Computability Theory
Domination, Measure, Randomness, and Reverse Mathematics

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The Game Plan

Basic Concepts
   Computability
   Cantor Space – Category and Measure

Domination
   and Measure, again

Randomness

Reverse Mathematics

(Given my contribution to this project please consider this as a survey talk not a research talk.)
Domination
Motivation and Definition

Definition
If $f(n) \geq g(n)$ for all but finitely many $n$, then $f$ dominates $g$, $f \geq^* g$. 
Theorem

Ackermann’s function dominates all primitive recursive functions.

Ackermann’s function is computable. Hence the primitive recursive functions do not capture the informal notion of computable.
Computable Functions

Basic Definitions

Definition (Turing Machines)

- \( \Phi^X_{e,s}(n) = \Phi_{e,s}(X, n) = \Phi^\emptyset_{e,s}(X, n) \) is the \( e \)th Turing machine with oracle \( X \) with input \( n \) run \( s \) stages. Determining if \( \Phi^X_{e,s}(n) \downarrow \) is computable in \( X \). The number and type of the inputs can vary.

- \( \Phi^X_e(n) \downarrow \) if there is a stage \( s \) such that \( \Phi^X_{e,s}(n) \downarrow \). \( \Phi_e \) need not be total. But from now on we will assume we are dealing with total \( \Phi \).

Definition (Turing Reducibility)

- \( f \leq_T X \) iff \( f = \Phi^X_e \), for some \( e \).
- \( Y \leq_T X \) iff \( \chi_Y \leq_T X \).
- \( Y \equiv_T X \) iff \( X \leq_T Y \) and \( Y \leq_T X \). We write this as the Turing degree, \( x \).
We will work in Cantor Space, $2^\omega$.

**Definition**

- Let $\sigma \in 2^\omega$. $[\sigma]$ is a basic clopen (open and closed) class.
- A *open* class is a countable union of basic open classes. The complement of a open class is *closed*.
- The countable intersection of (basic) open classes is a $G_\delta$ class.
- The countable union of closed classes is a $F_\sigma$ class.

Cantor Space is compact: if a closed class $\mathcal{X}$ can be covered by open classes, $\mathcal{X}$ can be covered by finitely open open classes.
Cantor Space
Computability and Category

Definition

- $X \subseteq 2^\omega$ is *computable* in $Z$ iff there is a total $\Phi$ such that $X \in X$ iff $\Phi^Z(X) \downarrow = 1$ and $X \notin X$ iff $\Phi^Z(X) \downarrow = 0$.

- $X$ is $\Pi^Z_1$ iff $X = \{X \mid \forall n (\Phi^Z(X, n) \downarrow = 1)\}$.

- $X$ is $\Pi^Z_2$ iff $X = \{X \mid \forall n \exists m (\Phi^Z(X, n, m) \downarrow = 1)\}$.

**Lemma**

- $X$ is clopen iff $X$ is computable. (Finite Use Principle)

- $X$ is closed iff $X$ is $\Pi^Z_1$, for some $Z$. If $Z = \emptyset$ then $X$ is called a $\Pi^0_1$-class or effectively closed.

- A class $X$ is $G_\delta$ iff $X$ is $\Pi^Z_2$, for some $Z$. These classes are called boldface $\Pi^0_2$ or $\Pi^0_2$.

- Similarly a $F_\sigma$ class is $\Sigma^0_2$. 
Measure Theory on Cantor Space

Definition

- $\mu([\sigma]) = 2^{-|\sigma|}$. This determines the measure of $G_\delta$ classes and hence $F_\sigma$ classes.
- A class $X$ is measurable iff $\lim\inf \mu(G)$ exists and is equal to $\mu(X)$, where $G$ is a $G_\delta$ class containing $X$.

Definition

A Borel measure $\hat{\mu}$ is regular if every measurable (in terms of $\hat{\mu}$) class $P$ there is a $G_\delta$ class $Q \supseteq P$ and an $F_\sigma$ class $S \subseteq P$ such that $\hat{\mu}(S) = \hat{\mu}(P) = \hat{\mu}(Q)$.

Theorem

$\mu$ is regular.
(Uniformly) Almost Everywhere Dominating

Definition (Dobrinen and Simpson)

- A Turing degree \( a \) is *almost everywhere* (a.e.) **dominating** if for almost all \( Z \) for all \( g \leq_T Z \) there is function \( f \) of degree \( a \) which dominates \( g \).
- A Turing degree \( a \) is *uniformly almost everywhere* (u.a.e.) **dominating** if there is function \( f \) of degree \( a \) such that

\[
\mu \left( \{ Z \in 2^{\omega} : (\forall g)[g \leq_T Z \Rightarrow g \leq^* f] \} \right) = 1.
\]

We also call such a function \( f \) **uniformly a.e. dominating**.

Lemma

*U.a.e. dominating implies a.e dominating.*
Existence

Theorem (Martin)
An uniformly a.e. dominating function $f$ dominates all computable functions and hence must be high. I.e. $f' \equiv_T 0''$, where $f' = \{e | \Phi_e^f(e) \downarrow \}$.

Theorem (Kurtz)
There is a uniformly a.e. dominating function of degree $0'$.

Theorem (Cholak, Greenberg, Miller)
There is an incomplete (c.e.) uniformly a.e. dominating degree.
Positive Measure Dominating

Definition (Kjos-Hanssen)

- \(\text{Tot}(\Phi) = \{X \mid \forall n \exists s \Phi_s(X, n) \downarrow \text{ is total}\}\)
- \(\Phi < a\) iff either \(\mu(\text{Tot}(\Phi)) = 0\) or there is an \(f \leq_T a\) such that
  \[
  \mu(\{X \in \text{Tot}(\Phi) \mid f \geq \Phi(X)\}) > 0.
  \]
- If, for all \(\Phi\), \(\Phi < a\) then \(a\) is positive measure (p.m.) dominating.

Lemma

\textit{U.a.e. dominating implies a.e dominating implies p.m. dominating.}

Theorem (Binns, Kjos-Hanssen, Miller, Soloman)

\textit{Converse holds. P.m. dominating implies a.e dominating which implies u.a.e. dominating.}
Theorem (Dobrinen and Simpson)

A Turing degree $a$ is u.a.e. dominating iff for every $\Pi^0_2$ class $Q \subseteq 2^\omega$ there is a $\Sigma^a_2$ class $S \subseteq Q$ such that $\mu(S) = \mu(Q)$.

$$(\Rightarrow) \quad Q = \{X | \forall n (\Phi_e(X, n) \downarrow)\},$$

for some $e$. Let $\Psi$ be such that $\Psi^X(n)$ is the least $s$ where $\Phi_{e,s}(X, n) \downarrow$. $f$ dominates $\Psi^X$ for almost all $X \in Q$.

$$S = \{X : \exists k \forall n (\Phi_{e,f(n)+k}(X, n) \downarrow)\}.$$
**P.m. Domination and Measure**

**Theorem (Kjos-Hanssen after Dobrinen and Simpson)**

A Turing degree $\alpha$ is p.m. dominating iff $\text{Tot}(\Phi)$ has a $\Pi^a_1$ subclass, $\mathcal{S}$, of positive measure.

$(\Leftarrow)$ By compactness, $\{\Phi(X, n) \mid X \in \mathcal{S}\}$ is finite for all $n$. Therefore $\{\langle n, m \rangle : \forall X (X \in \mathcal{S} \rightarrow \Phi(X, n) < m)\}$ is $\Sigma^a_1$. Hence by $\Sigma^a_1$ uniformization there is a function $f \leq \alpha$ such that $\forall n \forall X (X \in \mathcal{S} \rightarrow \Phi(X, n) < f(n))$.

**Goal Check I:** We have related domination and measure. Now lets add randomness to this mixture.
1-Random Reals

Want to miss all “effectively null classes”.

Definition (Martin-Löf)

- A *Martin-Löf test (relative to X)* is a computable (in X) collection of $\Sigma^0_1$ open classes $\{U_e\}$ with $\mu(U_e) \leq 2^{-e}$.
- $R$ misses a test, $\{U_e\}$, iff $R \notin \bigcap_e U_e$.
- $R$ is *1-random (relative to X)* iff $R$ misses all Martin-Löf tests (relative to X).

Theorem (Martin-Löf, Solovay, Levin, Chaitin, Kolmogorov)

*The definition of 1-randomness is very robust.*
Low for 1-Random

Definition

- $A$ is *low for 1-random* iff the class of 1-randoms is the class of 1-randoms relative to $A$.
- $A$ is *low for 1-random over Z* iff the class of 1-randoms relative to $Z$ is the class of 1-randoms relative to $A$ and $Z$ (or equivalently $A \oplus Z$) iff $A \leq_{LR} Z$.

Theorem (Downey, Hirschfeldt, Nies, Solovay, Stephan, Terwijn)

*The class of $A$ such that $A$ is low for 1-random is a nontrivial robust class. Furthermore for all such $A$, $A' \leq \emptyset'$.*
Lowness and Domination

Theorem (Binns, Kjos-Hanssen, Lerman, Solomon)
If $B$ is a.e. domainating then $0' \leq_{LR} B$.

Theorem (Kjos-Hanssen)
$A \leq_{LR} 0'$ iff every $\Pi^0_2$ class of positive measure has a $\Pi^A_1$ subclass of positive measure iff $A$ is positive measure dominating.

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)
If $A \leq_T B'$ and $A \leq_{LR} B$ then every $\Sigma^A_2$ class has a $\Sigma^B_2$ subclass of the same measure.

Question (Aside)
What is needed for this theorem in reverse math?
Lowness and Domination, II

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)

$B$ is a.e. domainating iff $0' \leq_{LR} B$.

Proof.

$(\Leftarrow)$ Let $P$ be any $\Pi^0_2$ class. $P$ is a $\Sigma^0_3$ class, so it has a $\Sigma^0_2$ subclass $Q$ of the same measure (Kurtz, 1981). But, by the above theorem, $Q$ has a $\Sigma^B_2$ class of the same measure.

Corollary

U.a.e. domainating iff a.e domainating iff p.m. domainating.
Weakly 2-random

- A Martin-Löf test (relative to $X$) is a computable (in $X$) collection of $\Sigma^0_1$ open classes $\{U_e\}$ with $\mu(U_e) \leq 2^{-e}$.
- We will remove the restriction that $\mu(U_e) \leq 2^{-e}$.
- Consider a computable (in $X$) collection of $\Sigma^0_1$ open classes $\{U_e\}$ with $\lim_e \mu(U_e) = 0$.
- $\bigcap_e U_e$ is a measure zero $\Pi^0_2$ class ($G_\delta$).
- $R$ is weakly 2-random (relative to $X$) iff $R$ misses all measure zero $\Pi^0_2$ ($\Pi^X_2$) classes.

Theorem (Kurtz)

Weakly 2 random implies 1-random but converse does not hold.
Low for weakly 2-random

- $A$ is *low for weakly 2-random* if every weak 2-random is weak 2-random over $A$.
- $A$ is *low for weak 2-tests* iff every $\Pi^A_2$ nullclass is covered by a $\Pi^0_2$ nullclass.

**Lemma**

*Low for weak 2-tests implies low for weak 2-random*. (Shortly we will see the converse also holds)

**Theorem (Downey, Nies, Weber, Yu)**

- *If $A$ is low for weakly 2-random then $A$ is low for random.*
- *There is a noncomputable c.e. $A$ is that is low for weak 2-tests.*
Low for random and weakly $2$-random

Theorem (Binns, Kjos-Hanssen, Nies, Miller, Solomon)

If $A$ is low for random then $A$ is low for weak $2$-tests.

Corollary

$A$ is low for random iff $A$ is low for weak $2$-tests iff $A$ is low for weakly $2$-randoms.
Statement \((G_\delta\text{-REG})\)
For every \(G_\delta\) class \(\mathcal{Q} \subseteq 2^\omega\) there is a \(F_\sigma\) class \(\mathcal{S} \subseteq \mathcal{Q}\) such that \(\mu(\mathcal{S}) = \mu(\mathcal{Q})\).

Theorem (Dobrinen and Simpson)
\(\text{ACA}_0\) implies \(G_\delta\text{-REG}\).

Theorem (Kjos-Hanssen)
\(\text{RCA}_0 + G_\delta\text{-REG}\) does not imply \(\text{ACA}_0\).
$G_\delta$-REG and traditional systems

$G_\delta$-REG seems to be “orthogonal" to the traditional systems.

Theorem (Referee of Dobrinen and Simpson)
WKL$_0$ does not imply $G_\delta$-REG.

Theorem (Cholak, Greenberg, Miller)
RCA$_0$ + $G_\delta$-REG does not imply DNR$_0$.

Theorem (Cholak, Greenberg, Miller)
WKL$_0$ + $G_\delta$-REG does not imply ACA$_0$; WWKL$_0$ + $G_\delta$-REG does not imply WKL$_0$.

Question
Does DNR$_0$ + $G_\delta$-REG imply WWKL$_0$?