INVARIANCE AND NONINVARIANCE IN THE LATTICE OF Π^0_1 CLASSES

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ABSTRACT. We prove that there are two minimal Π^0_1 classes that are not automorphic.

1. INTRODUCTION

A (computably bounded) Π^0_1 class C can be defined as the set of infinite paths through a computable tree $T \subseteq 2^{<\omega}$. The study of Π^0_1 classes has a long and interesting history, and many applications. We refer the reader to the surveys Cenzer [1], Cenzer-Remmel [5], and Cenzer-Jockusch [3].

This paper continues the study of the lattice of Π^0_1 classes along the lines of Cenzer, Downey, Jockusch, and Shore [2]. In particular, we are interested in the class of Π^0_1 classes introduced in Downey [8], but first constructed under duality in Martin and Pour-El [14]. These are the *thin* classes, where an infinite class P is called thin if, for all Π^0_1 subclasses $P' \subseteq P$ there is a clopen set C such that $C \cap P = P'$. Of particular interest to us in the present paper, is the case where the thin class has a unique nonisolated (rank one) point in it, and hence every Π^0_1 subclass is either finite or cofinite in P. These were first introduced in [2] and are called *minimal* classes.

Thin classes more or less correspond to hyperhypersimple sets in the lattice of computably enumerable sets. This intuition was made clear in Cholak, Coles, Downey, and Herrmann [6], where it is proven that an infinite class P is thin iff the lattice of subclasses forms a Δ^0_2 Boolean algebra, and every Δ^0_2 Boolean algebra is isomorphic to the lattice of subclasses of some thin class. For example, minimal classes correspond to the Boolean algebra of finite and cofinite sets. This characterization can be viewed as the analog of Lachlan's result [13] that the collection of computably enumerable supersets of a hyperhypersimple set is a Σ^0_3 Boolean algebra, and every such Boolean algebra can be realized.

The main result of [6] is that if S and T are *perfect* thin classes, then there is an automorphism of the lattice of Π^0_1 classes taking S to T. Here the class is perfect iff the lattice of subclasses is isomorphic to the free Boolean algebra, or, equivalently the class has no isolated points.

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The question we address in the present paper, is to determine whether the analogous result holds for minimal classes, a question raised in all of the surveys above and in [6]. Ultimately, the hope was that if the Boolean algebras of subclasses of two thin classes were isomorphic, then the classes would be automorphic.

Unfortunately, our main result is to prove that this hope is forlorn even for minimal classes.

We need some definitions.

Definition 1. *We say that a minimal class S is* cohesively minimal *if it obeys the following property $\mathcal{P}$ below. (All quantifiers range over that lattice of Π^0_1 classes.)*

For all $F \supseteq S$ one of the following holds.

(i) There exists a finite $G \subseteq S$ and a clopen $C \subseteq F$, such that, for $\widehat{S} = S - G$, $\widehat{S} \subseteq C$.
(ii) There is a finite $G \subseteq S$, such that, for $\widehat{S} = S - G$, and for all isolated $I \subseteq \widehat{S}$, there is no clopen C such that

$$(I \subseteq C) \wedge (C \subseteq F).$$

Theorem 2. *There is a cohesively minimal Π^0_1 class.*

Theorem 3. *There is a minimal Π^0_1 class that is not cohesively minimal.*

Theorems 2 and 3 give us our main result:

Corollary 4. *There exist two minimal Π^0_1 classes that are not automorphic.*

Proof. It is enough to observe that property $\mathcal{P}$ is definable by an infinitary formula. To see this note that being isolated (i.e. an atom of the lattice of Π^0_1 classes) is elementarily definable and hence being finite is definable via a nonelementary formula [1]. Being clopen is elementarily definable since it happens iff the class is complemented, as observed in Cholak, Coles, Downey, Herrmann [6]. This gives the result. □

We prove a little more than Theorem 3. To get Theorem 3, we actually prove the following. Define a rank one class $[T]$ to be *tame* if its rank one point has degree computable from $\emptyset'$ and there is a Δ^0_2 function m enumerating pairs $\langle \sigma_{m(e)}, \phi_{m(e)} \rangle$ consisting of a string $\sigma_{m(e)}$ and a partial computable function $\phi_{m(e)}$ such that :

(i) $\phi_{m(e)}$ is the characteristic function of an isolated point $I_{m(e)}$ of $[T]$, with $\sigma_{m(e)}$ an initial segment of $I_{m(e)}$.
(ii) $\{I_{m(e)} : e \in \omega\}$ list all the isolated points of $[T]$, and
(iii) $\{\sigma_{m(e)} : e \in \omega\}$ form an antichain of strings.

[1] Cenzer and Jockusch asked if the property of finiteness is elementarily definable. Recently, Cenzer and Nies [4] has answered this affirmatively. This means that our technique actually constructs an *elementary difference* between two minimal Π^0_1 classes.

For instance, any rank one class generated by $[T]$, where T has no dead ends is tame[2].

Theorem 5. *Suppose that S is a tame minimal class. Then S does not satisfy $\mathcal{P}$.*

We remark that it would be enough to construct a thin class S satisfying an apparently weaker property than cohesive minimality. Say a class is weakly cohesive if in place of (i) in Definition 1, we had instead used the following.

$(i)'$ There exists a finite $G \subseteq S$ such that, for $\widehat{S} = S - G$, for all isolated $I \subseteq \widehat{S}$, there is a clopen C such that

$$(I \subseteq C) \wedge (C \subseteq F).$$

This is because the proof of Theorem 5 demonstrates that classes with rank one points computable from $\emptyset'$ are not even weakly cohesively minimal. We do not know if the two properties are distinct.

On the other hand, Cenzer and Remmel did show that sometimes minimal classes are automorphic. They gave an effectiveness condition based on the complexity of the representation of the lattices of subclasses, that guaranteed automorphism of the classes. The condition is a special case of tameness. Specifically, (see [3], page 55) they defined a Π^0_1 class to be *standard* if its rank one point P is Δ^0_2 and for any $\nu\widehat{\ }i \prec P$, $\nu\widehat{\ }\langle 1-i\rangle$ is an initial segment of at most one isolated member of $[T]$. Clearly this condition implies tameness. We mildly generalize their result to prove:

Theorem 6 (essentially Cenzer and Remmel)**.** *Suppose that S and T are tame minimal classes. Then S and T are automorphic.*

Theorem 6 was established jointly with Carl Jockusch, before we learned of the Cenzer-Remmel work. It has a nice corollary. Recall that the degree of a class C is the degree of the set of strings that are not initial segments of members of C. In [2], a minimal class is constructed as the paths through a tree with no nonextendible nodes, so the degree of the minimal class is $\mathbf{0}$, and hence the class is tame. It is routine to construct a tame minimal class that is noncomputable. (For instance, if a class has its rank one point of degree $\leq \emptyset'$ and the class has low Turing degree, then it will be tame. We can see this using the following reasoning. First we will eventually find out if a given string is *not* an initial segment of a rank one point by the limit lemma. Then we can use lowness and hence $\emptyset'$ to ask if a node σ on T has two or more isolated extensions, assuming that the node is *not* an initial segment of the rank one point.) This gives the following.

Theorem 7. *(i) Any two computable minimal classes are automorphic.*
(ii) Being computable is not definable in the lattice of Π^0_1 classes.

The following question is more or less completely open. We know that if two thin classes are automorphic then their respective lattices of subclasses must be isomorphic Δ^0_2 boolean algebras. When can this be reversed? When does the isomorphism type of the Δ^0_2

[2] Suppose that we have ν as an initial segment of the rank one point, and the tree splits at ν. Then simply count which of $\nu\widehat{\ }0$ or $\nu\widehat{\ }1$ has the most split extensions to determine which way the rank one point goes. The q are the shortest strings which are initial segments of the isolated points, with the property that they are not initial segments of any other isolated points. You can order them, first by length, and then lexicographically.

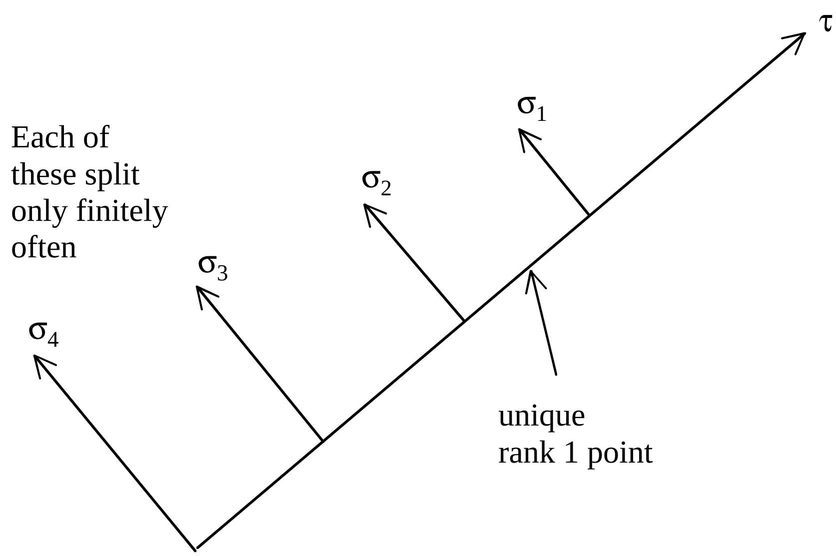

FIGURE 1. A rank one class

boolean algebra of subclasses of a thin class completely determine its automorphism type in the sense that all thin classes with the same Δ_2^0 boolean algebra B of subclasses will be automorphic? We have seen that, by [6], B being atomless is enough, and the proof will generalize to B having a finite number of atoms. By this paper, B being the Boolean algebra of finite and cofinite sets is not enough. We offer the following conjecture.

Conjecture 8. *The only Boolean algebras B, which have the property that any two thin classes with B as their lattices of subclasses are automorphic, are ones with a finite number of atoms.*

2. Preliminaries

We let $\{[V_e] : e \in \omega\}$ and $\{[U_e] : e \in \omega\}$ be enumerations of the collection of Π_1^0 classes. (We use two notations for the same collection to avoid confusion between the thinness requirements and the cohesiveness requirements.) For simplicity, we choose the following representation of these classes. Let P_e denote the e-th real time computable (i.e. $\sigma \in P_e$ is decidable in time $|\sigma|$) tree of strings. Then we will denote the stage s approximation to $[V_e]$ by $V_{e,s}$ and mean the strings of length s in P_e. Other notation more or less follows Soare [17].

3. Proof of Theorem 2

The proof that there is a Π_1^0 class which is both minimal and has property P, is relatively delicate and uses some of the technology developed in Downey [7]. In particular, it will be a kind of "full approximation" argument. We will need to construct S so that it is homeomorphic to the paths through the rank one tree of figure 1.

The reader should keep in mind that we will be constructing a tree of strings $T = \lim_s T_s$, so that $S = [T]$ will be the desired class. At any stage s, T_s will consist of a collection of strings which have been irrevocably declared *terminal*, as well as a collection of strings of length s that we currently believe are reasonable as possible initial segments of $[T]$. In the transition from stage s to $s+1$ we will either terminate such a string or extend it to one (or more) of length $s+1$.

An important picture the reader should have in mind for the present construction is that at any stage s we will have a current "view" as to the length s initial segment of the (unique) rank one point in $[T]$. This is the only string we are allowed to split in the transition from stage s to $s+1$. The other strings on T_s of length s might be extended but they cannot be split *and* extended.

Now in the construction, we may change our mind as to where the rank one point really lies, and the guessed initial segment may prove wrong. Of course, we may well have to put extensions splitting that segment, based on the false belief that that segment was the correct one. The construction will ensure that almost all such splittings will become terminal, so that all such "false splittings" with the "wrong guess" will die. (More on this later.)

Turning to the details of the proof, we must meet the requirements below.

$\mathcal{M}_e$: $[U_e] \subseteq S \to \exists$ clopen $D(D \cap S = [U_e])$.
$\mathcal{R}_e$: $S \subseteq [V_e] \to (i)$ or (ii) below holds
$(i)\exists$ finite $G \subset S(\exists$ clopen $C \subseteq [V_e])(S - G \subseteq C \wedge C \subseteq [V_e])$.
$(ii)\exists$ finite $G \subset S(\forall$ isolated $I \subset S - G)(\forall$ clopen $C)$
$(I \subseteq C \to (C \nsubseteq [V_e]))$.

Additionally, there will be implicit requirements asking that $[T]$ have a unique rank one path.

We begin by reviewing the standard procedure for meeting the $\mathcal{M}_e$, the thinness requirements. The basic idea is the following. Suppose that $[U_e] \subset S$. Then at some stage of the construction, for some string σ of length s in T_s, we will have $\sigma \notin U_{e,s}$. (That is, there can be no extension of σ in $[U_e]$.) Then we can win the requirement $\mathcal{M}_e$ by simply making sure that the rank one point in $[T]$ extends σ, and hence almost all members of S will extend σ. A simple priority argument will ensure that all the $\mathcal{M}_e$ get met.

Meeting the $\mathcal{R}_e$ requirements is much more difficult. Fix some e. At some stage, we have some belief as to some τ which we would like to be the initial segment of the rank one point of S. On the other hand, we will need to get the cohesiveness on the isolated paths of S. We do this by trying to verify (ii) of the definition of property P, on almost all paths.

Specifically, for the basic module, suppose that at some stage s_1 we had the belief that we should have τ as the initial segment of the rank one point, and 4 other strings $\sigma_1, \cdots, \sigma_4$, which we think ought to be initial segments of isolated paths in S, as in figure 1. We would like to know that there is no relevant clopen set for each of the σ_i, in terms of the last item of (ii) of the definition of property $\mathcal{P}$, and $\mathcal{R}_e$. The idea is the following.

We will (perhaps) temporarily abandon τ (and simply *directly* extend it as an isolated point till we return), and begin by making σ_1 the initial segment of the rank one point. This entails perhaps splitting strings above σ_1 in stages $t \geq s_1$ and directly extending σ_j for $j \neq 1$. We will do this until a stage s_2 is found where there exists a string γ with $\sigma_1 \prec \gamma$ and γ terminal in V_{e,s_2}. At this stage, we regard $\mathcal{R}_e$ as verified for σ_1, and will terminate all, save one, of the extensions of σ_1. See figure 2.

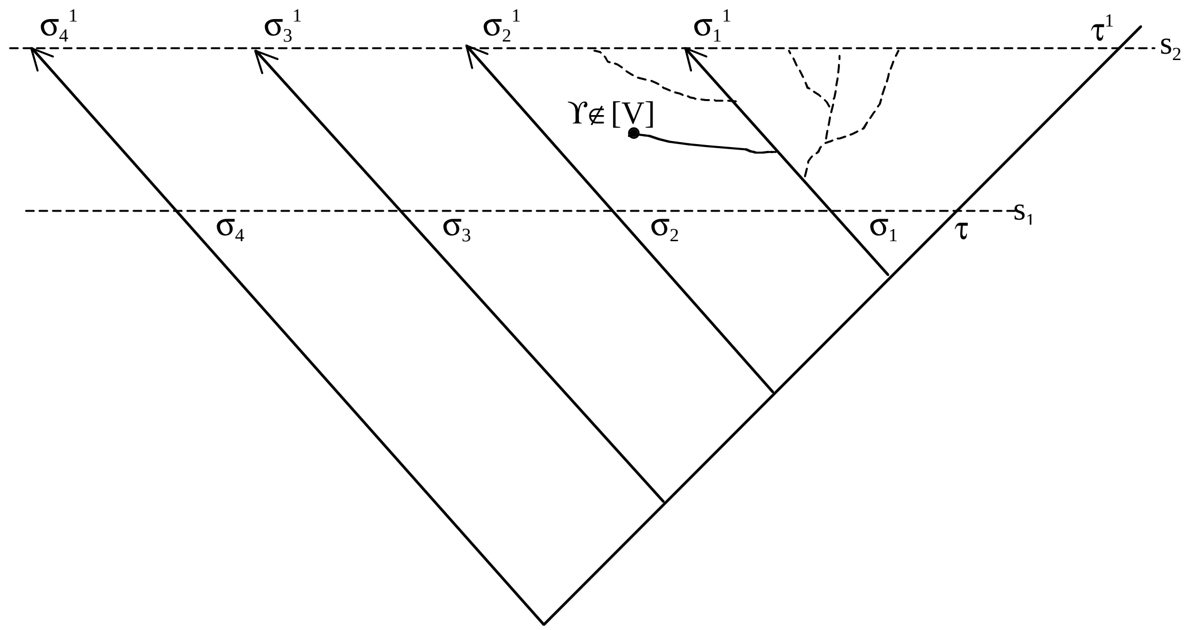

FIGURE 2. Termination of all but one extension

Remark 9. In the actual construction this termination is controlled by some $2e$-state machinery. All of the strings extending σ_1 will be given $2e$-state of the form $\beta\widehat{\ }f$. (This is for the appropriate β.) When we abandon the cone above σ_1 as the place where most of the construction lives, we terminate all *but one* of the strings ν with $2e$-states weaker than $\beta\widehat{\ }\infty$, which will terminate all, but one, of those strings with $2e$-state $\beta\widehat{\ }f$. The one we do not abandon, we will raise this strings $2e$-state to $\beta\widehat{\ }\infty$. We remark that $2e$ states(rather than e states) are used here since odd lengths will be devoted to thinnness requirements.

Now we repeat this process with all the other σ_j for $j = 2, 3, 4$[3]. Suppose first that we repeat this cycle through all the σ_j infinitely often. It will follow that for all initial segments $\mu_i : i = 1, \cdots, 4$, of the isolated paths $P_1, \cdots, P_4$ beginning with $\sigma_1, \cdots, \sigma_4$, there is always some isolated path D_i extending μ_i with $D_i \not\subset [V_e]$, and hence it is not possible for there to be a clopen C containing P_i with $C \subseteq [V_e]$.

On the other hand, suppose that we eventually get stuck in the cycle on some σ_j. That will entail the rank one point being a strict extension of $\sigma_j[t]$ hence hence being *above*

[3]Strictly speaking in the construction we may not go in this exact order, but we do ensure that all are considered. To wit, we will have a collection $F(\beta, s)$ of strings awaiting *verification* as above, here $F(\beta, s)$ would be $\{\sigma_1, \ldots \sigma_4\}$, and we would route the construction through the rightmost unverified σ_i until all are verified.

$\sigma_j[t]$ for all stages $u \geq t$. Thus we will force every string of length u in this clopen set to be in $V_{e,s}$. This means that above $\sigma_j[t]$, $[V_e]$ is clopen. Hence, with the exception of the finite number of isolated points we miss, $[V_e]$ is a clopen set containing $[T]$. Therefore, in this case, we satisfy (i) of the requirement $\mathcal{R}_e$.

The above describes the situation for one requirement. The situation becomes more complex with two requirements, as we now see.

Suppose that we are considering two requirements $\mathcal{R}_e$, and $\mathcal{R}_h$, with $e < h$. Consider the situation where we are extending σ_1, pretending it is the initial segment of the rank one point, but now seeking verification of both $\mathcal{R}_e$ and $\mathcal{R}_h$.

Now we are imagining that we are seeking verification above σ_1 begun at stage s. Of course if neither of the requirements are verified then we are in good shape. However, if one of the requirements becomes verified, what should we do? It would be tempting to stay in the cone above σ_1 when we see $\mathcal{R}_e$ verified also waiting for $\mathcal{R}_h$. This is because the cone above σ_1 looks like a very nice place to build the rank one point from the point of view of $\mathcal{R}_h$. However, if we do this then we must realize that the other paths extending σ_j for $j \neq 1$ are never again verified for e, and neither are extensions of τ. This seems no problem as there are only a finite number of them, but the reader can see that this process could repeat with infinitely many other $\mathcal{R}_{h'}$ (as for $\mathcal{R}_h$), causing us to lose on $\mathcal{R}_e$, because we *fail* to e-verify infinitely many isolated points in $[T]$, and hence $[V_e]$ could be as capricious as it desired.

Note that there is no problem if $\mathcal{R}_h$ is verified yet $\mathcal{R}_e$ is not, since the latter has higher priority than the former, so we should stay in the cone above σ_1.

The solution to the dilemma above is the following. Suppose that τ is the place above which we will be building an initial segment of the rank one point on the assumption that e is always verified. Then only nodes in the cone above τ can be assigned $2e$-states [4] $\beta^\frown\infty$.

Hence, if we abandon the cone above σ_1 for the sake of $\mathcal{R}_e$'s verification of the nodes σ_j, and τ, all such nodes will have $2e$ states too low, and hence it is reasonable to abandon them anyway.

However, iff we are working in the nodes above the node τ (where we agree that we will build the rank one point if the e-verification happens infinitely often) then this cancellation action is not reasonable since such a node ν_i, with $\tau \preceq \nu_i$, will have an $2h$-state extending τ's $2e$-state. We will designate a particular node τ' extending τ to be the prefer ered place to build the rank one point should *both* $\mathcal{R}_h$ and $\mathcal{R}_e$ be infinitely often verified. Thus, we will need to *temporarily* abandon τ' for other nodes extending τ and the σ_j, for the sake of e-verification, while *acknowledging* that from $\mathcal{R}_h$'s point of view the cone above τ' is the *preferred place* to build the rank one point. The reason that this is the place, preferred by $\mathcal{R}_h$ to build the rank one point, is that it looks like we have a clopen subset $C \subset [V_h]$ where almost all of the minimal class can be placed, and hence we can win $\mathcal{R}_h$ by the Σ^0_2 outcome, that is, (i) of Definition 1.

[4]Remember that $2e$ states are used here since odd lengths will be devoted to thinnness requirements.

Hence, until we get verification of $\mathcal{R}_h$ above τ', we will cycle through the extensions of the other nodes σ_j and τ, verifying $\mathcal{R}_e$, but making sure that we will add no new (permanent) *splittings* above these nodes. Again this is all forced by the assignment of $2e$ states to all nodes on the tree T. All save one of the potential paths of the tree added in the cone above, say, σ_3 while we are waiting for τ', will have $2e$-states $\beta^\frown f$, and hence will be canceled when we return. The nodes above τ' will have $2h$-state of the form $\beta^\frown\infty^\frown\gamma^\frown f$, for an appropriate $2h-1$-state $\beta^\frown\infty^\frown\gamma$. For the node τ' we ensure that $\tau'^\frown 0$ and $\tau'^\frown 1$ are on the tree, picking the right one as the preferred place for the state $\beta^\frown\infty^\frown\gamma^\frown\infty^\frown 1^\frown\infty$. (This is the preferred place for the $\mathcal{R}_{h+1}$ on the assumption that the $\mathcal{M}_{h+1}$ fails to act, which is the meaning of the $^\frown 1$ in the state.) All save one of these wrong guess strings extending $\tau'^\frown 0$ and $\tau'^\frown 1$ will be terminated, should we eventually h-verify τ', *and* e-verify all of the appropriate extensions of τ', at some stage $s' > s$. We will raise the state of the one remaining string for each extension. Whilst we are awaiting, say at stage s'' some verification for h above τ', we'd work on the assumption that it will not happen. See Figure 3 for a typical situation.

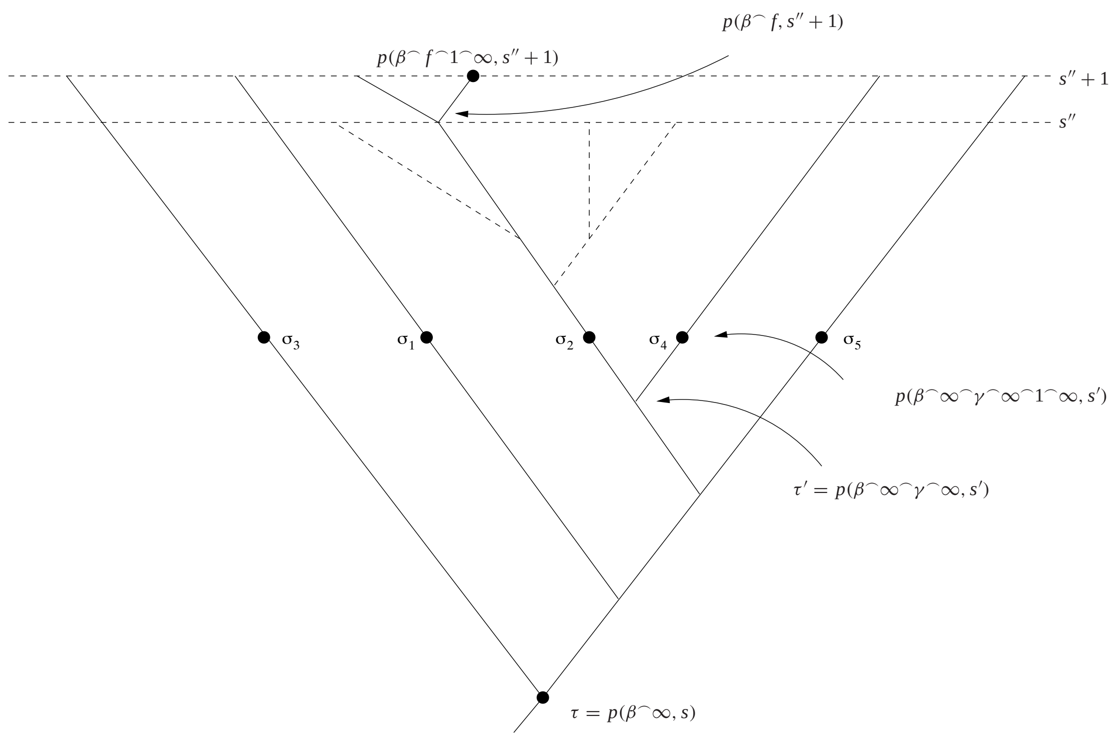

FIGURE 3. Preferred places – The function p refers to the "preferred places".

That is, suppose that we are working waiting for verification of $\mathcal{R}_e$, above, say, σ_3. Of course while we are waiting for verification of $\mathcal{R}_e$ we will put up splittings etc based on the belief that $\mathcal{R}_e$ is permanently stuck there. However should $\mathcal{R}_e$ be verified, then all but one of these paths will be terminated, and we will move to the next node, completely ignoring whether or not σ_3 is verified for $\mathcal{R}_h$.

Now should we return to some extension $\widehat{\nu}_i$, of ν_i to be $\mathcal{R}_e$ verified, we will "restart" the verification process for $\mathcal{R}_h$ and ν_i, whilst working in the cone above $\widehat{\nu}_i$ for $\mathcal{R}_e$. The difference between the nodes above σ_3 and those above ν_i is that when we leave σ_3 we terminate almost all of them. When we leave ν_i, while $\mathcal{R}_h$ is not yet verified, we will not terminate them. Of course should it turn out that we eventually verify $\mathcal{R}_h$ above σ_1, then we may well decide to cancel all such extensions except one.

The only thing we must ensure is that there is some place on the tree where we are building splittings based upon the belief that *both* $\mathcal{R}_e$ and $\mathcal{R}_h$ are always verified. It is clear that, above such a node, we should not terminate ever for the sake of $\mathcal{R}_e$ or $\mathcal{R}_h$. This process is controlled by the nodes with the $\alpha\widehat{\ }\infty$ states.

In summary, there will be a tree of strategies based on whether certain guesses are verified. At any stage for each guess there will be a cone above which is the *preferred place* to build the rank one point as far as that guess is concerned. The strings which are the bases of such cones will change from time to time, but we will argue that for the true path guesses, the bases come to a limit and will cohere.

In more detail, we will work with a priority tree $PT \subseteq \{\infty, f, 0, 1\}^{<\omega}$, such that if the length of α is even then α has two children $\infty <_L f$, and if the length of α is odd then α has two children $0 <_L 1$. (And thus PT is a complete binary tree.) We extend this ordering as lexicographic ordering on the tree. Members of PT will be called *guesses*. Such guesses can be assigned to nodes on the tree as *states*. A t state will be a string of length $t+1$ reflecting an outcome of a guess of length t. A state α is *higher* than a state β iff $\alpha <_L \beta$. To avoid confusion, we will use $\alpha, \beta, \gamma, \cdots$ (Greek letters from the beginning of the alphabet) for guesses, and $\sigma, \tau, \nu, \mu \cdots$ for members of $2^{<\infty}$, the nodes used to build the underlying tree T of the Π^0_1 class $[T]$. In both cases λ denotes the empty string, for no confusion will arise. We assign $\mathcal{R}_e$ to the guesses α of length $2e$, we write $e(\alpha) = e$ and refer to ∞, f as the outcomes of α. We write $\mathcal{R}_\alpha$ as the version of $\mathcal{R}_e$ identified with guess α. Similarly we assign $\mathcal{M}_e$ to those β (so $e(\beta) = e$) of length $2e+1$. All of this is more or less standard.

Now we can be more precise. For a guess α, $p(\alpha, s)$ will denote the current place, if any, above which we will build an initial segment of the rank one point should α be on the true path (TP). For $\alpha \prec \beta \prec TP$, we will ensure that $p(\alpha, s)$ and $p(\beta, s)$ each have a limit with $p(\alpha) \prec p(\beta)$.

As usual we will use the phrase *initialize*, which will mean that all parameters etc become undefined, strings terminated, etc. As usual, any parameter not explicitly initialized at some stage or substage retains its current value till the next stage or substage of the construction. We will be using the following parameters in the construction.

$TP(s, t)$: a string in PT that appears correct at substage t of stage s, which we write as stage (s, t).

Q_t : the value of a parameter Q at the beginning of stage (s, t).

$F(\alpha, s)$: a set of strings in T_s awaiting α-verification at stage s.

$\delta(\alpha, s)$: a split associated with α should $|\alpha|$ be odd and $\alpha \prec TP$. Should $\alpha \prec TP$ then this will come to a limit and $\delta(\alpha)$ is an initial segment of the rank one point, and both $\delta(\alpha)\widehat{\ }0$ and $\delta(\alpha)\widehat{\ }1$ will be initial segments of members of $[T]$.

$\mu(s, t)$: a string in T_s that appears to be the initial segment of the rank one point in T of length s.

We now turn to the formal details, although they are straightforward in view of the notes above.

Definition 10. We say that $\mathcal{M}_\alpha$ (with $|\alpha| = 2e + 1$) *requires attention* at guess α at stage $(s + 1, t)$, if $s + 1$ is a α-stage, and

(i) $\mathcal{M}_\alpha$ is not currently declared satisfied,
(ii) there is some string $\sigma \in T_s$ of length s (so it appears to be an initial segment of a member of $[T]$), such that $\sigma \notin U_{e,s}$, and $p(\alpha, s + 1)_t \prec \sigma$.

The Construction

Stage 1 Set $F(\lambda, 0) = \{1, 0\}$, assign the strings $1, 0$ to have each have state f. Set $p(f, 0) = p(f\widehat{\ }1, 0) = 0$, and set $p(\infty, 0) = \delta(\infty, 0) = \delta(f, 0) = \lambda$. Set $\mu(0) = 0$.

Stage $s+1, s > 0$ This will consist of up to $s+1$ many substages. We declare $(s+1, -1)$ to be a λ-stage. Let $p(\lambda, s) = \lambda$. We suppose that stage $(s + 1, t - 1)$ is an α-stage.

Substage $t = 2e$. (Attend $\mathcal{R}_\alpha$)

$(T1)$: Fact: $F(\alpha, s + 1)_t$ is currently defined unless either this is the first α stage, or α has been initialized since the last α stage.

Case A If $F(\alpha, s+1)_t$ is not defined, extend all strings currently alive to have length $s+1$ by direct extension. Additionally, let σ be the rightmost string in $p(\alpha, s)$ of length s. Put us another length $s + 1$ extension of σ to ensure that it is split, and hence $\sigma\widehat{\ }0$ and $\sigma\widehat{\ }1$ will be in T_{s+1}. Make the node $\sigma = p(\alpha\widehat{\ }\infty, s + 1) = \delta(\alpha\widehat{\ }f, s + 1) = \delta(\alpha\widehat{\ }\infty, s + 1)$. Now define $p(\alpha\widehat{\ }f, s + 1) = \sigma\widehat{\ }0$, and put $F(\alpha, s + 1) = \{\sigma\widehat{\ }0, \sigma\widehat{\ }1\}$.

Give both strings $t + 1$ states $\alpha\widehat{\ }f\widehat{\ }1$ Set $\mu(s + 1) = \sigma\widehat{\ }0$. Give every other string ν of length $s + 1$ not yet assigned a state, the same state as $\nu \upharpoonright s$. Go to stage $s + 2$. Declare $TP(s + 1) = \alpha\widehat{\ }f$.

Case B $F(\alpha, s + 1)_t$ is currently defined.

For all $\rho \in F(\alpha, s)$ not yet α-verified if we see some ν with $\rho \prec \nu$ and $\nu \notin V_{e,s}$, declare that ρ is α-verified. If $p(\alpha\widehat{\ }f, s + 1)_t = \rho$, then declare $p(\alpha\widehat{\ }f, s + 1)$ to be now undefined. Choose the first case below to pertain.

Case 1 There is some $\sigma \in F(\alpha, s + 1)_t$ such that, for all ν with $\sigma \prec \nu$, $|\nu| \le s$, $\nu \in V_{e,s}$.

Action For the rightmost such σ, set $p(\alpha\widehat{\ }f, s + 1) = \sigma$. (That is, σ replaces ρ.) For all $\tau \in F(\alpha, s + 1)_t$ which are already α-verified, by direct extension and by termination, if necessary[5] make sure that there is exactly one string of length $s+1$ extending τ and give it, (and all initial segments ν with $p(\alpha\widehat{\ }\infty) \preceq \nu \preceq \tau$) $t+1$-state $\alpha\widehat{\ }\infty$. Let $\mu(s+1,t) = \sigma (= p(\alpha\widehat{\ }f, s+1))$. Set $TP(s+1,t) = \alpha\widehat{\ }f$. Declare that $(s+1,t)$ is a $\alpha\widehat{\ }f$ stage.

Subcase 1 If $p(\alpha\widehat{\ }f, s+1)$ has just become defined (because its previous value ρ was α-verified), then proceed as follows. Set $\delta(\alpha\widehat{\ }f, s+1) = \sigma$, and add $\sigma\widehat{\ }0, \sigma\widehat{\ }1$ to $T_{s+1,t}$. Give them both $t+2$ state $\alpha\widehat{\ }f\widehat{\ }1\widehat{\ }\infty$ and set $p(\alpha\widehat{\ }f\widehat{\ }1\widehat{\ }\infty, s+1) = p(\alpha\widehat{\ }f\widehat{\ }1, s+1) = \sigma$. Set $F(\alpha\widehat{\ }f\widehat{\ }1, s+1) = \{\sigma\widehat{\ }0, \sigma\widehat{\ }1\}$. Set $\mu(s+1) = \sigma\widehat{\ }0$, $TP(s+1) = \alpha\widehat{\ }f\widehat{\ }1\widehat{\ }f$ and initialize as strings of lower state, making sure that the ones of not lower state are extended to length $s+1$ with the same state, if they are not terminal. Declare that $(s+1,t)$ is a $\alpha\widehat{\ }f\widehat{\ }1\widehat{\ }f$-stage. Go to stage $s+2$.

Subcase 2 If $p(\alpha\widehat{\ }f, s+1) = p(\alpha\widehat{\ }f, s+1)_t$ simply go to stage $(s+1, t+1)$ after initializing, where appropriate, strings of lower t-state than $\alpha\widehat{\ }f$, and extending strings to length $s+1$ to ones with the same state outside of the cone above σ.

Case 2 Not Case 1. Hence we know that all the strings in $F(\alpha, s+1)_t$ are now α-verified.

Action Set $TP(s+1,t) = \alpha\widehat{\ }\infty$. Declare $(s+1,t)$ to be a $\alpha\widehat{\ }\infty$-stage. Since we know that all the strings in $F(\alpha, s+1)_t$ are e verified, and hence the ∞ outcome looks correct.

$(T2)$ Fact: $p(\alpha\widehat{\ }\infty, s+1)_t$ is currently defined.

Initialize by termination all strings of lower t-state than $\alpha\widehat{\ }\infty$, making sure to leave one extension of length s for any currently nonterminated strings with t-state $\alpha\widehat{\ }\infty$. No matter which subcase below pertains, at the end of stage $s+1$, set $F(\alpha, s+2)$ to be the collection of strings of length $s+1$ (i. e. alive at the end of the stage) *extending* $p(\alpha\widehat{\ }\infty, s+1)_t$. For simplicity, we split one of these length $s+1$ extensions, say, σ, so that $\sigma\widehat{\ }0, \sigma\widehat{\ }1$ are on T_{s+1}, and declare $p(\alpha\widehat{\ }f, s+1) = \delta(\alpha\widehat{\ }f, s+1) = \sigma$, and note that we have initialized all guesses weaker than, or extending $\alpha\widehat{\ }f$. (The point if this, should the next α stage be an $\alpha\widehat{\ }f$ stage, we will have somewhere to go.)

Subcase 1 $p(\alpha\widehat{\ }\infty\widehat{\ }i\widehat{\ }\infty, s+1)_t$ is currently defined for some $i \in \{0,1\}$. In this subcase, simply go to the next substage.

Subcase 2 Otherwise. In this subcase, extend all of the remaining strings to have length $s+1$, and make the rightmost one have a split extension at the end, so that it equals $\nu\widehat{\ }0$ with $\nu\widehat{\ }1$ also on the tree. For $i = 0, 1$, give both strings $t+2$ state $\alpha\widehat{\ }\infty\widehat{\ }1\widehat{\ }\infty$, and set $\delta(\alpha\widehat{\ }\infty\widehat{\ }1\widehat{\ }\infty, s+1) = p(\alpha\widehat{\ }\infty\widehat{\ }1, s+1) = \nu$. Set $p(\alpha\widehat{\ }\infty\widehat{\ }1\widehat{\ }\infty, s+1) = \nu$, $F(\alpha\widehat{\ }\infty\widehat{\ }1) = \{\nu\widehat{\ }0, \nu\widehat{\ }1\}$, and $\mu(s+1) = p(\alpha\widehat{\ }\infty\widehat{\ }1\widehat{\ }\infty, s+1) = \nu\widehat{\ }0$. Go to stage $s+2$.

Substage $t = 2e+1$**.** (Attend $\mathcal{M}_e$) Let $|\alpha| = 2e+1$.

$(T3)$ Fact : We may assume that $\mu(s+1, t-1) = p(\alpha, s+1)_t$.

Case 1 $\mathcal{M}_\alpha$ is not currently declared satisfied, and we see any string σ through which it requires attention.

[5] Termination would be needed if τ had become verified since the last α stage.

Action Set $TP(s+1) = \alpha\widehat{\ }0$. Initialize all other requirements, of weaker guesses, and set $\mu(s+1) = \sigma\widehat{\ }0$ and add this to T_s. Also add $\sigma\widehat{\ }1$ to T_s. Set $p(\alpha\widehat{\ }0, s+1) = p(\alpha\widehat{\ }0\widehat{\ }\infty, s+1) = \delta(\alpha\widehat{\ }0, s+1) = \sigma$. Set $p(\alpha\widehat{\ }0\widehat{\ }f) = \sigma\widehat{\ }0$. Give both strings $\sigma\widehat{\ }i$ $t+2$ state $\alpha\widehat{\ }0\widehat{\ }\infty$. $\mathcal{M}_\sigma$ is declared satisfied. Set $F(\alpha\widehat{\ }0, s+1) = \{\sigma\widehat{\ }0, \sigma\widehat{\ }1\}$. To then form T_{s+1}, as usual initialize terminate strings remembering to make sure that ones with states not lower than $\alpha\widehat{\ }0$ are extended to have length $s+1$ by a single extension. Go to stage $s+2$.

Case 2 $\mathcal{M}_\sigma$ is currently satisfied.

Action Set $TP(s+1, t) = \alpha\widehat{\ }0$ and set $\mu(s+1, t) = p(\alpha\widehat{\ }0, s+1)_t$. Go to substage $t+1$. Declare that $(s+1, t)$ is a $\alpha\widehat{\ }0$-stage.

Case 3 Otherwise.

Action Set $TP(s+1, t) = \alpha\widehat{\ }1$ and $\mu(s+1, t) = p(\alpha\widehat{\ }1, s+1)_t$. Go to substage $t+1$. Declare that $(s+1, t)$ is a $\alpha\widehat{\ }1$-stage. **End of Construction**

Verification This runs along the lines of the intuitive discussion. It is easy to verify by induction the statements $T1 - T3$ of the proof. Furthermore, we claim,

(i) There is a leftmost path TP of strings in PT visited infinitely often and its length (measured by lim inf) is unbounded.
(ii) For $\alpha \prec TP$, $\lim_s p(\alpha, s) = p(\alpha)$ exists, and there are at most finitely many paths in $[T]$ that do not extend initial segments of $p(\alpha)$.
(iii) For $\alpha \prec TP$, $|\alpha| = 2e+1$, $\mathcal{M}_\alpha$ requires attention at most finitely often, and is met.
(iv) For $\alpha \prec TP$ of odd length, $\lim_s \delta(\alpha, s) = \delta(\alpha)$ exists, and both $\delta(\alpha)\widehat{\ }0$ and $\delta(\alpha)\widehat{\ }1$ are initial segments of paths in $[T]$.
(v) Suppose that $|\alpha| = 2e$ and $\alpha \prec TP$. If $\alpha\widehat{\ }\infty \prec TP$, then the initial segments of paths extending $p(\alpha)$ all have final $2e$-state $\alpha\widehat{\ }\infty$. Furthermore, all initial segments of paths in $p(\alpha)$ have extensions that are are e-verified at some stage.
(vi) Suppose that $|\alpha| = 2e$ and $\alpha\widehat{\ }f \prec TP$. Then $\lim_s F(\alpha, s) = F(\alpha)$ exists and there is some node $\sigma \in F(\alpha)$ (the rightmost) which equals $p(\alpha\widehat{\ }f)$. Furthermore no node in the cone above σ is ever e-verified.

The proof of the items above is very straightforward, and proceeds by way of induction. Suppose that we have the list of claimed statements for all $\beta \preceq \alpha$. Further, suppose that we are at a stage s_0 where for all $s \geq s_0$ $TP(s) \not\leq_L \alpha$, and we assume that $\alpha \prec TP$. Go to the next α-stage s_1 after s. Then we know that $p(\alpha, s_1)$ is set at, or before, that stage, since it is defined at α-stages. Note also that it is only reset through the action of β of stronger priority, and hence, by hypothesis it is now at a limit. Also $\delta(\alpha, s)$ is now at a limit. Let $p(\alpha) = p(\alpha, s_1)$ and $\delta(\alpha) = \delta(\alpha, s_1)$.

Now suppose that $|\alpha| = 2e+1$. At every stage $s \geq s_1$ where we pass through $p(\alpha)$ we consider extensions of $p(\alpha)$ and see if they leave $[U_e^s]$. If we see some σ extending $p(\alpha)$ such that (ii) of definition 10 pertains we will use subcase 1 of substage $2e+1$ of the construction. Hence we reroute the construction through that extension $\sigma \notin U_e$, and declare the requirement $\mathcal{M}_\alpha$ as met. The outcome will change to $\alpha\widehat{\ }0$ and this will be the outcome each time we visit α, since this declaration cannot be initialized except through the agency of requirements of stronger priority than α, which won't happen by hypothesis. In this case, $\alpha\widehat{\ }0 \prec TP(s)$ for all α stages after the stage s_1 we declare $\mathcal{M}_\alpha$ as satisfied. Note that at that stage in the subcase 1 action of the construction for $\mathcal{M}_\alpha$, we will define $\delta(\alpha\widehat{\ }0, s_1) = \sigma$, put both the extensions of σ on the tree, give them the high state $\alpha\widehat{\ }0$. Furthermore $F(\alpha\widehat{\ }0)$ is set to be the cone above σ. For the same reasons we have seen above these actions will never be initialized. Thus because we always extend a string of a nonitialized state, we see that both of the extensions will have at least one path extending them in $[T]$. Finally every string in the cone above $p(\alpha)$ guessing the low state $\alpha\widehat{\ }1$ is initialized, (save the finite number with state σ not in the cone above $p(\alpha\widehat{\ }0)$ which will be directly extended), and whilst we may put up splittings for guesses β right of $\alpha\widehat{\ }0$, we know, by hypothesis that the cones $p(\beta, t)$ extending $p(\alpha)$ will be initialized, as $\beta \not\prec TP$. Note that every string in $p(\alpha\widehat{\ }0)$ will have, as its initial segment, a string with state $\alpha\widehat{\ }0$. Again, it is true that false splitting can be constructed at some stages during the construction, extending these strings, because of β's right of α. But these splittings will be initialized, and we only directly extend strings γ with state extending $\alpha\widehat{\ }0$. Thus in the limit all strings in the cone $p(\alpha\widehat{\ }0)$ have $2e+1$ state $\alpha\widehat{\ }0$.

The argument for $\alpha\widehat{\ }1$ is even easier. Things are now at their limits. Hence we have the above for the case $|\alpha| = 2e+1$.

Finally consider the case that $|\alpha| = 2e$. The argument is very similar. We note that, at the first α stage after s_0 (If not earlier) we will set $p(\alpha\widehat{\ }\infty, s)$ and $F(\alpha, s)$. Thus, by priorities, are both now at their limits. Hence it follows that

(i) either we play this outcome infinitely often, and each time we play this outcome we initialize lower priority stuff; or

(ii) $\alpha\widehat{\ }f \prec TP$ because we get stuck verifying some $\sigma \in F(\alpha) = \lim_s F(\alpha, s)$, in which case $p(\alpha\widehat{\ }f, s)$ comes to a limit.

The argument that everything comes to a limit in the cone (and that $\delta(\alpha\widehat{\ }f, s)$ come to a limit) is virtually identical. The facts that we always reset $F(\alpha, s)$ to all the live strings of length s in the cone above $p(\alpha\widehat{\ }\infty)$ each time we play $\alpha\widehat{\ }\infty$, and that we always verify the current members of $F(\alpha, s)$ whenever we are at an α-stage and we see it possible to verify the member, mean that all potential initial segments of isolated points are systematically examined and the theorem follows.

4. Proof of Theorem 5

This argument is much more straightforward. Let $[T]$ be the relevant tame class presented by the computable tree $T = \lim_s T_s$. We suppose that the unique rank one point P is Δ_2. By the definition of tameness, it follows that there is a Δ_2 collection $I_0, I_1, \cdots$ of all of the isolated members of $[T]$, ordered using initial segments $\sigma_i = \lim_s \sigma_{i,s} \prec I_i$, ($\sigma_i$ is *not* an initial segment of P). Then $\{\sigma_i : i \in \omega\}$ forms an antichain of strings. We may suppose that enumeration of I_j has the property that, after some stage $s(j)$, I_j is constructed as an ever increasing string in a uniformly computable fashion.

It is therefore easy to construct a class $[V]$ which demonstrates the failure of property $\mathcal{P}$. $[V]$ will be constructed as a collection of paths through a computable tree $[V]$. On the even I_{2e}, we will, in a stage by stage basis, put all the extensions of $\sigma_{2e,s}$ of length s, which have not been terminated, into V_s. On I_{2e+1}, for each initial segment τ with $\sigma_{2e+1,s} \preceq \tau \prec I_{2e+1,s} \upharpoonright s$ (the current approximation to I_{2e+1}), if $|\tau|$ is divisible by 4, terminate all extensions of $\tau\frown\langle i-1\rangle$ given $\tau\frown i \preceq I_{2e+1,s} \upharpoonright s$. If $|\tau|$ is not divisible by 4, put all extensions of $\tau\frown\langle i-1\rangle$ of length s, not extending strings already terminated at a previous stage, into V_s.

For each j, if $\sigma_{j,s+1}$ changes (we change our mind on I_j), terminate all strings in V_{s+1} extending $\sigma_{j,s}$ of length s, except ones extending strings in T_s of length s. Finally at every stage, terminate all strings of length s not extending some $\sigma_{i,s}$ that are not initial segments of T_s.

The point is that, after some stage, I_j becomes stable. From that point on, *either* j has even length, and hence we will add every string extending some initial segment of I_j (hence $[V]$ is clopen there, and contains I), *or* j is odd and hence from some point onwards, we force any $U \subseteq V$ containing I_j to miss out on the paths extending $\tau\frown\langle i-1\rangle$ for infinitely many $\tau\frown i \prec I_j$.

Hence $[T]$ does not satisfy property $\mathcal{P}$, or even weak cohesiveness.

5. **Proof of Theorem 6.**

Let $[T_1]$ and $[T_2]$ be two tame minimal classes. By tameness, as in the previous section, we can have Δ^0_2 lists of the isolated points of the respective classes

$$I^j_0, I^j_1, \cdots \text{ for } j \in \{1,2\},$$

with corresponding strings

$$\sigma^j_0, \sigma^j_1, \cdots,$$

together with a Δ^0_2 collection of strings τ^j_i for $j \in \{1,2\}$, such that

$$\lim_s \tau^j_{i,s} \prec \lim_s \tau^j_{i+1,s} \prec P_j,$$

where P_j denotes the rank one point in $[T_j]$.

From [6], we recall that a *basis* for $\overline{[T_j]}$ is a computably enumerable set of strings $\{\gamma^j_i : i \in \omega\}$ generating the filter of strings with no extension in $[T]$, and which form an antichain. Given a Π^0_1 class one can easily construct such a basis.

The extension lemma (Theorem 8.1 of [6]), interpreted in the present setting of minimal classes, says the following:

Theorem 11 ([6], Theorem 8.1, special case)**.** *Suppose that there is a computable permutation p and computably enumerable bases γ^j_i as above, such that additionally*

$$\forall k \exists s \forall i > s((\sigma^1_k \prec \gamma^1_i \textit{ iff } \sigma^2_k \prec \gamma^2_{p(i)}) \wedge (\tau^1_k \prec \gamma^1_i \textit{ iff } \tau^2_k \prec \gamma^2_{p(i)})).$$

Then $[T_1]$ is automorphic to $[T_2]$.

It is now clear how to finish the proof. We assume that we are given two computable bases as above, and we define the p so that we try to force $\sigma^1_k \prec \gamma^1_i$ iff $\sigma^2_k \prec \gamma^2_{p(i)}$ and $\tau^1_k \prec \gamma^1_i$ iff $\tau^2_k \prec \gamma^2_{p(i)}$, noting that there is no conflict between these two for almost all strings. We do this in a stage by stage fashion, and use the approximations to the σ^j_k at each stage. The fact that the approximations are Δ^0_2 means that it all settles down, giving the result.

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