth module does not have permitting number n. Instead, the first of β's modules to reach State 5 between initializations gets permitting number 1 for its witnesses; the next one gets permitting number 2; and so on. The assignment of modules to permitting numbers may change, depending on which modules reach State 5 and when. If the kth module has permitting number 1, and some jth module ($j < k$) reaches State 5 later, then the kth module is initialized and the jth module gets permitting number 1. In general, if some jth module reaches State 5 *after* some kth module, $k > j$, has reached its own State 5, then all such kth modules are initialized and the jth module gets the lowest permitting number that any of them had.

It is this seemingly peculiar permitting arrangement that allows us to use the "multiply-inductive permitting" argument of Downey and Shore [1996] in the verification in Section 4. This type of permitting is, in fact, necessary for this construction; ordinary or even multiple permitting (as in Downey, Jockusch, and Stob [1990]) will not suffice. Ordinary permitting does not work because, as noted earlier, the number of permissions needed by each module is not only greater than 1 but also *dependent on* V_e. So A might fail to get a permission from D, but this failure might come after a number of permissions it *does* get from D; thus the failure of $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ (cofinitely often) to get its permissions does not imply that D is computable. It can be argued that this failure makes $A \oplus V_e \geq_{\mathrm{T}} D$, so that $A \oplus V_e$ is nonlow$_2$ and thus noncontiguous anyway, but in this case the $\mathcal{Q}_{e,i}$-node may enumerate infinitely many numbers into A, thus initializing nodes below it infinitely often; and *that* is something we have to avoid.

Thus we require that D be nonlow$_2$, so that all the permissions *are* received and the $\mathcal{Q}_{e,i}$-node quits enumerating. This cannot even be guaranteed below all *array noncomputable* degrees, since we don't know "ahead of time" (i.e., computably) how many witnesses will be associated with each permitting number.

Remark 12. The functions $\texttt{priv}(\beta, n, s+1)$ and $\texttt{thr}(\beta, \delta, n, s+1)$ are the privilege and threshold described on page 8 of Section 2.2. Recall that each module's place in the privilege ordering depends on when the module was started; and each module's threshold with respect to a contiguity node δ tells which of δ's computations the module may injure. We have set things up so that lower (i.e., stronger) privilege implies lower threshold.

We now give the full construction of the set A whose degree, $\mathbf{a}$, is posited by Theorem 6. We will first set up, as in Downey and Shore [1996], the machinery to achieve the multiply-inductive permitting from the given nonlow$_2$ degree $\mathbf{d}$. To this end, we define a class of computably approximable functions:

$$\mathcal{F} = \left\{ \tilde{f} : \begin{array}{l} \tilde{f}(0) = 1 \ \wedge \ (\exists \text{ computable } f(n,s))\,(\forall n > 0) \\ [\lim f(n,s) = \tilde{f}(n) \ \wedge \ |\{s : f(n,s+1) \neq f(n,s)\}| \leq \tilde{f}(n-1)\,] \end{array} \right\}.$$

It is not hard to see that $\mathcal{F}$ can be generated uniformly computably in $\mathbf{0}'$. From $\mathcal{F}$ we define a class of c.e. sets:

$$\mathcal{V} = \left\{ W : (\exists \tilde{f} \in \mathcal{F})\,(\forall n)\,\left[\,\left|W^{[n]}\right| \leq \tilde{f}(n)\,\right] \right\}.$$

This class $\mathcal{V}$ is also uniformly computable in $\mathbf{0}'$, as is the final class of functions:

$$\mathcal{G} = \left\{ g : (\exists W \in \mathcal{V})\,(\forall n) \left[\begin{array}{c} g(n) \text{ is the last stage at which a} \\ \text{number is enumerated into } W^{[n]} \end{array} \right] \right\}.$$

Since the class $\mathcal{G}$ is uniformly computable in $\mathbf{0}'$, we can specify a function $\widehat{g} \leq_{\mathrm{T}} \mathbf{0}'$ that dominates every $g \in \mathcal{G}$. Next, we choose a c.e. set $D \in \mathbf{d}$ and note that, since D is nonlow$_2$, there is a function $\tilde{h} \leq_{\mathrm{T}} D$ that is not dominated by $\widehat{g}$. Finally, we choose a computable approximation $h(n,s)$ to $\tilde{h}$, with the following specifications: For all n and s,

- $h(n,n) > \widehat{g}(n,n)$;
- if $h(n,s+1) \neq h(n,s)$, then $h(n,s+1) > \widehat{g}(n,s+1)$;
- $h(n+1,s) \geq h(n,s)$;
- $h(n,s+1) \geq h(n,s)$; and
- if $h(n,s+1) \neq h(n,s)$, then $h(n,s+1) = s+1$.

This approximation h will be used (as in Downey and Shore [1996]) for permitting during the construction, to help ensure that $A \leq_{\mathrm{T}} D$.

At the beginning of the construction, all thresholds are 0, all restraints are 0, and all kick set markers are equal to the numbers that they are supposed to mark (i.e., $\Lambda_{\pi(\alpha),0}(n) = n$). These parameters retain their values unless explicitly reset. Further, the privilege list is ω at the beginning of the construction, and no module for any $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$-node is in *any* State unless the construction explicitly directs that it be started. At stage 0, do nothing.

Construction, stage $s+1$. We will build nodes $\beta_{s+1,k}$ by induction on k $(0 \leq k \leq s+1)$, and define the approximation to the true path as $f_{s+1} = \beta_{s+1,s+1}$.

First, $\beta_{s+1,0} = \lambda$. Now let $\beta = \beta_{s+1,k}$.

Case 1 (β **is a noncontiguity node):** Suppose $|\beta| = 4m = 4\langle e, i\rangle$, so that β is a noncontiguity node working on $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$.

1. First, if $i = 0$ (so that β is a parent), then for all $n \leq s+1$ reset the kick set markers $\Lambda_{\beta,s+1}(n)$ as follows. For all n such that $\Lambda_\beta(n)[s]\downarrow$, let $\Lambda_{\beta,s+1}(n) = \Lambda_\beta(n)[s]$. Now let y be least such that $\Lambda_\beta(y)[s]\uparrow$. Set $\Lambda_{\beta,s+1}(y)$ to be large, and for z such that $s \geq z \geq y$, set $\Lambda_{\beta,s+1}(z+1) = \Lambda_{\beta,s+1}(z) + 1$, thus preserving the properties on page 6.
2. Run the strategy described on pages 14 to 16 for $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$.
3. If $k \leq s$, set $\beta_{s+1,k+1} = \beta^\frown 0$. If $k = s+1$, go to the end-of-stage directions instead.

Case 2 (β **is a contiguity node):** Suppose $|\beta| = 4m+1$, so that β is a contiguity node working on $\mathcal{N}_{\mathbf{m}}$.

1. If there exist any positive node α and any $n \leq s+1$ such that $\beta^\frown 0 \subset \alpha$ and
$$\min \mathsf{G}_{\alpha,n,s} < U(\beta, \mathtt{thr}(\alpha,\beta,n,s), s+1),$$
then cancel the witness set $\mathsf{G}_{\alpha,n,s+1}$.
2. If $k \leq s$, then let the outcome of β be determined as follows (if $k = s+1$, go to the end-of-stage directions): If $L(\beta, s+1) > \nu(\beta, s)$, then set $\beta_{s+1,k+1} = \beta^\frown 0$; otherwise set $\beta_{s+1,k+1} = \beta^\frown 1$.

Case 3 (β **is a Sacks coding node):** Suppose $|\beta| = 4m+2$, so that β is a Sacks coding node working on $\mathcal{P}_{\mathbf{m}}$. Let $t < s+1$ be the most recent β-stage, and let p be the smallest length of agreement between $\Phi_\beta(C)$ and A since stage t, that is,
$$p = \min_{v:\, t \leq v \leq s+1} \{l(\Phi_\beta(C), A, v)\}.$$

1. Cancel $\mu_{\beta,s+1}(z)$ for all $z > p$.

2. Let n be least such that the $(n+1)$st module $\mathsf{H}_{\beta,n+1}[s]$ is undefined. If there is some contiguity node δ such that $\delta\frown 0 \subset \beta$ and $L(\delta, s+1) < \mathtt{thr}(\beta, \delta, n, s+1)$, go directly to step 8 below.
3. If $p \in \mathsf{H}_{\beta,n}[s]$, then let $\mathsf{H}_{\beta,n,s+1} = \mathsf{H}_{\beta,n}[s] \cap \{0, 1, \dots, p\}$. Otherwise, for $j \le n$, let $\mathsf{H}_{\beta,j,s+1} = \mathsf{H}_{\beta,j}[s]$.
4. If the witness set $\mathsf{G}_{\beta,n,s+1}$ is incomplete (that is, if $\left|\mathsf{G}_{\beta,n,s+1}\right| < \left|\mathsf{H}_{\beta,n,s}\right|$), then go directly to step 8 below and, *at the end of the present stage*, attempt to choose an x to put in $\mathsf{G}_{\beta,n,s+1}$. Otherwise (if $\left|\mathsf{G}_{\beta,n,s+1}\right| = \left|\mathsf{H}_{\beta,n,s}\right|$) proceed to step 5 to define the $(n+1)$st module.
5. Define $\mathsf{H}_{\beta,n+1,s+1}$ as $\{z : \max \mathsf{H}_{\beta,n,s+1} < z \le p\}$.
6. Set $\mathtt{priv}(\beta, n+1, s+1)$ equal to the lowest privilege number still on the privilege number list, and then remove $\mathtt{priv}(\beta, n+1, s+1)$ from the privilege number list.
7. For every contiguity node δ such that $\delta\frown 0 \subset \beta$: Set $\mathtt{thr}(\beta, \delta, n+1, s+1)$ to be 1 plus the maximum of the following set:

$$\{L(\delta, s+1)\} \bigcup \left\{ \mathtt{thr}(\alpha, \delta, m, v) : \begin{array}{l} \alpha \text{ is any positive} \\ \text{node, } m, v \le s+1 \text{, and} \\ \mathtt{thr}(\alpha, \delta, m, v)\downarrow \end{array} \right\}.$$

8. If $k \le s$, then set $\beta_{s+1,k+1} = \beta\frown p$. If $k = s+1$, then go instead to the end-of-stage directions.

Case 4 (β is an incompleteness node): Suppose $|\beta| = 4m+3$, so that β is an incompleteness node working on $\mathcal{R}_{\mathbf{m}}$.

1. If there is some x such that $\Phi_{\beta,s+1}(A_{s+1} \oplus C_{s+1}; x)\downarrow \ne 1$ and $x \in K_{s+1}$, then set

$$r(\beta, s+1) = u(\Phi_\beta(A \oplus C), A, x, s+1) + 1$$

for the least such x. For any positive node α that extends β, and any witness set $\mathsf{G}_{\alpha,n,s+1}$ such that $\min \mathsf{G}_{\alpha,n,s+1} \le r(\beta, s+1)$, cancel $\mathsf{G}_{\alpha,n,s+1}$.
If there is no such x, then proceed as follows. If $s+1$ is a β-expansionary stage, set

$$r(\beta, s+1) = u(\Phi_\beta(A \oplus C), A, l, s+1) \; + \; 1,$$

where $l = l(\Phi_\beta(A \oplus C), K, s+1)$; also, for any positive node α that extends β, and any witness set $\mathsf{G}_{\alpha,n,s+1}$ such that $\min \mathsf{G}_{\alpha,n,s+1} \le r(\beta, s+1)$, cancel $\mathsf{G}_{\alpha,n,s+1}$. But if $s+1$ is not a β-expansionary stage, do nothing.

2. Finally, if $k \le s$, then let $\beta_{s+1,k+1} = \beta\widehat{\ }0$; if $k = s + 1$, then go instead to the end-of-stage directions.

At the end of stage $s + 1$:

1. Set $f_{s+1} = \beta_{s+1,s+1}$.
2. Initialize all nodes to the right of f_{s+1}; that is, cancel all parameters, witnesses (even those in the permitting bin), and functionals associated with such nodes, and cancel the set Q_τ and the kick set markers of any parent node τ to the right of f_{s+1}.
3. *Witness appointment:* Among all the modules that were instructed to appoint witnesses at the end of this stage, let α and n be such that $\texttt{priv}(\alpha, n, s + 1)$ is least.

   If α is a noncontiguous bounding node working on some requirement $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$, then simply choose a large number x and put x into $\mathsf{G}_{\alpha,n,s+1}$. If $\left|\mathsf{G}_{\alpha,n,s+1}\right| < v_n + 1$ (where v_n is the challenge number associated with this module for this node), then declare that the witness set $\mathsf{G}_{\alpha,n,s+1}$ is *incomplete* and return the module to State 3 at the next β-stage; otherwise declare that $\mathsf{G}_{\alpha,n,s+1}$ is *complete* and put the module into State 4 at the next β-stage.

   If α is a Sacks coding node working on some requirement $\mathcal{P}_{\mathbf{e}}$, then let y be the least number such that $\mu_{\alpha,s+1}(y)\uparrow$. Set $\mu_{\alpha,s+1}(y)$ large, and put $\mu_{\alpha,s+1}(y)$ into $\mathsf{G}_{\alpha,n,s+1}$. If $y \ne \max \mathsf{H}_{\alpha,n,s+1}$, so that $\left|\mathsf{G}_{\alpha,n,s+1}\right| < \left|\mathsf{H}_{\alpha,n,s+1}\right|$, then declare that the witness set $\mathsf{G}_{\alpha,n,s+1}$ is *incomplete*; otherwise declare that $\mathsf{G}_{\alpha,n,s+1}$ is *complete*.
4. Any parameter not canceled or explicitly defined or redefined during stage $s + 1$ has the same value as it did at stage s.
5. *For the Sacks coding:* Let $z = \min(D_{s+1} - D_s)$. Let k be the least such that for some Sacks coding node γ and some n,

   $z \in \mathsf{H}_{\gamma,n,s+1}$; $\texttt{priv}(\gamma, n, s + 1) = k$; $\mu_{\gamma,s+1}(z)$ is still defined; and either $\gamma <_{\mathrm{L}} f_t$ or $\gamma \subseteq f_t$.

   (That is, z is in γ's nth module, and this is the strongest-priority uncancelled module that z is in.) Enumerate $\mu_{\gamma,s+1}(z)$ into A. Remove all numbers greater than z from the module $\mathsf{H}_{\gamma,n,s+1}$, and remove their markers from the witness set $\mathsf{G}_{\gamma,n,s+1}$. Initialize all nodes that lie to the right of γ, or that extend $\gamma\widehat{\ }m$ for $m \ge z$.
6. *Permitting via the function h:* Let n be least such that $h(n, s + 1) \ne h(n, s)$. Enumerate into A any number x still in the permitting bin that has a permitting number greater than or equal to n.

   For any such x, let α and m be such that $x \in \mathsf{G}_{\alpha,m,s+1}$. Put α's mth module back into State 4. Cancel α's m'th module, for all $m' > m$. Initialize all nodes to the right of or extending α.

7. *Cancellation of (some) witnesses:* For every y that entered A at stage $s+1$, proceed as follows. The module for which y was a witness had some privilege number k. If there is any $z > y$ such that z is a witness for a module with privilege number $k' > k$, then cancel the witness set containing z. Further, if z's module is a module for a noncontiguity node, then put the module back into State 3.
8. *Kick set maintenance:* For every parent node $\tau \subseteq f_{s+1}$ that is working on a requirement $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ and is currently uninitialized, find the least y, if there is one, such that $y \in (V_\tau \oplus A)_{s+1} - (V_\tau \oplus A)_s$ and $\Lambda_{\tau,s+1}(y)\downarrow$. Enumerate $\Lambda_{\tau,s+1}(y)$ into Q_τ, and declare that $\Lambda_{\tau,s+1}(z)\uparrow$ for all $z \geq y$.
9. For any contiguity node $\delta \subseteq f_{s+1}$, set $\nu(\delta, s+1)$ equal to

$$\max_{\sigma : \delta^\frown 0 \subset \sigma} \left\{ x : \begin{array}{l} \sigma \text{ is a positive node and} \\ x \text{ is a witness for } \sigma \end{array} \right\}.$$

End of Construction

Remark 13. Notice that, in step 7 of the end-of-stage directions, we do not cancel a witness set unless it (a) has weaker privilege than the enumerated number y, *and* (b) some of its members are greater than y. The idea is that we want to be able to ascertain, computably in A, *whether* a witness set gets canceled. Both of these conditions are necessary: condition (b) ensures that A gets some record of the cancellation, and condition (a) (as we will argue) ensures that such cancellation stops eventually.

Remark 14. Recall that the value $\nu(\delta, s+1)$, set at the end of stage $s+1$ for each contiguity node δ active at stage $s+1$, is what makes the "confirmation" part of the contiguity strategy work. Namely, $\nu(\delta, s+1)$ is the maximum of all witnesses chosen by stage $s+1$ for nodes extending $\delta^\frown 0$; and δ will not acknowledge an expansionary stage until a stage t such that $L(\delta, t) > \nu(\delta, s+1)$. Thus at such a stage t, all witnesses for nodes extending $\delta^\frown 0$ are, implicitly, "confirmed" in the sense of Ladner and Sasso [1975] or Stob [1983]; and witnesses to the right of $\delta^\frown 0$ (that is, those that have been assigned since the last expansionary stage) are canceled.

§4. The verification. Our first few lemmas, 15 through 18, are short in both statement and proof:

Lemma 15. *$A \leq_T D$.*

Proof. Immediate (as nothing is enumerated into A except by parts 5 and 6 of the end-of-stage directions). ⊣

LEMMA 16. *Suppose τ is a parent node that is active at infinitely many stages and is initialized at only finitely many stages. Then $Q_\tau \equiv_T V_\tau \oplus A$.*

PROOF. Let s_0 be a stage such that τ is active at stage s_0 and τ is not initialized at any stage $s \geq s_0$.

($Q_\tau \leq_T V_\tau \oplus A$.) Given $x \in \omega$, suppose $x \notin Q_{\tau,s_0}$ (otherwise we are done). Wait until the first τ-stage s_1 after stage x. By the basic strategy, since markers are chosen large, if x has not yet been chosen as a marker $\Lambda_{\tau,s_1-1}(y)$ for some y, it never will be; then $x \notin Q_\tau$. So suppose $x = \Lambda_{\tau,s_1}(y)\ \big(= \Lambda_{\tau,s_1-1}(y)\big)$ for some y. If $y \in (V_\tau \oplus A)_{s_1-1}$ or $y \notin V_\tau \oplus A$, then $x \notin Q_\tau$. Otherwise, let $s_2 \geq s_1$ be the stage at which y enters $V_\tau \oplus A$. Then $x \in Q_\tau$ if and only if $x = \Lambda_{\tau,s_2}(y)$.

($V_\tau \oplus A \leq Q_\tau$.) Given $x \in \omega$, suppose $x \notin (V_\tau \oplus A)_{s_0}$. Wait until the first τ-stage s_1 after stage x. By the basic strategy, a marker $\Lambda_{\tau,s_1}(x)$ will be chosen at stage s_1 if not before, and $\Lambda_{\tau,s_1}(x) = \Lambda_{\tau,t}(x)$ for all $t \geq s_1$ unless a marker $\Lambda_{\tau,s_1'}(y)$, $y < x$, enters Q_τ at some stage $s_1' > s_1$. By consulting Q_τ we can tell whether such a marker enters; then a new marker for x will be chosen at the first τ-stage s_2 after stage s_1'. Then $\Lambda_{\tau,s_2}(x) = \Lambda_{\tau,t}(x)$ for all $t \geq s_2$ unless another marker enters A, and so on. The process repeats until a τ-stage s_n such that for all $y < x$, $\Lambda_{\tau,s_n}(y) \notin Q_\tau$. Then $x \in V_\tau \oplus A$ if and only if $\Lambda_{\tau,s_n}(x) \in Q_\tau$. ⊣

LEMMA 17. *For any positive nodes σ_1 and σ_2; any contiguity node δ such that $\delta^\frown 0 \subset \sigma_i$ for $i = 1, 2$; all n, and all s:*

$$\texttt{thr}(\sigma_1, \delta, n, s) < \texttt{thr}(\sigma_2, \delta, n, s) \text{ iff } \texttt{priv}(\sigma_1, n, s) < \texttt{priv}(\sigma_2, n, s),$$

assuming of course that these thresholds and privileges are defined.

PROOF. Immediate, by the procedure for picking thresholds. ⊣

LEMMA 18. *Let $k \in \omega$. Only finitely many witnesses associated with privilege number k are ever chosen; and in particular, only finitely many are ever enumerated into A.*

PROOF. By induction on k. Any module with privilege number $k+1$ attempts to choose, and thus may enumerate, only finitely many witnesses for its witness set. Of course the witness set can be canceled, allowing new choices for witnesses and more enumerations; but we claim that such cancellation cannot happen infinitely often. For the witness set is not canceled unless

1. the module itself is canceled (in which case $k+1$ is never again used as a privilege number); or
2. some number with stronger privilege than $k + 1$ is enumerated into A (part 7 of the end-of-stage directions); or

3. some contiguity node δ increases its use relative to the module's threshold (part 1 of the contiguity node directions).

We say the use has *increased* in this last case because the witnesses were chosen, at some stage s, to be greater than whatever the contiguity node's restraint was at s; but the restraint is now greater than the witnesses, or else they would not have been canceled. The increase in restraint implies an enumeration into A by some module that was not restricted below the same threshold as the module with privilege number $k+1$. This module had to have stronger privilege than $k+1$, by Lemma 17.

Case 1 obviously stops enumerations with privilege number $k+1$, and Cases 2 and 3—since each involves enumeration into A by a stronger-privilege module—happen only finitely often by the induction hypothesis. ⊣

Our next two lemmas are concerned with the workings of positive nodes "along the true path." The problem is that, since a $\mathscr{P}_{\mathbf{e}}$-node may have any natural number as its outcome at any stage, there is no guarantee that the true path even exists. Thus Lemmas 19 and 20 do not assume the existence of a true path; they merely assert that *if* certain true-path–like hypotheses are true, *then* the nodes involved do what they are supposed to do. Lemma 21 below will, simultaneously, assert the existence of the true path and show that the requirements are met by nodes along the true path, thus proving Theorem 6.

LEMMA 19. *Let σ be a noncontiguous bounding node that is active at infinitely many stages and is initialized at only finitely many stages, and such that the restraint for every noncontiguity or incompleteness node $\alpha \subset \sigma$ is finite.*

Then for all n, σ's nth module enters State 6 of the basic strategy only finitely often.

PROOF. If σ's nth module is not canceled, it retains its challenge number v_n. Since the module does not enter State 6 unless V_σ changes below $v_n + 1$, this entry can happen only finitely often as long as v_n does not change. (Note that it does not matter how often the nth module's witness set is canceled.)

Now we just need to argue that σ's nth module is canceled only finitely often. Since σ itself is initialized only finitely often, the cancellation of the nth module can come about only because the n'th module, for some $n' < n$, enters its own State 6. In particular, the first module is canceled only finitely often and thus enters State 6 only finitely often; the lemma follows by induction. ⊣

LEMMA 20. *Let σ be either a noncontiguous bounding node for which only finitely many of σ's modules ever enter State 6, or a Sacks coding node. Suppose σ is active at infinitely many stages and is initialized at only finitely many stages, and suppose that the restraint for every noncontiguity or incompleteness node $\alpha \subset \sigma$ is finite. Suppose further that infinitely many modules for σ are started. Let $k_1 < k_2 < \cdots$ be the sequence of privilege numbers chosen for these modules, and suppose that the m_ith module has privilege k_i. (It is possible that $m_i = m_j$ for $i \neq j$ since privilege numbers can be canceled.)*

Then the procedure for finding a stage t_i such that

> *EITHER the privilege number k_i is canceled by stage t_i OR the witness set with privilege k_i is complete at t_i and is never canceled after stage t_i*

is A-computable uniformly in i.

PROOF. Let t_0 be a stage at which σ is active, after which σ is never initialized again, after which no node above σ ever increases its restraint, and—if σ is a noncontiguous bounding node—after which (by Lemma 19) *none* of σ's modules ever enters State 6. We describe an A-computable procedure for finding t_{i+1}, assuming (by induction) that we have used the same procedure already to find t_j for $j \leq i$. We note without further comment that if the privilege number k_{i+1} is canceled at any stage t during the process, we simply set $t_{i+1} = t$.

Wait until a stage t_i' greater than all t_j ($j \leq i$) and such that, for all $j \leq i$ such that the privilege number k_j has not been canceled by t_j, we have

$$A_{t_i'} \upharpoonright \max \mathsf{G}_{\sigma,m_j,t_i'} = A \upharpoonright \max \mathsf{G}_{\sigma,m_j,t_i'} \ (\subseteq A \upharpoonright \max \mathsf{G}_{\sigma,m_j,t_j}).$$

Clearly such a t_i' exists, as none of the witness sets $\mathsf{G}_{\sigma,m_j,t_j}$ is canceled after t_j.

Next, wait until a σ-stage t_i'' such that the witness set $\mathsf{G}_{\sigma,m_{i+1},t_i''}$ is complete. We claim that such a stage t_i'' exists, for at any σ-stage t at which $\mathsf{G}_{\sigma,m_{i+1},t}$ is incomplete, σ will attempt (as directed in State 3 of the basic strategy or part 4 of the Sacks coding directions) to choose another element for $\mathsf{G}_{\sigma,m_{i+1},t}$; it will be prevented from doing so only if a module of stronger privilege chooses a witness instead (as in part 3 of the end-of-stage directions). Further, the witness set $\mathsf{G}_{\sigma,m_{i+1},t}$ can be canceled only by enumerations from stronger-privileged modules; this follows as in the proof of Lemma 18 and by the fact that, if σ is a noncontiguous bounding node, none of its modules is ever again canceled by the entry of a lower-numbered module into State 6. It follows from Lemma 18 that, eventually, the stronger-privileged modules both stop choosing and stop enumerating witnesses; and after that, the witness set $\mathsf{G}_{\sigma,m_{i+1},t}$ will eventually be filled, and the claim follows.

Having reached such a stage t_i'', it is now easy to find a stage t_{i+1} after which the m_{i+1}th module's witness set $\mathsf{G}_{\sigma,m_{i+1},t_{i+1}}$ is complete and is never again canceled. For define u_1 as

$$\max\left(\mathsf{G}_{\sigma,m_{i+1},t_i''} \bigcup \left\{U(\delta, \mathtt{thr}(\sigma, \delta, m_{i+1}, t_i''), t_i'') : \delta^\frown 0 \subset \sigma\right\}\right).$$

If $A_{t_i''} \upharpoonright u_1 = A \upharpoonright u_1$, then the witness set $\mathsf{G}_{\sigma,m_{i+1},t_i''}$ can never be canceled, as such cancellation requires either an enumeration into A below $\max \mathsf{G}_{\sigma,m_{i+1},t_i''}$—hence below u_1—or an increase in $U(\delta, \mathtt{thr}(\sigma, \delta, m_{i+1}, t_i''), t_i'')$ for some δ with $\delta^\frown 0 \subset \sigma$, which again requires an enumeration below u_1.

If $A_{t_i''} \upharpoonright u_1 \neq A \upharpoonright u_1$, then wait until a stage s at which $A_s \upharpoonright u_1 = A \upharpoonright u_1$, and then find the first σ-stage $t_i''' > s$ such that the witness set $\mathsf{G}_{\sigma,m_{i+1},t_i'''}$ is complete. Again we claim that such a stage must exist. We define u_2 as

$$\max\left(\mathsf{G}_{\sigma,m_{i+1},t_i'''} \bigcup \left\{U(\delta, \mathtt{thr}(\sigma, \delta, m_{i+1}, t_i'''), t_i''') : \delta^\frown 0 \subset \sigma\right\}\right).$$

Again, if $A_{t_i'''} \upharpoonright u_2 = A \upharpoonright u_2$, then the witness set $\mathsf{G}_{\sigma,m_{i+1},t_i'''}$ can never be canceled, so we set $t_{i+1} = t_i'''$; and if $A_{t_i'''} \upharpoonright u_2 \neq A \upharpoonright u_2$, then we wait until a stage s at which $A_s \upharpoonright u_2 = A \upharpoonright u_2$, and then find the first σ-stage $t_i'''' > s$ such that the witness set $\mathsf{G}_{\sigma,m_{i+1},t_i''''}$ is complete. We continue until, for some u_j and $t_i^{(j+1)}$, $A_{t_i^{(j+1)}} \upharpoonright u_j = A \upharpoonright u_j$.

Note that the values $u_1, u_2, \ldots$ cannot keep increasing forever. Any increase from u_j to u_{j+1} implies either a cancellation of $\mathsf{G}_{\sigma,m_{i+1},t_i^{(j+1)}}$ or an injury, for some δ with $\delta^\frown 0 \subset \sigma$, to some δ-computation below $\mathtt{thr}(\sigma, \delta, m_{i+1}, t_i^{(j+1)})$. Either of these cases requires an enumeration into A of some witness of stronger privilege than k_{i+1}. But by Lemma 18, such enumerations must stop. ⊣

We are finally ready to prove, simultaneously, the existence of the true path and the success of the nodes that lie along it.

LEMMA 21. *For all* $m\ (= \langle e, i\rangle) \in \omega$,

(i) For $j \in \{0, 1, 2\}$, $\liminf_s (f_s \upharpoonright (4m + j))$ *exists.*

(ii) Let $\alpha = \liminf_s (f_s \upharpoonright 4\langle e, i\rangle)$. *Then* α *meets* $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$. *Further, only finitely many witnesses for* α *are enumerated into* A, *and* $\lim_s r(\alpha, s) < \infty$.

(iii) Let $\sigma = \liminf_s (f_s \upharpoonright (4m+1))$. *Then* σ *meets* $\mathcal{N}_{\mathbf{m}}$.

(iv) Let $\gamma = \liminf_s (f_s \upharpoonright (4m+2))$. *Then* γ *meets* $\mathcal{P}_{\mathbf{m}}$*; the lim inf of the outcomes of* γ *exists; and if* p *is this lim inf, then there are only finitely many stages at which any nodes extending* $\gamma\widehat{\ }p$ *are initialized on behalf of* γ.

(v) Let $\delta = \liminf_s (f_s \upharpoonright (4m+3))$. *Then* δ *meets* $\mathcal{R}_{\mathbf{m}}$, *and* $\lim_s r(\sigma, s) < \infty$.

Proof. By induction on m.

(i) By the induction hypothesis and the fact that nodes of length $4m-1$, $4m$, and $4m+1$ have no outcomes other than 0 and 1.

(ii) We assume that $\widehat{\Phi}_i(Q_{\pi(\alpha)}) = A$. Otherwise $\limsup_s l(\widehat{\Phi}_i(Q_{\pi(\alpha)}), A, s) < \infty$, since $\widehat{\Phi_i}$ is a *wtt*-reduction; then the result is immediate, as only finitely many modules are started.

By the induction hypothesis, wait until a stage s_0 after which α never gets initialized again. Let the challenge numbers picked after this stage be v_1, v_2, and so on; these challenge numbers can never again be canceled by initialization of α (though any v_n can be canceled if some mth module, $m < n$, reaches State 6). It follows as in the proof of Lemma 20 that each module eventually acquires a complete witness set, so each module manages to reach State 4; but then by the assumption that $\widehat{\Phi}_i(Q_{\pi(\alpha)}) = A$, each module also reaches State 5. For all n, let s_n be the stage at which the nth module reaches State 5, having previously been in State 4.

If we have $V_{e,s_n} \upharpoonright v_n + 1 = V_e \upharpoonright v_n + 1$ for cofinitely many n, then V is computable. In this case only finitely many modules put up a State 5 restraint or enumerate anything into A, so we are done.

Thus, assume that for infinitely many n we have $V_{e,s_n} \upharpoonright v_n + 1 \neq V_e \upharpoonright v_n + 1$. Note that, if α is still in State 5 when V_e changes below $v_n + 1$, α throws up a restraint and cancels witnesses for nodes extending α. Then the nth module's witness set can *never* be canceled—since such a cancellation would have to come from the enumeration of a smaller witness for a stronger-privileged module, which (by assumption that α is never initialized again) would have to be working for a node extending α. However, it is possible that the witness set could be canceled *before* the change came, so that α's nth module would not be in State 5 and thus would not be ready for the change. Of course the witness set for α's module may stop being canceled eventually; but perhaps $V_e \upharpoonright v_n + 1$ will have settled down by then.

Call the nth module *disappointed* in the latter situation. We will consider two cases: cofinitely many of α's modules *are* disappointed, and infinitely many of them are *not* disappointed.

If infinitely many of α's modules are *not* disappointed, then infinitely many modules reach State 6 to place at least one witness in the permitting bin and wait for permission from h. Note that, if such a permission comes, then the requirement is won unless V_e changes below v_n+1 again, in which case the module returns to State 6 to put something else in the permitting bin. Since (as noted above) the witness set can never be canceled once the module reaches State 6, the only thing that could prevent α from winning the requirement for good would be if h did not provide enough permissions at the right times.

Stepping back for a moment, we consider a function g_α such that for every n, $g_\alpha(n)$ bounds the number of permissions needed for witnesses with permitting number n that α places into the permitting bin after stage s_0. In particular we will consider the natural computable approximation $g_\alpha(n, s)$ to $g_\alpha(n)$ (the approximation that increases at each stage when α puts something into the permitting bin).

First, we investigate $g_\alpha(1)$. Let n_1 be such that the n_1th module is the first to reach State 6, and let t_1 be the stage at which it reaches State 6. Then we have $g_\alpha(1, t_1) = v_{n_1} + 1$, where v_{n_1} is the n_1th challenge number. Now, the placement of any witness in the permitting bin corresponds to a change in V_e from State 5. If V_e changes below $v_{(n_1-1)}$, then some n'_1th module takes over, where $n'_1 < n_1$ is least such that $V_e \upharpoonright n'_1 + 1$ changed, and *its* witnesses get permitting number 1. If this happens, then only $v_{n_1} - v_{n'_1}$ witnesses can be enumerated (and $v_{n_1} - v_{n'_1}$ permissions used) before the n'_1th module usurps the n_1th module's place. Similarly, if some n''_1th module later usurps the n'_1th module's place then only $v_{n'_1} - v_{n''_1}$ permissions can have been used in the meantime.

The result is that, no matter how many times we change our minds about *which* module's witnesses have permitting number 1, the total number of permissions needed by *all* those witnesses is no more than $v_{n_1} + 1$. Thus, once $g_\alpha(1, t_1) = v_{n_1} + 1$ is set, we do not need $g_\alpha(1, t) \neq g_\alpha(1, t_1)$ for *any* $t > t_1$. Thus the approximation to $g_\alpha(1)$ changes only once—from 0 to $v_{n_1} + 1$.

In general, the challenge number $v_{n_{i+1}}$ will be canceled only if V_e changes below $v_{n_{i+1}-1}$ at some stage t. This V_e-change will either cause some n'_{i+1}th module to usurp the n_{i+1}th module's place (in which case, as above, we do not have to make $g_\alpha(i+1, t) \neq g_\alpha(i+1, t_i)$), or it will be low enough to cancel the n_{i+1}th module by requiring action for a module whose witnesses have permitting number less than or equal to i. In the latter case we *would* eventually have to increase $g_\alpha(i+1, t')$ at some stage $t' > t$, when some other module reached State 6 and took permitting number $i+1$ for its witnesses. But this can happen no more than $g_\alpha(i)$ times, where $g_\alpha(i) = \lim_s g_\alpha(i, s)$.

Thus for all n,

$$|\{s : g_\alpha(n, s+1) \neq g_\alpha(n, s)\}| \leq g_\alpha(n-1).$$

But then g_α is in the collection $\mathcal{F}$ described on page 17.

Now, suppose we create an auxiliary set W_α that gets a number enumerated into its nth column whenever α puts a witness with permitting number n into the permitting bin. Then W_α is in $\mathcal{V}$. And if we define a function $\tilde{g}$ so that $\tilde{g}(n)$ is the last stage at which something goes into $W_\alpha^{[n]}$, then $\tilde{g} \in \mathcal{G}$. Then h is not dominated by $\tilde{g}$, so we reason as in Downey and Shore [1996]. Specifically, there is an $n > s_0$ such that $h(n) > \tilde{g}(n)$; say $h(n) = s$. Then $h(n, s) \neq h(n, s-1)$ by a convention on page 17. But then, by choice of $\tilde{g}$, *all* of α's witnesses with permitting number n have been put into the permitting bin by stage $\tilde{g}(n)$, thus before stage s; so the permission at stage s meets the requirement. After that, α never acts again, so it quits enumerating witnesses, initializing other nodes, and increasing its restraint.

Thus, there is a happy ending in the case where infinitely many of α's modules are *not* disappointed—to wit, the requirement is won and only finitely many modules are started after all! So suppose that cofinitely many of α's modules *are* disappointed. In particular, since none of those modules reaches State 6, none of them is ever again canceled. We claim that

CLAIM 22. *For any i such that the ith module is disappointed, the process of finding a stage t_i such that $V_{e,t_i} \restriction v_i = V_e \restriction v_i$ is A-computable uniformly in i.*

Proof of Claim. The proof is similar to that of Lemma 20. For the induction step, note that since the ith module and challenge number are never canceled, neither is the associated privilege number k_i. By Lemma 20 (which applies since only finitely many of α's modules enter State 6) we can find, uniformly in i, a stage t_i at which the ith module's witness set is complete and after which the witness set is never canceled. Since the ith module is disappointed, we see that, at the next stage t_i^* at which the ith module is in State 5, $V_{e,t_i^*} \restriction v_i = V_e \restriction v_i$, and we set $t_{i+1} = t_i^*$.

Since $V_e \leq_{\mathrm{T}} A$ by the Claim, the requirement $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ is met. Further, no disappointed module ever enumerates anything into A or increases α's restraint. Non-disappointed modules do so only finitely often, by Lemma 19. This completes the proof of part (ii).

(iii) Suppose that $\Psi_\sigma(A) = W_\sigma$ and $\Phi_\sigma(W_\sigma) = A$ (otherwise there is nothing to prove). We will show that $W_\sigma \equiv_{wtt} A$. First, by the induction hypothesis and parts (i) and (ii), let s_0 be a stage such that σ is active at stage s_0 and is never initialized after stage s_0.

($W_\sigma \leq_{wtt} A$:) Given x, let m_x be the least number such that, for some $s \geq s_0$, $u(\Phi_\sigma, W_\sigma, m_x, s) > x$. Let s_x be the first stage after s_0 such that $L(\sigma, s_x) \geq m_x$, and let $u = u(\Psi_\sigma, A, x, s_x)$. We claim that $A \upharpoonright u$ is all that will be needed to compute $W_\sigma(x)$.

If $x \notin W_\sigma$ or $x \in W_{\sigma,s_x}$, this is immediate. So suppose that $x \in \left(W_\sigma - W_{\sigma,s_x}\right)$. Since $W_\sigma = \Psi_\sigma(A)$ and $L(\sigma, s_x) \geq m_x$, it is clear that $A \upharpoonright u \neq A_{s_x} \upharpoonright u$, so let y be the *last* element enumerated into $A \upharpoonright u$, and let s_2 be the first σ-stage after it is enumerated. We claim that $A_{s_2} \upharpoonright u(\Psi_\sigma, A, x, s_2) = A \upharpoonright u(\Psi_\sigma, A, x, s_2)$, so that $x \in W_\sigma$ if and only if $x \in W_{\sigma,s_2}$.

Now, y was part of some module with privilege, say, k, and the module's σ-threshold must have been less than or equal to m_x. No witnesses less than y are enumerated after s_2 by choice of y. All witnesses greater than y *and* of privilege $k' > k$ are canceled when y is enumerated. All witnesses less than $u(\Psi_\sigma, A, x, s_2)$ and of σ-threshold greater than m_x are canceled by σ at s_2. Any witnesses chosen at or after stage s_2 will be chosen too large to injure the relevant computation for x. Thus the only witnesses that could possibly injure the computation are those that have been chosen by stage s_2; are greater than y; have stronger privilege than y; *and* have threshold less than or equal to m_x.

Suppose such a witness z is enumerated into A after s_2, injuring the computation. Let σ_z and σ_y be the nodes for which z and y are witnesses (clearly these nodes are different, since witnesses for the same module are enumerated in decreasing order and enumeration from one module cancels witnesses for higher-numbered modules of the same node). Since $z > y$, y was chosen before z. Since the choice of z did not initialize y, and the enumeration of y did not initialize z, neither of σ_y and σ_z is to the left of the other; so one extends the other.

If $\sigma_y \subset \sigma_z$, then σ_y is not a noncontiguity node, since in that case the enumeration of y would have initialized σ_z (canceling the privilege for z's module). So σ_y is a Sacks coding node, and y is the marker for some number p. Since the enumeration of y did not initialize the module for z, it must be that $\sigma_y\widehat{\ }q \subset \sigma_z$ for some $q < p$. But then z was chosen as a witness when the outcome of σ was q, and at that stage no number greater than q had a marker; thus y was chosen after z was, contradicting $y < z$.

Thus $\sigma_z \subset \sigma_y$, so that σ_z was active at the stage s_y at which y was chosen as a witness. Then z's module was active at that stage also, and since it has stronger privilege than y's module, z's module must have had a complete witness set $\mathsf{G}_{\sigma_z,n,s_y}$ at stage s_y; and z wasn't in it. Thus, this other witness set must have been canceled sometime after s_y. If this cancellation came from an initialization of σ_z, then σ_y would have been initialized also; if it came from an increase in restraint by a $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$- or $\mathcal{P}_{\mathbf{e}}$-node above σ_z, then σ_y's witness set would have been canceled as will. And if it came about because of some contiguity node δ such that $\delta\hat{\ }0 \subset \sigma_z$ then, since y's module has weaker privilege (and thus a higher δ-threshold) than z's, again y would have been canceled.

So there is no such z, and $A_{s_2} \restriction u(\Psi_\sigma, A, x, s_2) = A \restriction u(\Psi_\sigma, A, x, s_2)$, as desired.

($A \leq_{wtt} W_\sigma$.) For $x \in \omega$, let s_x be the first $\sigma\hat{\ }0$-stage after s_0 after x is chosen as a witness, and let $u_x = u(\Phi_\sigma, W_\sigma, x, s_x)$. We claim that $W_\sigma \restriction u_x$ is all the information needed to tell whether $x \in A$.

Since positive nodes above σ stop enumerating witnesses (by the induction hypothesis and part (i)), we may assume that x is a witness for a node $\sigma_x \supset \sigma$. Then $L(\sigma, s_x) > x$ since σ does not have outcome 0 unless its length of agreement is larger than all witnesses for nodes extending σ. If $W_{\sigma,s_x} \restriction u_x = W_\sigma \restriction u_x$ then we are done. Suppose $W_{\sigma,s_x} \restriction u_x \neq W_\sigma \restriction u_x$; then $A_{s_x} \restriction u(\Psi_\sigma, A, u_x, s_x) \neq A \restriction u(\Psi_\sigma, A, u_x, s_x)$. Let $y < u(\Psi_\sigma, A, u_x, s_x)$ be a number that enters A after stage s_x; note that once W_σ changes, σ does not have its next expansionary stage until after y is enumerated.

Note that, since $y < u(\Psi_\sigma, A, u_x, s_x)$, y was chosen as a witness before stage s_x, and y is a witness for a node $\sigma_y \supset \sigma\hat{\ }0$. Then $y < x$ since $\sigma\hat{\ }0$ is not active between the stage at which x is picked and the stage s_x. Then x was chosen as a witness after y, and since the choice of x did not cancel y, $\sigma_x \not<_{\mathrm{L}} \sigma_y$. Now, suppose that the enumeration of y does not cancel x (for if it does, then x is canceled by the first $\sigma\hat{\ }0$-stage after W_σ changes, and we're done). The argument from here proceeds *precisely* as in the $W_\sigma \leq_{wtt} A$ proof, with x substituted for z. The contradiction reached then proves that there is no such y; or rather, that if W_σ changes below u_x and any $y < u(\Psi_\sigma, A, u_x, s_x)$ enters A after s_x, then x is canceled by the next $\sigma\hat{\ }0$ stage after W_σ changes.

Thus if W_σ changes below u_x, *either* the corresponding A-change happens *below* x—so that x is canceled by the next $\sigma\hat{\ }0$-stage—or the corresponding A-change *is* x (and in this case, again, we will know it by the time the next $\sigma\hat{\ }0$-stage rolls around).

(iv) To show that γ meets $\mathscr{P}_{\mathbf{e}}$, we assume that γ fails, that is, $A = \Phi_e(C)$. We will show that $D \leq_{\mathrm{T}} A \oplus C = C$, contrary to hypothesis. By the induction hypothesis and parts (i) and (ii) above, let s_0 be a stage such that γ is not initialized at any stage $s \geq s_0$.

Given $x \notin D_{s_0}$, we describe how to tell, using $A \oplus C$ as an oracle, whether $x \in D$. Wait until the first γ-stage $s_1 \geq s_0$ such that $l(A, \Phi_e(C), s_1) > x$ and $C_{s_1} \upharpoonright u(\Phi_e, C, x, s_1) = C \upharpoonright u(\Phi_e, C, x, s_1)$. By Case 3 of the construction, at stage s_1 x will be put into a module $\mathsf{H}_{\gamma,n,s_1}$ with some privilege number k, if this has not been done already. As in the proof of Lemma 20, there will be a stage $s_2 > s_1$ at which the witness set $\mathsf{G}_{\gamma,n,s_2}$ will be complete; in particular there will be a marker $\mu_{\gamma,s_2}(x)$ for x.

Now, perhaps $\mu_{\gamma,s_2}(x) = \mu_{\gamma,t}(x)$ for all $t \geq s_2$; if not, then we claim that we can find, computably in A, a stage s_2' such that $\mu_{\gamma,t}(x) = \mu_{\gamma,s_2'}(x)$ for all $t \geq s_2'$. For by Lemma 20 we can find, computably in A, a stage t_n after which the witness set $\mathsf{G}_{\gamma,n,t_n}$ is complete and never canceled. Then we can also tell, computably in A, whether any $\mu_{\gamma,z}(t)$, $z \in \mathsf{H}_{\gamma,n,s_1}$, $z < x$, enters A after t_n. If no such number enters, go to the next paragraph; if *so*, then at the next γ-stage, x will be put into a new module that contains even fewer z's such that $z < x$, and we repeat the previous and the present paragraph and argue *a fortiori* for this new module.

Thus, let s_3 be the first γ-stage after these enumerations are finished. Then $\mu_{\gamma,s_3}(x) = \mu_{\gamma,t}(x)$ for all $t \geq s_3$. Then by Step 5 of the end-of-stage portion of the construction, $x \in D - D_{s_3}$ if and only if $\mu_{\gamma,s_3}(x) \in A$. Since x was arbitrary, $D \leq_{\mathrm{T}} A \oplus C = C$, contrary to assumption.

So $A \neq \Phi_e(C)$. Thus there is some least number p such that $A(p) \neq \Phi_e(C; p)$, so by definition of the outcomes of γ, the lim inf of these outcomes is p (as the stages t at which $l(A, \Phi_e(C), t) = p$ are shuffled in with the γ-stages forever).

Finally, note that there is a stage s_p by which $D_{s_p} \upharpoonright p = D \upharpoonright p$, so that at no stage $t \geq s_p$ does any $\mu_{\gamma,t}(z)$, for $z \leq p$, get enumerated into A. Since these are the only γ-related enumerations that can initialize nodes extending $\gamma\widehat{\ }p$ (as in part 5 of the end-of-stage directions), we see that such initialization has to stop.

(v) We show that if $K = \Phi_\delta(A \oplus C)$, then $K \leq_{\mathrm{T}} C$, contrary to choice of C. By the induction hypothesis and parts (i)—(iv), wait until a stage s_0 such that

- all nodes to the left of or above δ never again enumerate anything into A or increase their restraint after stage s_0;

- the approximation to the true path, f_t, never moves to the left of δ for $t > s_0$; and
- δ is active at s_0.

Suppose $K = \Phi_\delta(A \oplus C)$. Then there are infinitely many δ-expansionary stages. So, given x, wait until the first δ-expansionary stage $s_1 > s_0$ such that $l(\Phi_\delta(A \oplus C), K, s_1) > x$. At stage s_1, $r(\delta, s_1)$ is set equal to

$$u(\Phi_\delta, A \oplus C, l(\Phi_\delta(A \oplus C), K, s_1), s_1) + 1;$$

indeed, at every δ-stage s_n after s_1, $r(\delta, s_n)$ is set equal to or greater than

$$u(\Phi_\delta, A \oplus C, l(\Phi_\delta(A \oplus C), K, s_n), s_n) + 1.$$

By the induction hypothesis and the choice of s_0, after such a stage s_n nothing is ever enumerated into A below this restraint $r(\delta, s_n)$; so either $x \in K_{s_1}$, or the later enumeration of x into K (if it occurs at all) is recorded in C below some $u(\Phi_\delta(A \oplus C), C, l(\Phi_\delta(A \oplus C), K, s_n), s_n) + 1$. The number x was arbitrary, so $K \leq_T C$.

Thus, $K \neq \Phi_\delta(A \oplus C)$. Now let p be the least number such that $K(p) \neq \Phi_\delta(A \oplus C; p)$, and let s_2 be a stage such that for all $y < p$ and all $t \geq s_2$, $K_t(y) = \Phi_{\delta,t}(A \oplus C_t; y)$.

Suppose $p \in K$, and let s_2' be such that $p \in K_{s_2'}$. If $\Phi_{\delta,t}(A \oplus C_t; p)\uparrow$ for all $t \geq s_2'$, then δ never again imposes restraint. So suppose $\Phi_{\delta,t}(A \oplus C_t; p)\downarrow$ for some $t \geq s_2'$. Then δ does not increase its restraint at any stage t unless $\Phi_{\delta,t}(A \oplus C_t; p) \downarrow\neq 1$ or t is δ-expansionary. If t is δ-expansionary, then $l(\Phi_\delta(A \oplus C), K, t) > p$, so $\Phi_{\delta,t}(A \oplus C_t; p) = K_t(p) = 1$; but then the restraint preserves the $\Phi_\delta(A \oplus C)$-computation forever, so that $\Phi_\delta(A \oplus C; p) = K(p) = 1$, contradicting the choice of p. So if δ ever increases its restraint again at some $s_3 > s_2'$, it must be true that $\Phi_{\delta,s_1}(A \oplus C_{s_3}; p) \downarrow\neq 1$. At stage s_3, δ restrains $A \oplus C$ below $u(\Phi_\delta, A \oplus C, p, s_3)$, and since this restraint is respected, $\Phi_{\delta,t}(A \oplus C_t; p) \neq 1 = K_t(p)$ for all $t \geq s_3$, and δ never again increases its restraint.

If $p \notin K$, on the other hand, then there are no δ-expansionary stages after s_2, for at such a stage s_3 we would clearly have $\Phi_{\delta,s_3}(A \oplus C_{s_3}; p) = 0 = K_{s_3}(p)$, and the restraint imposed by δ at s_3 would ensure that $\Phi_\delta(A \oplus C; p) = 0 = K(p)$, contradicting the choice of p. ⊣

This completes the proof of Theorem 6.

§5. Notes on Theorem 9. We describe briefly how the maximal contiguity construction can be altered to prove Theorem 9, that is, to construct a noncontiguous array computable degree with no contiguous degrees above it. We refer the reader to Downey, Jockusch, and Stob [1990] for background on array (non)computable sets and degrees.

First, we fix a very strong array $\mathcal{F} = \{F_i\}_{i\in\omega}$; the array

$$\mathcal{F} = \{\{0\}, \{1, 2\}, \{3, 4, 5\}, \ldots\}$$

works nicely. We wish to build A so that no $B_e \leq_{\mathrm{T}} A$ is $\mathcal{F}$-array noncomputable; that is, so that for all Φ_e and B_e, there is a c.e. set U_e such that

$$\mathcal{N}_{\mathbf{e}} : \Phi_e(A) = B_e \Longrightarrow (\text{a.e. } n)\ \mathcal{N}_{\mathbf{e},\mathbf{n}},$$

where $\mathcal{N}_{\mathbf{e},\mathbf{n}}$ is

$$\mathcal{N}_{\mathbf{e},\mathbf{n}} : [U_e \cap F_n \neq B_e \cap F_n]\,.$$

Once $l(\Phi_e(A), B_e, s)$ is greater than max F_n, we start trying to meet $\mathcal{N}_{\mathbf{e},\mathbf{n}}$. The basic strategy, as in Downey, Jockusch, and Stob [1990], is as follows: If $B_e \cap F_n \neq \emptyset$, keep $U_e \cap F_n$ empty; the strategy is won. Otherwise, put something into $U_e \cap F_n$ so that $U_e \cap F_n$ has more elements than $B_e \cap F_n$, and restrain A below $u(\Phi_e, A, \max F_n, s)$ to try to keep elements out of $B_e \cap F_n$. Of course, higher-priority positive requirements may enumerate things into A and injure this restraint; but we will eventually win if we can put another element in $U_e \cap F_n$ *every* time the $\Phi_e(A; \max F_n)$ computation recovers from an injury. Clearly this will work if the number of injuries is less than $|F_n|$.

For the Theorem 9 construction the requirements are the $\mathcal{N}_{\mathbf{e}}$ and $\mathcal{N}_{\mathbf{e},\mathbf{i}}$ as above; the $\mathcal{Q}_{\mathbf{e}}$ and $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ from the maximal contiguous construction; and requirements to make deg(A) noncontiguous, that is,

$$\mathcal{P}_{\mathbf{e}} : A \neq \widehat{\Phi}_e(Q),$$

where Q is a kick set of A constructed the same way the Q_e's are constructed for the requirements $\mathcal{Q}_{\mathbf{e}}$. Each $\mathcal{P}_{\mathbf{e}}$ requires only two witnesses per witness set; the strategy is as described on page 7, and now we do not have to worry about complications from any V_e.

The construction is, overall, much simpler than that for our main theorem, and even simpler than the basic maximal contiguous construction described in Section 2; but then it should come as no surprise that array computability is easier to achieve than contiguity, since contiguity is a stronger condition. In particular, for none of our positive requirements (the $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ or the $\mathcal{P}_{\mathbf{e}}$) do the witnesses have to be chosen one by one; the numbers in a witness set can be chosen all at once. Also, the enumerations do not have to be done in reverse order; what is important for array computability is the *number* of enumerations.

The construction happens on a subtree of 2^ω (we do not have Sacks coding nodes to require outcomes other than 0 and 1). There are three types of nodes. The basic strategy for $\mathcal{Q}_{\mathbf{e},\mathbf{i}}$ (for the nth module of some node α) is much as before. In State 1 we choose a challenge number, and let $\texttt{priv}(\alpha, n, s)$ be the least number still on the privilege list, as in the maximal contiguous construction. Thresholds are handled thus: For each array computability node δ such that $\delta^\frown 0 \subset \alpha$, we would like $\texttt{thr}(\alpha, \delta, n, s)$ to be the least m such that F_m can absorb all the enumerations from this nth module *and* from stronger-privileged modules. Let j be the highest value of $\texttt{thr}(\sigma, \delta, i, t)$ set so far for any σ, i, and t. We will set $\texttt{thr}(\alpha, \delta, n, s)$ to be the least m such that

$$|F_m| > \left|F_j\right| + \prod \left(\left|\mathsf{G}\right|\right),$$

where the product ranges over all witness sets G that are currently active for positive nodes extending α. The $|F_j|$ is the number of possible enumerations from stronger-privilege modules; the product is the total number of enumerations this nth module may have to make, as a result of its witnesses being canceled because of enumerations by stronger-privilege modules below it. (Enumerations above it will initialize the entire node and thus do not have to be counted in the product.)

The only other change to the basic strategy is that when a witness set is needed, it can be chosen immediately (not at the end of a stage) and all at once.

The noncontiguity requirements $\mathcal{P}_{\mathbf{e}}$ use the strategy sketched on page 7, as noted above.

The array computability requirements $\mathcal{N}_{\mathbf{e}}$ work the same way the contiguity requirements worked in the construction for the main theorem. The condition for canceling witness sets is now

$$\min \mathsf{G}_{\alpha,n,s} < u(\Phi_e, A, \max F_m, s),$$

where $m = \texttt{thr}(\alpha, \beta, n, s)$; and the condition for acknowledging an expansionary stage has nothing to do with a function $\nu(\beta, s)$ but with the length of agreement $l(\Phi_e(A), B_e, s)$ and, in particular, whether

$$l(\Phi_e(A), B_e, s) > \max F_n > l(\Phi_e(A), B_e, t) \text{ for all } t < s.$$

Most of the end-of-stage directions can now be ignored as irrelevant; only one addition needs to be made, in that the "Kick set maintenance" step has to be set up to maintain the kick set Q in addition to all of the other kick sets Q_e. With these modifications, the construction yields the desired degree, proving Theorem 9.

§6. On a question of Li (Joint work with Andre Nies). Finally we note that, with Andre Nies, we were able to answer a question of Li's (see Slaman [1993, 3.10(a)]): "Given the language $L \subseteq \{0, 1, \cup, \cap\}$ and a property P in L. A *neighborhood* of $\mathbf{a}$ is an interval $[\mathbf{c}, \mathbf{b}]$ with $\mathbf{c} < \mathbf{a} < \mathbf{b}$. Property P is an *isolated* property of the recursively enumerable degree $\mathbf{a}$ if there is a neighborhood of $\mathbf{a}$ such that $\mathbf{a}$ is the unique element of the interval which satisfies P. Is there an isolated property in $\mathcal{R}$?"

There are several ways to understand (or properly phase) this question:

1. Are there a 0-definable Θ, an interval $[\mathbf{b}, \mathbf{c}]$, and a unique $\mathbf{a}$ such that $\mathbf{b} < \mathbf{a} < \mathbf{c}$ and $\mathcal{R}$ realizes $\Theta(\mathbf{a})$?
2. Are there a 0-definable Θ, an interval $[\mathbf{b}, \mathbf{c}]$, and a unique $\mathbf{a}$ such that $\mathbf{b} < \mathbf{a} < \mathbf{c}$ and $[\mathbf{b}, \mathbf{c}]$ realizes $\Theta(\mathbf{a})$?
3. Are there a Θ (with or without parameters), an interval $[\mathbf{b}, \mathbf{c}]$, and a unique $\mathbf{a}$ such that $\mathbf{b} < \mathbf{a} < \mathbf{c}$ and $[\mathbf{b}, \mathbf{c}]$ realizes $\Theta(\mathbf{a})$?

A positive answer to 1 (2) implies a positive answer to 2 (3), while a positive answer to 2 (3) says nothing about a positive answer to 1 (2).

Cooper [1996] answers 2: $\mathbf{a}$ is a *nonsplitting base* in $[\mathbf{0}, \mathbf{b}]$ iff $\mathbf{a} < \mathbf{b}$ and for all $\mathbf{c}$ and $\mathbf{d}$, if $\mathbf{c} \cup \mathbf{d} = \mathbf{b}$ then either $\mathbf{c} \cup \mathbf{a} = \mathbf{b}$ or $\mathbf{d} \cup \mathbf{a} = \mathbf{b}$. Cooper shows there is a $\mathbf{b}$ such that $[\mathbf{0}, \mathbf{b}]$ has a least nonsplitting base.

Downey also has an unpublished positive answer to the second version of Li's question: Let $\mathbf{a_0}$ and $\mathbf{a_1}$ be computably enumerable degrees such that $\mathbf{a_0}$, $\mathbf{a_1}$, and $\mathbf{a_0} \cup \mathbf{a_1}$ are contiguous, $\mathbf{a_0}$ does not bound half of a minimal pair, and every computably enumerable degree below $\mathbf{a_1}$ is the join of a minimal pair. Both $\mathbf{a_0}$ and $\mathbf{a_1}$ are definable in the cone below $\mathbf{a_0} \cup \mathbf{a_1}$. For example, $\mathbf{a_0}$ is the greatest degree $\mathbf{d}$ below $\mathbf{a_0} \cup \mathbf{a_1}$ such that $\mathbf{d}$ does not bound a half of a minimal pair.

The following theorems allow us, jointly with Andre Nies, to provide a positive answer to the first version of Li's question.

THEOREM 23 (with Andre Nies). *Let $\mathbf{d}$ and $\mathbf{b_1}$ be noncomputable degrees such that $\mathbf{d}$ is not low$_2$. Then there are an $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b}$ is a maximal contiguous degree, $\mathbf{a} < \mathbf{a} \cup \mathbf{b}$, $\mathbf{a} < \mathbf{d}$, $\mathbf{b} \leq \mathbf{b_1}$, and for all $\mathbf{v}$, if $\mathbf{a} \cup \mathbf{v}$ is contiguous then $\mathbf{a} \cup \mathbf{b} \geq \mathbf{v}$ (all the degrees here are computably enumerable).*

PROOF SKETCH. We modify the basic construction of a maximal contiguous degree (Theorem 6) as follows. Rather than making $\mathbf{a}$ a maximal contiguous degree we will make $\mathbf{a} \cup \mathbf{b}$ a maximal contiguous degree. Balls enter $A \in \mathbf{a}$ only with permission from $\mathbf{d}$ (as in Theorem 6). We must add nodes to force $A \not\equiv_T A \oplus B$ (where B is in $\mathbf{b}$). To act, these nodes need only one permission from $\mathbf{b_1}$ and to restrict an initial segment of A. So they interact without a problem with the nodes working to make $\mathbf{a} \cup \mathbf{b}$ a maximal contiguous degree. To make $\mathbf{a} \cup \mathbf{v}$ noncontiguous we use the same strategy employed in the basic construction; if we fail to make $\mathbf{a} \cup \mathbf{v}$ noncontiguous then $\mathbf{a} \cup \mathbf{b} \geq \mathbf{v}$. ⊣

Now $\mathbf{a} \cup \mathbf{b}$ is maximal contiguous, and every contiguous degree in $[\mathbf{a}, \mathbf{0}']$ is below $\mathbf{a} \cup \mathbf{b}$. Then $\mathbf{a} \cup \mathbf{b}$ is the only maximal contiguous degree in $[\mathbf{a}, \mathbf{0}']$, so that

COROLLARY 24. *Maximal contiguity is an isolated property (as in the first version of isolated).*

Another isolated property is evident when we apply Theorem 23 to the degrees $\mathbf{d}$ and $\mathbf{b_1}$ from the following theorems:

THEOREM 25 (Downey, Lempp, and Shore [1993]). *There is a high$_2$ nonbounding degree* $\mathbf{d}$*, that is, a degree below which there are no minimal pairs.*

THEOREM 26 (Downey, Lempp, and Shore [1993]). *There is high superbounding degree* $\mathbf{b_1}$*, that is, a degree below which every degree is the join of a minimal pair.*

Notice that the classes of nonbounding and superbounding degrees are closed downwards. Thus $\mathbf{a}$ is nonbounding and $\mathbf{b}$ is superbounding, both of which are definable properties. In addition, for all $\mathbf{v}$, if $\mathbf{a} \cup \mathbf{v}$ is contiguous then $\mathbf{a} \cup \mathbf{b} \geq \mathbf{v}$. Let $\Theta(\mathbf{x})$ be the formula

> "$\mathbf{x}$ is nonbounding and there is a superbounding $\mathbf{b}$ such that $\mathbf{x} \cup \mathbf{b}$ is contiguous and for all $\mathbf{v}$, if $\mathbf{x} \cup \mathbf{v}$ is contiguous then $\mathbf{x} \cup \mathbf{b} \geq \mathbf{v}$."

Now Θ is 0-definable and $\Theta(\mathbf{a})$ holds (in $\mathcal{R}$) for some $\mathbf{a}$ with its corresponding $\mathbf{b}$. Let $\mathbf{a}^*$ be any degree in $[\mathbf{0}, \mathbf{a} \cup \mathbf{b}]$ such that $\Theta(\mathbf{a}^*)$ holds (with a corresponding $\mathbf{b}^*$—note it need not be that $\mathbf{b}^* \leq \mathbf{a} \cup \mathbf{b}$). Since $\mathbf{a} \cup \mathbf{b}$ is contiguous and $\mathbf{a}^*$ is nonbounding, $\mathbf{a}^* \leq \mathbf{a}$ (otherwise, by local distributivity of $\mathbf{a} \cup \mathbf{b}$, $\mathbf{a}^*$ is the join of a nonbounding degree and a superbounding degree and hence *not* nonbounding). Let $\mathbf{v} = \mathbf{a} \cup \mathbf{b}$. By the last clause of $\Theta(\mathbf{a}^*)$, $\mathbf{a}^* \cup \mathbf{b}^* \geq \mathbf{a} \cup \mathbf{b}$. Now by the same reasoning as above $\mathbf{a} \leq \mathbf{a}^*$. Hence $\mathbf{a} = \mathbf{a}^*$. (Note the roles of nonbounding and superbounding can be switched.)

This shows that Θ is isolated. Notice that this time we have a *lower* cone in which the isolable degree is the unique degree satisfying Θ, whereas with maximal contiguity we had an *upper* cone.

A big open question is whether there is a definable degree, or in other words, whether there is a formula that is isolated in the interval $[\mathbf{0}, \mathbf{0}']$.

The above results on isolated formulas seem to be as close as anyone has gotten to resolving this question.

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