ATOMLESS r-MAXIMAL SETS

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Abstract. We focus on $\mathcal{L}(A)$, the filter of supersets of A in the structure of the computably enumerable sets under the inclusion relation, where A is an atomless r-maximal set. We answer a long standing question by showing that there are infinitely many pairwise non-isomorphic filters of this type.

1. Introduction

Let $\mathcal{E}$ be the collection of computably enumerable sets ordered via the inclusion relation. A main question concerning this structure is to classify the orbits. That is, given a computably enumerable set A, determine all the other computably enumerable sets B such that there is an automorphism Φ of $\mathcal{E}$ with $\Phi(A) = B$. (It is understood that from this point on all sets are computably enumerable.) There has been some success in this area. For example, the maximal sets form an orbit [Soare, 1974] and the hemi-maximal sets (Friedberg splittings of maximal sets) form an orbit [Downey and Stob, 1992].

It is easy to see that if A and B are in the same orbit then $\mathcal{L}(A)$ is isomorphic to $\mathcal{L}(B)$. (By work of Cholak [1995] and Soare [1974], and some unpublished work of Herrmann, the converse fails unless $\mathcal{L}^*(A)$ is finite.) $\mathcal{L}(A)$ is the principal filter that A determines in $\mathcal{E}$; ie. $\mathcal{L}(A) = \{\{A \cup W_e\}_{e\in\omega}; \subset\}$ and $\mathcal{L}^*(A)$ is $\mathcal{L}(A)$ modulo the ideal of finite sets. Hence we will turn our attention to classifying the various different principal filters that are possible.

Principal filters of $\mathcal{E}$ are in a correspondence with intervals of $\mathcal{E}$. Fix infinite computably enumerable sets $A \subseteq B$. Define $\mathcal{E}(X) = \{\{W_e \cap X\}_{e\in\omega}; \subset\}$, where X need not be computably enumerable; for example $X = B - A$. Now one can find an one-to-one computable function f whose range is B. The principal filter formed by the pullback of A, $f^{-1}(A)$, is isomorphic to $\mathcal{E}(B - A)$. Similarly, if one has a principal filter one can easily find a corresponding interval. The only intervals of

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$\mathcal{E}$ which have been (at least partially) classified in terms of isomorphism are either isomorphic to $\mathcal{E}$, $\mathcal{M}$ (the interval formed by $A \subset_m B$, for more see below), a Σ_3 Boolean algebra, or the principal filter formed by a r-maximal set. We will explore what is known.

By work of Soare [1982], it is known that if A is low then $\mathcal{L}^*(A) \approx \mathcal{E}^*$. (And then we can use another result of Soare to show $\mathcal{L}(A) \approx \mathcal{E}$.) This has been improved to low$_2$, in yet to be published work of Harrington, Lachlan, Maass and Soare. Also, in Cholak [1995], it is shown this remains true if A is semi-low$_2$ and has the outer splitting property. The point is that a large class of sets determine principal filters which are isomorphic to $\mathcal{E}$.

It is useful to recall the definition of a major subset:

Definition 1.1. A is a *major* subset of B, $A \subset_m B$, if $B - A$ is infinite and for every computably enumerable set W, $\overline{B} \subseteq^* W \implies \overline{A} \subseteq^* W$.

Let $A \subset_m B$ and $C \subset_m D$, then by Maass and Stob [1983], we know that $\mathcal{E}(B - A) \approx \mathcal{E}(D - C)$. We define $\mathcal{M} = \mathcal{E}(B - A)$.

By Lachlan [1968], a set H is hhsimple iff $\mathcal{L}^*(H)$ is isomorphic to a Σ_3 Boolean algebra. A maximal set M is a hhsimple set; $\mathcal{L}^*(M)$ is the two element Boolean algebra. Since there are infinite many such Boolean algebras, we know that hhsimple sets break up into infinitely many orbits. Slaman and Woodin (unpublished) used this result of Lachlan to show:

Theorem 1.2 (Slaman-Woodin). *The set $\{\langle h_1, h_2\rangle :$ where h_i is an index for a hhsimple set H_i and $\mathcal{L}(H_1) \approx \mathcal{L}(H_2)\}$ is Σ^1_1 complete.*

The proof goes something like this: Build a uniformly computable collection of computable Boolean algebras $\{\mathcal{B}_i\}_{i\in\omega}$ such that the set

$$\{\langle i, j\rangle : \mathcal{B}_i \text{ is isomorphic to } \mathcal{B}_j\}$$

is Σ^1_1 complete. Now to this computable collection apply Lachlan's construction of a hhsimple set to get a computable collection of computably enumerable sets $\{H_i\}_{i\in\omega}$ where $\mathcal{L}^*(H_i)$ is isomorphic to $\mathcal{B}_i$.

With these results in mind, we turned towards the r-maximal sets. Recall the following definition and lemmas: (The proofs of the lemmas can be found in Soare [1987, X.4].)

Definition 1.3. (i) An infinite set C is *r-cohesive* if there is no computable set R such that $R \cap C$ and $\overline{R} \cap C$ are both infinite.
(ii) A computably enumerable set A is *r-maximal* if $\overline{A}$ is r-cohesive. (Clearly if A is r-maximal and $A \subseteq B$ then B is r-maximal.)

Lemma 1.4 (Lachlan). *Assume $A \subset_\infty B$ and B is r-maximal. Then $A \subset_m B$ iff A is r-maximal.*

Corollary 1.5. *If B is r-maximal and $A \subset_m B$ then A is r-maximal but not maximal.*

If A and B are r-maximal sets with maximal supersets then $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$ are isomorphic. Fix two such sets. If M_1 and M_2 are two maximal supersets of A then either $M_1 =^* M_2$ or $M_1 \cup M_2 =^* \omega$. The latter cannot occur. So modulo the ideal of finite sets, A can only have one maximal superset M. Similarly, B can only have one maximal superset M'. So $\mathcal{L}(M)$ and $\mathcal{L}(M')$ are isomorphic. $A \subset_m M$ and $B \subset_m M'$ by Lemma 1.4. By Maass and Stob [1983], we know that $\mathcal{E}(M - A) \approx \mathcal{E}(M' - B) \approx \mathcal{M}$. Then we can piece together an isomorphism between $\mathcal{L}(A)$ and $\mathcal{L}(B)$.

Definition 1.6. A coinfinite computably enumerable set A is *atomless* if A has no maximal superset.

Much less is known about atomless r-maximal sets. It is known that there are two atomless r-maximal sets A and B such that $\mathcal{L}(A)$ and $\mathcal{L}(B)$ are *not* isomorphic [see Soare, 1987, X.5.7-8]. It has been a long standing open question if there were infinite many such sets. This is our main result:

Theorem 1.7. *There are infinitely many atomless r-maximal sets such that the principal filters formed by these sets are pairwise non-isomorphic.*

Hence the atomless r-maximal sets also break up into infinitely many orbits. In the next two sections we will focus on providing a proof of this result. First, in Section 2, we give a general construction of an atomless r-maximal set and then, in Section 3, we use this construction to get the desired result.

So far we were unable to classify all the possible principal filters which can be formed by an atomless r-maximal set. To prove the above theorem, we came up with infinitely many such principal filters. But we are on the fence as to whether this list of principal filters (or any finite modification of it) is inclusive of all such principal filters. We would like to think that our construction of an atomless r-maximal set is the canonical construction of an atomless r-maximal set as Lachlan's [1968] construction provided for the hhsimple sets. But we have no evidence to support this claim.

However, in our desire to find a classification, we were able to prove the following:

Theorem 1.8. *There is a satisfiable definable property P (in the language $\{\subset\}$) such that if $P(A)$ and $P(B)$ then A and B are atomless r-maximal sets and $\mathcal{L}^*(A) \approx \mathcal{L}^*(B)$*

A proof of this can be found in Section 4. There are other examples of definable classes of sets $\{A : \mathcal{L}^*(A) \approx \mathcal{L}\}$, where $\mathcal{L}$ is *not* a Boolean algebra. The set $\{A : \exists M(A \subset_m M \ \&\ M \text{ is a maximal set})\}$ is such a definable class. For all k, by Soare [1974], we know that $\{A : \mathcal{L}^*(A) \approx \mathcal{L}\}$ is definable, where $\mathcal{L}$ is the Boolean algebra of size 2^k. In Harrington and Nies [n.d.], there are other examples where $\mathcal{L}$ is an infinite Boolean algebra. For example, $\mathcal{L}$ might be the dense Boolean algebra or the Boolean algebra formed by the finite and cofinite sets – the completely atomic Boolean algebra.

Our notation is standard and follows Soare [1987].

2. The Basic Construction of an Atomless r-Maximal Set

Theorem 2.1 ([Robinson, 1967; Lachlan, 1968])**.** *There exists an atomless r-maximal set A.*

The goal of this section is to provide a proof of the above theorem. The basic construction we present is a modified version of John Norstad's construction (unpublished) which can be found in Lerman and Soare [1980] and Soare [1987]. In general, this section follows the course of Soare [1987, Section X.5] However, the construction we present will have two major modifications. Briefly, the modifications consist of laying out the markers on a tree and making each set built *simple* w.r.t. to its successors over its predecessor on the tree (to understand how we are using simple in this context we refer the reader to Definition 2.4).

Definition 2.2. A sequence of computably enumerable sets $\{H_n\}_{n\in\omega}$ forms a tower if $\cup_n H_n = \omega$ and for all n, $H_n \subset_\infty H_{n+1}$.

Lemma 2.3. *If A is a computably enumerable set and $\{H_n\}_{n\in\omega}$ is a tower such that $A = H_0$ and the requirements*

$$P_n \qquad W_n \subseteq^* H_n \text{ or } \overline{A} \subseteq^* W_n$$

holds for all $n > 0$, then A is atomless and r-maximal.

Proof. If $A \subseteq W_n$, then $A \subseteq W_n \subseteq H_n \subset_\infty H_{n+1}$ and hence W_n is not maximal. So A is atomless. Assume R splits $\overline{A}$. Let $R = W_i$ and $\overline{R} = W_j$, $i > j$. By P_i and P_j, $W_i \subseteq^* H_i$ and P_j, $W_j \subseteq^* H_j$. But then $W_i \cup W_j = \omega \subseteq^* H_i$, a contradiction. So A is r-maximal. □

Herrmann (unpublished) observed that the converse of the above lemma is true. In particular, if A is an atomless r-maximal set then there exists a tower; furthermore, this tower is computable in $\mathbf{0^{(3)}}$. Given H_n choose H_{n+1} such that $H_n \subset_\infty H_{n+1} \subset_\infty \omega$ and if $\overline{A} \not\subseteq^* W_{n+1}$ then $W_{n+1} \subseteq H_{n+1}$.

Let $T \subseteq \omega^{<\omega}$ be an infinite computable tree. For this section, we will restrict all lower Greek letters to T. Let $\alpha \to i_\alpha$ be a one-to-one onto computable function from T to ω such that $\alpha \prec \beta \implies i_\alpha < i_\beta$. Note that $i_\lambda = 0$, where λ is the empty string. For $\alpha \neq \lambda$, let α^- be α's immediate predecessor in the tree T. The trees will be used in conjunction with the simplicity requirements. In later sections we will vary the trees that we use.

Let $\{\Gamma_{\langle i_\alpha, j\rangle}\}_{\alpha \in T-\{\lambda\}, j\in\omega}$ be a collection of markers arranged along T. d_n^s will denote the element associated with Γ_n at stage s. We must ensure that for all $n > 0$,

N_n $$d_n =_{\text{dfn}} \lim_s d_n^s \text{ exists.}$$

We let $\overline{A}_s = \{d_n^s\}_{n\in\omega}$, $\overline{A} = \{d_n\}_{n\in\omega}$,

$$C_\alpha^s = C_{i_\alpha}^s = \{d^s_{\langle i_\alpha, j\rangle}\}_{j\in\omega},$$

$$C_\alpha = C_{i_\alpha} = \{d_{\langle i_\alpha, j\rangle}\}_{j\in\omega},$$

and

$$H_\alpha = H_{i_\alpha} = A \cup (\bigcup\{C_i : i \leq i_\alpha\}).$$

To ensure that A is an atomless r-maximal set (and to do slightly more) we will meet the following requirements for all α,

R_α $\quad A \cup \bigcup\{C_\beta : \beta \preceq \alpha\}$ is a computably enumerable set

and N_n, for all n. In Definition 2.9 we will define a computably enumerable set A_α and Lemma 2.11 we will show that $A_\alpha =^* A \cup \bigcup\{C_\beta : \beta \preceq \alpha\}$. Given that we meet these requirements and the definition of i_α, it is easy to see that the $\{H_n\}_{n\in\omega}$ form a tower. These requirements will also allow us to show that $A_\alpha \cap A_\beta = A_{\alpha\cap\beta}$, see Lemma 2.10.

To meet the above requirements it is enough to do a slightly modified Friedberg construction maximizing e-states measured w.r.t. to

$$U_{n,s} = \{x : (\exists t \leq s)[x = d^t_{\langle i,j\rangle} \;\&\; x \in W_{n,t} \;\&\; n < i\}.$$

Of course, $U_n = \bigcup\{U_{n,s} : s \in \omega\}$. The e-state of x at stage s, $\sigma(e, x, s) = \{i \leq e : x \in U_{i,s}\}$. The construction will ensure that d_i is in the highest possible i-state.

Definition 2.4 (Simplicity)**.** We say that A is *simple* w.r.t. a superset B over a subset C iff for every computably enumerable set W, if $W \cap (B - A)$ is infinite then $W \cap (A - C)$ is infinite. (For example, a simple set A, is simple w.r.t. to ω over $\emptyset$.) Clearly this is definable in $\mathcal{E}$ given parameters for A, B, and C.

Our goal is to make A_α simple w.r.t. A_β over A_{α^-} iff $\lambda \neq \alpha \prec \beta \in T$. Assuming we meet the requirements R_α, if $\beta|\alpha$ then A_α cannot be simple w.r.t. A_β over A_{α^-}; by Lemma 2.10, $A_\alpha \cap A_\beta = A_{\alpha\cap\beta}$, so A_β is not a superset of A_α. Hence it is enough to ensure that if $\lambda \neq \alpha \prec \alpha\widehat{\ }j \in T$ then A_α is simple w.r.t. $A_{\alpha\widehat{\ }j}$ over A_{α^-}. Therefore, it is enough to meet the following requirements for all α, j and n such that $\alpha\hat{\ }j \in T$ and $\alpha \neq \lambda$,

$$Q_{\alpha,j,n} \qquad |W_n \cap C_{\alpha\hat{\ }j}| = \infty \implies |W_n \cap C_\alpha| \geq 1$$

If we meet $Q_{\alpha,j,n}$, for all n, then $|W \cap C_{\alpha\widehat{\ }j}| = \infty$ implies $|W \cap C_\alpha| = \infty$, for all W.

The markers $\{\Gamma_{\langle i_\alpha, \langle j,n,l\rangle\rangle}\}_{l\in\omega}$ will be used to meet $Q_{\alpha,j,n}$. We only allow $\Gamma_{\langle i_\alpha, \langle j,n,l\rangle\rangle}$ to pull elements y at stage s which are in $C^s_{\alpha\hat{\ }j} \cap W_{n,s}$, assuming $Q_{\alpha,j,n}$ is not already met. If we pull y for the sake of $Q_{\alpha,j,n}$, we will call y a *witness*. y will remain a witness until its position in terms of the markers changes (in which case it is either pulled for some higher priority requirement or dumped into A).

There is a slight conflict between the requirements $Q_{\alpha,j,n}$ and R_α. To meet R_α we will pull to maximize d_m's m-state. We must be careful that we do not lose *every* witnesses $y = d^s_m$ for the sake of R_α. In this regard, we will find witnesses for $Q_{\alpha,j,n}$ which are in the maximum $\langle i_\alpha, j, n\rangle$-state. Hence we will only allow $\Gamma_{\langle i_\alpha, \langle j,n,l\rangle\rangle}$ to pull elements y at stage s if $y \in C^s_{\alpha\hat{\ }j} \cap W_{n,s}$ and y and $d^s_{\langle i_\alpha, \langle j,n,l\rangle\rangle}$ have the same $\langle i_\alpha, j, n\rangle$-state, assuming $Q_{\alpha,j,n}$ is not already met. And if d^s_m is a witness for $Q_{\alpha,j,n}$, we will only pull to increase its $\langle i_\alpha, j, n\rangle$-state (not its m-state).

To this end, we define $\tilde{e}_s$, $\tilde{e}_s = e$ unless d^s_e is a witness for $Q_{\alpha,j,n}$ at stage s in which case $\tilde{e}_s = \langle i_\alpha, j, n\rangle$.

The Construction 2.5. *Stage* $s = 0$. Let $d^0_n = n$.

Stage $s + 1$.

Part I: Find the least e such that for some i, $e < i \leq s$, $\sigma(\tilde{e}_s, d^s_i, s) > \sigma(\tilde{e}_s, d^s_e, s)$. Choose such an i with $\sigma(\tilde{e}_s, d^s_i, s)$ as large as possible and define $\hat{d}^s_e = d^s_i$ (Γ_e *pulls* d^s_i for $R_{\tilde{e}_s}$). Enumerate d^s_k, $e \leq k \leq s$, $k \neq i$, into A (the standard dump). Let $\hat{d}^s_j = d^s_j$, for all $j < e$, and $\hat{d}^s_{e+k} = d^s_{s+k}$, for all $k > 0$.

Part II: Find the least e, j, n, and i (in that order) such that for some l, $e = \langle i_\alpha, \langle j, n, l\rangle\rangle$; $e < i \leq s$; for some k, $i = \langle i_{\alpha\hat{\ }j}, k\rangle$ [note that $i_\alpha < i_{\alpha\hat{\ }j}$]; $\sigma(\langle i_\alpha, j, n\rangle, \hat{d}^s_i, s) = \sigma(\langle i_\alpha, j, n\rangle, \hat{d}^s_e, s)$; for all $l' \leq l$, $\hat{d}^s_{\langle i_\alpha, \langle j,n,l'\rangle\rangle} \notin W_{n,s}$ [hence there are no smaller witnesses that $Q_{\alpha,j,n}$ is already met]; and $\hat{d}^s_i \in W_{n,s}$. Define $d^{s+1}_e = \hat{d}^s_i$ (Γ_e *pulls* the *witness* $\hat{d}^s_i$ for $Q_{\alpha,j,n}$ at stage s; we also say that $Q_{\alpha,j,n}$ acts at stage s). Enumerate

$\hat{d}^s_k$, $e \le k \le s$, $k \ne i$, into A (another dump). Let $d^{s+1}_k = \hat{d}^s_k$, for all $k < e$, and $d^{s+1}_{e+k} = \hat{d}^s_{s+k}$, for all $k > 0$.

Lemma 2.6. *For every e, $d_e = \lim_s d^s_e$ exists and $\overline{A} = \{d_e\}_{e\in\omega}\}$.*

Proof. By induction on e. Assume the lemma holds for $i < e$. Choose s such that for all $i < e$, $d_i = d^s_i$. Part I of the construction can only apply to Γ_e at most 2^e times after stage s. Choose a stage t such that Part I never applies to Γ_e after stage t. Part II can only apply to Γ_e at most once after stage t. □

Lemma 2.7. *For all n, $U_n \cap \overline{A}$ is finite or $\overline{A} \subseteq^* U_m$.*

Proof. By induction on n. Using the construction and the definition of U_0, it is easy to see that U_0 is infinite iff $\overline{A} \subseteq U_0$. Now fix $n > 0$ and suppose for all $m < n$ that

$$U_m \cap \overline{A} \text{ is finite or } \overline{A} \subseteq^* U_m. \tag{2.1}$$

Let $\sigma = \{m < n : \overline{A} \subseteq^* U_m\}$ (the $(n-1)$-state of $\overline{A}$). Let $U_\sigma = \bigcup\{U_m : m \in \sigma\}$. If $U_n \cup U_\sigma$ is finite then $U_n \cap \overline{A}$ is finite and $W_n \subseteq^* H_n$, by the definition of U_n. By the definition of $\tilde{e}_s$, we have that for almost all e and for all s, $\tilde{e}_s > n$. So if $U_n \cap U_\sigma$ is infinite then, by the construction, $\overline{A} \subseteq^* U_n$. Thus we can continue the induction. □

Lemma 2.8. *Assume that $\alpha\hat{\ }j \in T$ and $\alpha \ne \lambda$. Then for all n, $Q_{\alpha,j,n}$ is met. Furthermore, $Q_{\alpha,j,n}$ only acts finitely often.*

Proof. Assume not. Fix some n such that $|W_n \cap C_{\alpha\hat{\ }j}| = \infty$ but $|W_n \cap C_\alpha| = \emptyset$. Let σ be the $\langle i_\alpha, j, n\rangle$-state of $\overline{A}$ (see Equation 2.1). Choose n' such that $\sigma(\langle i_\alpha, j, n\rangle, d_m) = \sigma$, for all $m \ge n'$. Find the least l such that $m = \langle i_\alpha, \langle j, n, l\rangle\rangle \ge n'$. Choose the least s such that $d^s_m = d_m$. Since $|W_n \cap C_{\alpha\hat{\ }j}| = \infty$, there must exist an i and a stage $t > s$ such that $m < i \le t$, $i = \langle i_{\alpha\hat{\ }j}, k\rangle$, for some k, $\sigma(\langle i_\alpha, j, n\rangle, \hat{d}^t_m, t) = \sigma(\langle i_\alpha, j, n\rangle, \hat{d}^t_i, s) = \sigma$ and $\hat{d}^t_i \in W_{n,t}$. Γ_m will pull $\hat{d}^s_i$ for $Q_{\alpha,j,n}$ at stage t. This contradicts the choice of s. □

Definition 2.9. Fix $\alpha \in T$. Let $\sigma = \{i \le i_\alpha : \overline{A} \subseteq^* U_i\}$ (by 2.1 this is the i_α-state of $\overline{A}$). Choose n' such that $\sigma(i_\alpha, d_m) = \sigma$, for all $m \ge n'$. Choose s' such that for $d_m = d^{s'}_m$, for all $m < n'$ and if $\langle i_\beta, j, n\rangle < i_\alpha$ then $Q_{\beta,j,n}$ never acts after stage s'. Define a computably enumerable set $A_\alpha = A_{i_\alpha}$ as follows: $y = d^t_k$ enters A_α at stage $t+1 > s'$ if either y enters A at stage $t+1$ or $n' \le k \le t$,

$$\text{there is a } \beta, j \text{ such that } k = \langle i_\beta, j\rangle \text{ and } \beta \preceq \alpha,$$

$$\text{for all } m, \text{ if } n' \le m \le k \text{ then } \sigma(i_\alpha, d^t_m, t) = \sigma, \text{ and}$$

$$\text{for all } m \leq k, d_m^t = d_m^{t+1}.$$

Lemma 2.10. *Let $y = d^t_{\langle i_\alpha, l\rangle}$ enter A_α at stage $t+1$. Then for all stages $s \geq t+1$, either $y \in A_s$ or $y = d^s_{\langle i_\beta, l\rangle}$, for some $\beta \preceq \alpha$. Furthermore, if $y = d^s_{k'}$ then for all m, if $n \leq m \leq k'$ then $\sigma(i_\alpha, d^s_m, s) = \sigma \restriction (i_\alpha + 1)$.*

Proof. By induction on s. If y's position in terms of the markers does not change from stage $s-1$ to stage s then, by the definitions of A_α and U_n, the lemma holds at stage s. By the induction hypothesis, the construction, the definitions of σ, n', s' and A_α, no Γ_m will pull y for Part I at stage s. However, some Γ_m could pull y for $Q_{\beta^-,j,l}$. In this case, clearly $\beta^- \prec \beta$ and, by the definitions of U_n and n', the i_α state of the balls less than y which are not dumped into A does not change. The only other possible movement of y is into A. □

Lemma 2.11. *For all α, $A_\alpha =^* A \cup \bigcup\{C_\beta : \beta \preceq \alpha\}$. Hence for all e, H_e is a computably enumerable set.*

Proof. By the definition of A_α and Lemma 2.7, $A \cup \bigcup\{C_\beta : \beta \preceq \alpha\} \subseteq^* A_\alpha$. Assume y enters A_α at stage $t+1$. Then, by the above lemma and the definition of i_α, either $y \in A$ or for some $\beta \preceq \alpha$, y in C_β. □

We will wrap up this section by noting that the construction presents proves the following theorem:

Theorem 2.12. *Let $T \subseteq \omega^{<\omega}$ be an infinite computable tree. Let $\alpha \to i_\alpha$ be a one-to-one onto computable function from T to ω such that $\alpha \prec \beta$ implies $i_\alpha < i_\beta$. Then there is an atomless r-maximal set A and computably enumerable sets $\{A_\alpha\}_{\alpha \in T}$ containing A such that*

(i) the computably enumerable sets $\{H_e\}_{e\in\omega}$ form a tower, where $H_e = \bigcup\{A_\alpha : i_\alpha \leq e\}$,

(ii) if $\beta^- = \alpha$ and $\alpha^- = \gamma$, then the sets $A_\beta - A_\alpha$ and $A_\alpha - A_\gamma$ are infinite and A_α is simple w.r.t. to A_β over A_γ, and

(iii) $A_\alpha \cap A_\beta = A_{\alpha\cap\beta}$.

(iv) Let $A_{\mathcal{C}} = \bigcup\{A_\alpha : \alpha \in \mathcal{C}\}$. $A_{\mathcal{C}}$ is computably enumerable iff $\mathcal{C}$ is finite and closed under initial segments or $\mathcal{C} = T$ (and $A_{\mathcal{C}} =^* \omega$).

(For (iv) note that every coinfinite computably enumerable superset of A is contained in some H_e. Hence if $\mathcal{C} \neq T$ and is infinite then $A_{\mathcal{C}}$ is not a computably enumerable set.)

3. Infinitely many non-isomorphic $\mathcal{L}^*(A)$

Consider an atomless r-maximal set A built in the fashion of the last section using the computable tree T. The sets $\{A_\alpha : \alpha \in T\}$ form a substructure of $\mathcal{L}^*(A)$. We will let A^T denote this atomless r-maximal set and $\{A^T_\alpha : \alpha \in T\}$ the corresponding substructure.

In this section we will explore how this substructure can distinguish A from other atomless r-maximal sets built in the same fashion. We will explore how different trees give rise to atomless r-maximal sets whose principal filters are non-isomorphic.

In the next section we will discuss some partial results about how similar trees give rise to isomorphic principal filters and the relationship between the atomless r-maximal sets we construct and the ones which were constructed previously.

Notation 3.1. Let F and T be trees. The tree $F(T)$ is formed by adding T above any node in F; $F(T) = F \cup \{\beta\frown\alpha : \beta \in F \;\&\; \alpha \in T\}$. Let $T^0 = \{0^n : n \in \omega\}$, $T^1 = \{0^n, 0^n\frown j : n, j \in \omega\}$, and $T^{n+1} = T^1(T^n)$. Let $T^\triangle = \{i : i \in \omega\}$ (this is all the strings of length 1 in $\omega^{<\omega}$). We will call $T^\triangle$ the *triangle* tree and T^0 the *chain*. The atomless r-maximal sets constructed using $T^\triangle$ and T^n are called $A^\triangle$ and A^n respectively and the corresponding substructures are called $\{A^\triangle_\alpha : \alpha \in T^\triangle\}$ and $\{A^n_\alpha : \alpha \in T^n\}$.

Definition 3.2. Let A be an atomless r-maximal set. We call the sets $\{V_i : i \in \omega\}$ a *triangle over* V *in* A if $A \subseteq V$; for all i, $V \subset_\infty V_i$; for all i, j, if $i \neq j$ then $V_i \cap V_j =^* V$; and for all computably enumerable sets W, if W is coinfinite then there is a largest n such that $(V_n - V) \cap W$ is nonempty.

If $\{\alpha\frown j : j \in \omega\} \subseteq T$ then the sets $\{A^T_{\alpha\frown j} : j \in \omega\}$ form a triangle over A^T_α in A^T. This follows from the fact that the sets $\{A^T_{\alpha\frown j} : j \in \omega\}$ are part of the tower used to show A^T is an atomless r-maximal set. Hence the sets $\{A^\triangle_i\}$ form a triangle over $A^\triangle$ in $A^\triangle$. If the sets $\{V_i : i \in \omega\}$ are a triangle over V in A and Φ is an isomorphism, taking $\mathcal{L}^*(A)$ to some other $\mathcal{L}^*(B)$, then the sets $\{\Phi(V_i) : i \in \omega\}$ form a triangle over $\Phi(V)$ in B.

The last clause in the above definition is necessary: Assume $A \subset_m B$. Then we can split B into infinitely many sets B_i such that $A \subset_\infty B_i$ by using the Owings Splitting Theorem infinitely many times. These sets will satisfy all but the last clause in the above definition.

Definition 3.3. Let A be an atomless r-maximal set. We call the sets $\{S_i : i \in \omega\}$ a *spine in* A if for all i, $A \subset_\infty S_i \subset_\infty S_{i+1}$; for all $i > 0$ and for all computably enumerable sets W, if $W \cap (S_{i+1} - S_i)$ is infinite then $W \cap (S_i - S_{i-1})$ is infinite; and for all computably enumerable sets W, if W is coinfinite then there is a largest n such that $(S_n - S_{n-1}) \cap W$ is nonempty.

If the strings $\{\alpha_i : i \in \omega\}$ form an infinite path through a tree T then the sets $\{A^T_{\alpha_i} : i \in \omega\}$ form a spine in A^T (see Definition 2.4 and Lemma 2.8). Hence the sets $\{A^0_{0^n} : n \in \omega\}$ form a spine in A^0. If the sets $\{S_i : i \in \omega\}$ are a spine in A and Φ is an isomorphism, taking $\mathcal{L}^*(A)$ to some other $\mathcal{L}^*(B)$, then the sets $\{\Phi(S_i) : i \in \omega\}$ form a spine in B. If $i \neq 0$ then S_i is simple w.r.t. S_{i+1} over S_{i-1}.

The following lemma contains the key idea for this paper! Informally, it says that one cannot use a triangle to build a spine and a spine to build a triangle.

Lemma 3.4. *Let A be an atomless r-maximal set. Assume that $\{V_i : i \in \omega\}$ forms a triangle over V in A and that $\{S_i : i \in \omega\}$ a spine in A. Then there are n and k such that for all m and all l, if $(S_m - S_{m-1}) \cap (V_l - V)$ is infinite then $m \leq n$ and $l \leq k$. Furthermore, $V \cap \bigcup_{i\in\omega} S_i \subseteq^* S_n$.*

Proof. There is a largest n' such that $(S_{n'} - S_{n'-1}) \cap V$ is infinite. So

$$V \cap \bigcup_{i\in\omega} S_i \subseteq^* S_{n'}. \tag{3.1}$$

There is a largest k such that $S_{n'+1} \cap (V_k - V)$ is infinite. Therefore

$$S_{n'+1} \cap \bigcup_{i\in\omega} V_i \subseteq^* \bigcup_{j\leq k} V_j. \tag{3.2}$$

There is a largest n such that $(S_n - S_{n-1}) \cap \bigcup_{j\leq k} V_j$ is infinite. So $n \geq n'$ and

$$\bigcup_{j\leq k} V_j \cap \bigcup_{i\in\omega} S_i \subseteq^* S_n.$$

Hence the lemma holds for $l \leq k$.

Fix $l > k$. Suppose that $(S_m - S_{m-1}) \cap (V_l - V)$ is infinite. Then, by the choice of k, $m > n'$. By the definition of a spine, $(S_{n'+1} - S_{n'}) \cap V_l$ is infinite. Then, by Equation 3.1, $(S_{n'+1} - S_{n'}) \cap (V_l - V)$ is infinite. But, by Equation 3.2, $(S_{n'+1} - S_{n'}) \cap (V_l - V) \subseteq^* \bigcup_{j\leq k} V_j$ Since, by the definition of a triangle, $(V_l \cap \bigcup_{j\leq k} V_j) - V$ is finite, $(S_{n'+1} - S_{n'}) \cap (V_l - V)$ is finite. Contradiction. □

Let Φ be an embedding of $\{A^n_\alpha\}_{\alpha\in T^n}$ into some $\mathcal{L}^*(A)$. We say that Φ is a T^n-*embedding* if Φ preserves triangles and spines. If Φ is an isomorphism between $\mathcal{L}^*(A^n)$ and $\mathcal{L}^*(A^T)$ then Φ is a T^n-embedding into $\mathcal{L}^*(A^T)$.

Assume that Φ is a T^n-embedding of $\{A^n_\alpha\}_{\alpha\in T^n}$ into some $\mathcal{L}(A^T)$. For ease of notation, we let $\Phi(A^n_\alpha) = \tilde{A}^n_\alpha$, for all $\alpha \in T^n$. Let $I(\Phi) =$

$\{\beta : \exists\alpha[(A^T_\beta - A^T_{\beta^-}) \cap \tilde{A}^n_\alpha$ is infinite$]\}$. Note that $\tilde{A}^n_\alpha \subseteq^* \bigcup\{A^T_\beta : (A^T_\beta - A^T_{\beta^-}) \cap \tilde{A}^n_\alpha$ is infinite$\}$ and the set of such β's for any fixed α is finite.

Lemma 3.5. *Assume that Φ is a T^{n+1}-embedding into $\mathcal{L}(A^T)$. Then for all finite trees F, $I(\Phi) \not\subseteq^* F(T^n)$.*

Proof. By induction on n. Assume that we have a T^{n+1} embedding Φ, where $I(\Phi) \subseteq F(T^n)$ for some finite tree F. We will derive a contradiction. γ will always be used to denote a node of F. Note that F is a subtree of $F(T^n)$ and without loss of generality we can assume that $F \subset T$.

The Base Case: $n = 0$. For all γ, there is a n_γ such that either $\gamma^\frown 0^{n_\gamma+1} \notin T$ or the sets $\{A^T_{\gamma^\frown 0^m}\}_{m\in\omega}$ form a spine in $\mathcal{L}(A^T)$. The sets $\{\tilde{A}^1_j\}_{j\in\omega}$ form a triangle over $\tilde{A}^1$. By Lemma 3.4, for all γ (where n_γ is not already defined), there is a n_γ and k_γ such that if $(A^T_{\gamma^\frown 0^m} - A^T_{\gamma^\frown 0^{m-1}}) \cap (\tilde{A}^1_l - \tilde{A}^1))$ is infinite then $m \leq n_\gamma$ and $l \leq k_\gamma$. Hence for all j, $\tilde{A}^1_j - \tilde{A}^1 \subseteq^* \bigcup_\gamma A^T_{\gamma^\frown 0^{n_\gamma}}$. Contradiction (this embedding does not preserve triangles).

The Inductive case: Suppose the lemma holds for n. In this case, we will use Φ to construct a T^n-embedding which violates the lemma. Briefly the idea is find a node δ in T^{n+1} such that the tree above δ is isomorphic to T^n and the images of the sets constructed above δ violates the lemma for n.

Now the sets $\{\tilde{A}^{n+1}_j\}_{j\in\omega}$ form a triangle over $\tilde{A}^{n+1}$ in A^T. For all γ, let f_γ be the rightmost (0 is to the right of 1 in our trees) infinite branch extending γ in $I(\Phi) \cap T$, if such a branch exists. We will now restrict γ to those nodes in F where f_γ exists. The sets $\{A^T_\beta\}_{\beta\in f_\gamma}$ form a spine in $\mathcal{L}^*(A^T)$.

By Lemma 3.4, for all γ there is a n_γ and k_γ such that if $\beta \subset f_\gamma$ and $(A^T_\beta - A^T_{\beta^-}) \cap (\tilde{A}^{n+1}_l - \tilde{A}^{n+1})$ is infinite then $|\beta| \leq n_\gamma$ and $l \leq k_\gamma$. Let $k = \max\{k_\gamma, n_\gamma\} + 1$.

Assume $k \subseteq \alpha$ and $(A^T_\beta - A^T_{\beta^-}) \cap \tilde{A}^{n+1}_\alpha$ is infinite. We will show by induction on α that $f_\gamma \restriction k \not\subseteq \beta$. Assume otherwise. Then, by the simplicity requirements, $\tilde{A}^{n+1}_\alpha \cap (A^T_{f_\gamma\restriction k} - A^T_{f_\gamma\restriction(k-1)})$ is infinite. Clearly this is false for $\alpha = k$. Hence the base case of our induction holds. By the inductive hypothesis and the choice of k, $\tilde{A}^{n+1}_{\alpha^-} \cap (A^T_{f_\gamma\restriction k} - A^T_{f_\gamma\restriction(k-1)})$ is finite. So $(\tilde{A}^{n+1}_\alpha - \tilde{A}^{n+1}_{\alpha^-}) \cap (A^T_{f_\gamma\restriction k} - A^T_{f_\gamma\restriction(k-1)})$ is infinite. Now, again by the simplicity requirements, $(\tilde{A}^{n+1}_k - \tilde{A}^{n+1}) \cap A^T_{f_\gamma\restriction k}$ is infinite. This contradicts the choice of k.

Let $F_1 = F \cup \{\beta : \gamma \subseteq \beta \subseteq f_\gamma \restriction (k-1)\} \cup \{\beta : \tilde{A}^{n+1}_k \cap (A^T_\beta - A^T_{\beta^-})$ is infinite$\}$. At this point we will lift the above restrictions on γ and replace them with the restriction that $\gamma \in F_1$.

The sets $\{\tilde{A}^{n+1}_{k\frown 0^m}\}_{m\in\omega}$ form a spine in A^T. For all γ either there is a largest k_γ such that $\gamma\frown(k_\gamma+1) \notin T$ or the sets $\{A^T_{\gamma\frown j}\}_{\gamma\frown j \in T}$ form a triangle over A^T_γ in A^T.

By Lemma 3.4, for all γ (where k_γ is not already defined), there is a n_γ and k_γ such that if $(\tilde{A}^{n+1}_{k\frown 0^m} - \tilde{A}^{n+1}_{k\frown 0^{m-1}}) \cap (A^T_{\gamma\frown l} - A^T_\gamma)$ is infinite then $m \leq n_\gamma$ and $l \leq k_\gamma$. Let $q = \max\{n_\gamma, k_\gamma\} + 1$.

Assume $k\frown 0^q \subseteq \alpha$ and $(A^T_\beta - A^T_{\beta^-}) \cap \tilde{A}^{n+1}_\alpha$ is infinite. We will show by induction on α that, for all $l > q$, $\gamma\frown l \not\subseteq \beta$. Assume otherwise. Then, by the simplicity requirements, $\tilde{A}^{n+1}_\alpha \cap (A^T_{\gamma\frown l} - A^T_\gamma)$ is infinite. By the inductive hypothesis and the choice of q and F_1, $\tilde{A}^{n+1}_{\alpha^-} \cap (A^T_{\gamma\frown l} - A^T_\gamma)$ is finite. So $(\tilde{A}^{n+1}_\alpha - \tilde{A}^{n+1}_{\alpha^-}) \cap (A^T_{\gamma\frown l} - A^T_\gamma)$ is infinite. Clearly this is false for $\alpha = k\frown 0^q$. Now, again by the simplicity requirements, $(\tilde{A}^{n+1}_{k\frown 0^q} - \tilde{A}^{n+1}_{k\frown 0^{q-1}}) \cap A^T_{\gamma\frown l}$ is infinite. By Lemma 3.4, $(\tilde{A}^{n+1}_{k\frown 0^q} - \tilde{A}^{n+1}_{k\frown 0^{q-1}}) \cap A^T_\gamma$ is finite. But this implies that $(\tilde{A}^{n+1}_{k\frown 0^q} - \tilde{A}^{n+1}_{k\frown 0^{q-1}}) \cap (A^T_{\gamma\frown l} - A^T_\gamma)$ is infinite which contradicts the choice of q.

Let $F' = F \cup F_1 \cup \{\gamma\frown l : l \leq q\}$. Let $\delta = k\frown 0^q$. If either $\alpha \subseteq \delta$ or $\delta \subset \alpha$ and $\tilde{A}^{n+1}_\alpha \cap (A^T_\beta - A^T_{\beta^-})$ is infinite then β extends some node of F' or is in F'. By the definition of T^{n+1}, $\{\beta : \beta \in I(\Phi)\ \&\ \beta$ extends some node in $F'\} \subseteq F'(T^{n-1})$.

Now clearly there is a T^n-embedding Ψ into $\mathcal{L}(A^{n+1})$ such that $I(\Psi) = \{0^p, 0^p\frown k, \alpha : \delta \subseteq \alpha\}$. (By the definition of T^{n+1}, $\{\alpha : \delta \subseteq \alpha\} \approx T^n$. Use this to get Ψ.) So the composition $\Phi \circ \Psi$ is a T^n-embedding into $\mathcal{L}(A^T)$. But $I(\Phi \circ \Psi) \subseteq F'(T^{n-1})$. Contradiction. □

It is possible to improve the above lemma to show that $I(\Phi)$ cannot be embedded (as a tree) into $F(T^n)$. The following is our main result:

Theorem 3.6. *For all n, $\mathcal{L}^*(A^{n+1})$ is not isomorphic to $\mathcal{L}^*(A^n)$. Hence there are infinitely many principal filters formed by atomless r-maximal sets.*

Proof. If $\mathcal{L}^*(A^{n+1})$ is isomorphic to $\mathcal{L}^*(A^n)$, we would have a T^{n+1}-embedding Φ into $\mathcal{L}^*(A^n)$ such that $I(\Phi) \subseteq T^n$. Contradiction. □

4. Towards a classification

4.1. **The case of $A^\triangle$.** We claim that $\mathcal{L}^*(A^\triangle)$ is isomorphic to $\mathcal{L}^*(A)$, where A is the atomless r-maximal set constructed in Soare [1987, Theorem X.5.4]. In fact, we will show something much stronger.

Definition 4.1. We say an atomless r-maximal set A is a *triangle* set if there are infinitely many computably enumerable sets A_i such that $A \subset_\infty A_i$, if $i \neq j$, then $A_i \cap A_j =^* A$ and for all computably enumerable sets W, if W is coinfinite then there is some n such that $W \subseteq^* \widehat{A}_n =_{\mathrm{dfn}} \bigcup_{i \leq n} A_i$.

It is clear that triangle sets are atomless r-maximal sets. The $\{\widehat{A}_n\}_{n\in\omega}$ form a tower. No superset of a triangle set A is simple w.r.t. to any other superset of A. $A^\triangle$ is a triangle set and so is the atomless r-maximal set constructed in Soare [1987, Theorem X.5.4]. If the sets $\{A_i\}_{i\in\omega}$ form a triangle over A in $\mathcal{L}(A)$ and $\bigcup_i A_i =^* \omega$, then A is a triangle set. If B is a coinfinite superset of a triangle set then B is a triangle set. Furthermore, if A and B are triangle sets then $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$ are isomorphic: Map A_i to B_i. Since $A \subset_m A_i$ and $B \subset_m B_i$, by Maass and Stob [1983], we know that $\mathcal{E}(A_i - A) \approx \mathcal{E}(B_i - B)$ via Φ_i. If W is cofinite then map W to ω. Otherwise find the least n such that $W \subseteq^* \bigcup_{i\leq n} A_i$ and map W to $\bigcup_{i\leq n} \Phi_i(W \cap A_i)$. This induces an isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$.

Theorem 4.2. *The set $\{A : A$ is a triangle set$\}$ is definable in $\mathcal{E}$.*

Proof. We will show that A is a triangle set iff the following property holds for A in $\mathcal{E}$:

$$P(A) \qquad \begin{gathered} A \text{ is atomless } r\text{-maximal set } \& \\ (\forall V \supseteq A)(\forall W \supseteq A)(\exists B \supseteq A) \\ [V \cap B = A \;\&\; W \subseteq^* V \cup B], \end{gathered}$$

where the quantifiers range over all computably enumerable coinfinite sets.[1]

If: Given V find the least $m > n$ such that $V \subseteq^* \bigcup_{i\leq m} A_i$. Given W find p such that $W \subseteq^* \bigcup_{i\leq p} A_i$. Let $B = A \cup \bigcup_{m<i\leq p} A_i$.

Only if: We will construct the desired A_i's by induction. Assume that we have $\{A_i\}_{i<n}$. Let $V = A \cup \bigcup_{i<n} A_i$. Let m be the least such that $V \subset_\infty V \cup W_m \subset_\infty \omega$. Let $W = W_m$ and A_n the corresponding B. □

Note that if $P(A)$ holds then for all $C \supseteq A$, $P(C)$ holds. Also the sets $\{A_i\}$ are computable in $\mathbf{0^{(4)}}$. Hence if A and B are two triangle sets then $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$ are isomorphic via an isomorphism computable in $\mathbf{0^{(4)}}$.

[1]Originally our property was slightly more complex. This simplified version is due to Mike Stob.

4.2. **r-maximal sets with spines.** Let A be an atomless r-maximal set with a spine. Then, by Lemma 3.4, no superset of A is a triangle set. (Let $\{A_i\}_{i\in\omega}$ form a triangle over some superset B of A and $\{S_i\}_{i\in\omega}$ form a spine in A. Let n be as given in Lemma 3.4 then $S_{n+1} \not\subseteq^* A \cup \bigcup_i A_i$. Hence the sets $\{A_i\}_{i\in\omega}$ do not witness that B is a triangle set.) Therefore, the property P does not holds for any superset of A.

Theorem 4.3. *(Stob) An atomless r-maximal set A has a spine iff*

$$(\forall B \supseteq A)(\exists C \supseteq B)(\forall W)[W \cap \overline{C} \neq^* \emptyset \rightarrow W \cap (C - B) \neq^* \emptyset],$$

where the quantifiers range over all computably enumerable sets.

Proof. Suppose that $V_0 \subset V_1 \subset V_2 \subset \cdots$ is the spine. Given B let n be such that $B \subseteq V_n$. Let $C = V_{n+1}$. Given W such that $W \cap \overline{C} \neq^* \emptyset$ either $\overline{C} \subseteq^* W$ in which case $W \supseteq (C - B) \neq^* \emptyset$ (by Lemma 1.4) or $W \cap (V_{k+1} - V_k) \neq^* \emptyset$ some $k > n$. But then $W \cap (V_{n+1} - V_n)$ is infinite and contained in $C - B$.

Suppose on the other hand the condition. Let $A = V_0$. Given V_n, let m be least such that $V_n \subset_\infty V_n \cup W_m \subset_\infty \omega$, let $B_{n+1} = V_n \cup W_m$ and let V_{n+1} be the set C_{n+1} given by the condition with $B = B_{n+1}$. Obviously, for all n, either $W_n \supseteq \overline{A}$ or $W_n \subseteq V_{n+1}$. Suppose that W is an computably enumerable set such that $W \cap (V_{n+1} - V_n)$ is infinite. Now $V_n = C_n$ as defined by our recursion. Thus $W \cap \overline{C}_n$ is infinite. Thus we have that $W \cap (C_n - B_n)$ is infinite. But $W \cap (C_n - B_n) \subseteq V_n - V_{n-1}$. ☐

4.3. **Another definable class.** We can use $P(A)$ to define another class of atomless r-maximal sets: Let $R(C)$ be the formula which says that C is an atomless r-maximal set, $\neg P(C)$ and there exists A such that $C \subset A$ and $P(A)$.

Let $T = \{0, 0^\frown j : j \in \omega\}$. In this case, A_0 is a triangle set but A_0 is simple w.r.t. $A_{0^\frown j}$ (over A^T), for all j (see Definition 2.4 and Lemma 2.8). Hence $R(A^T)$ holds. The atomless r-maximal set C constructed in Soare [1987, Exercises X.5.8] also satisfies R.

A property similar to $R(C)$ was used to found the first example of two atomless r-maximal sets A and B such that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$ are not isomorphic [see Soare, 1987, Exercises X.5.8]. We conjecture that if $R(C)$ and $R(\tilde{C})$ hold then $\mathcal{L}^*(C)$ and $\mathcal{L}^*(\tilde{C})$ are isomorphic. In fact, it is open whether $\mathcal{L}^*(A^T)$ and $\mathcal{L}^*(C)$ are isomorphic (where T is as above and C is the atomless r-maximal set constructed in Soare [1987, Exercises X.5.8]).

We will spend the rest of this section exploring what sets satisfy the property R. We would like to classify those trees T where $R(A^T)$ holds.

If T has an infinite branch then, by the work in the above subsection, we know that $R(A^T)$ does not hold (and does not hold for any superset of A^T).

Lemma 4.4. *Let T be an infinite computable tree T such that T has no infinite branch (with at least one infinitely branching node other than the empty string). Then T has only finitely many infinitely branching nodes iff $R(A^T)$ holds.*[2]

Proof. If: $\mathcal{C}$ be the set of all nodes α in T such that α is an infinitely branching node and no extension of α in T is infinitely branching. By our assumption $\mathcal{C}$ is non-empty. Let $A = \bigcup_{\alpha\in\mathcal{C}} A^T_\alpha$. The sets

$$\{\bigcup_{\gamma\in\mathcal{C}}\{\bigcup_{\gamma^\frown i\subseteq\gamma'} A_{\gamma'}\}\}_{i\in\omega}$$

(where it is understood that $\gamma' \in T$) witness that A is a triangle set. (By Theorem 2.12 (iv) and our assumption about T, A and all the above mentioned sets are computability enumerable.) $P(A^T)$ fails but A witnesses that $R(A^T)$ holds.

Only If: Let A be a superset of A^T such that A is a triangle set. By Theorem 2.12 (iv), there is some non-empty finite set of nodes in T, $\mathcal{C}$, such that $A \subseteq^* A_{\mathcal{C}}$. Since every computably enumerable coinfinite superset of a triangle set is a triangle set, $A_{\mathcal{C}}$ is a triangle set. So WLOG we can assume that $A = A_{\mathcal{C}}$. Let $\{V_i\}_{i\in\omega}$ be the sets which witness the fact A is a triangle set.

Let γ be an infinitely branching node $\gamma \in T$ which is not in $\mathcal{C}$. So $A_\gamma - A$ is infinite. Hence there is a least n such that $A_\gamma \subseteq^* \widehat{A}_n =_{\text{defn}} \bigcup_{i\leq n} V_i$. By simplicity, if γ' is any proper extension of γ in T then $A_{\gamma'} \subseteq^* \widehat{A}_n$. (Assume for some i of the least m such that $A_{\gamma^\frown i} \subseteq^* \widehat{A}_m$ is greater than n. But then $V_m \cap (A_{\gamma^\frown i} - A_\gamma)$ is infinite and $V_m \cap (A_\gamma - A_{\gamma'})$ is finite. The argument is by induction for other extensions of γ.) But, by Theorem 2.12, there is a finite set $\mathcal{C}'$ such that $\widehat{A}_n \subseteq^* A_{\mathcal{C}'}$. Contradiction. Hence every infinite branching node is in $\mathcal{C}$ and there are most finitely such nodes. □

4.4. **Spines and beyond.** We have no good ideas how to classify atomless r-maximal sets with spines. The following is an outline of a plan to come up with a classification of principal filters formed by atomless r-maximal sets (with spines).

First show that there is some (hopefully with some degree of effectiveness – this might be needed below) way to go from an atomless r-maximal set to the substructures (the $\{A_\alpha\}$'s) we used to show

[2]The "only if" direction is due to Mike Stob

these sets form infinitely many non-isomorphic principal filters. Perhaps identifying all the triangles and spines and how they are arranged together is enough. In this regard, we do not know if every atomless r-maximal set has a triangle or spine of supersets. Under some additional assumption (see below) this would say that our construction constructs an atomless r-maximal set of every type. One thing to consider in this is the effect that turning off the simplicity requirements at various levels in the tree has on the construction (ie. not using $Q_{\alpha,j,n}$ for α of various lengths). We conjecture this just collapses these levels of the substructure.

Then show that atomless r-maximal sets with isomorphic substructures (maybe with an additional effectiveness condition on the isomorphism) form isomorphic principal filters. This should be possible since we know, by Maass and Stob [1983], the intervals of supersets of an atomless r-maximal set are isomorphic to $\mathcal{M}$. Hence one should be able to combine this and the isomorphism between the substructures together in one argument to build the desired isomorphism between the principal filters (it is here that we may need a good deal of effectiveness). A good first step in this direction would be to show that if one builds two atomless r-maximal set A and B using our construction and the chain as the tree then these sets form isomorphic principal filters. A next step would be to show that if two atomless r-maximal sets A and B are built using the same tree then they form isomorphic principal filters. Perhaps this remains true if A and B are built using different but very "similar" trees. For example, for all n, and for all finite trees F, $\mathcal{L}^*(A^n)$ and $\mathcal{L}^*(A^{F(T^n)})$ may be isomorphic and in which case we would consider T^n and $F(T^n)$ as "similar" trees.

Even if work along the lines we outlined above does not lead to a classification of the principal filters formed by atomless r-maximal sets, it should be enough to prove the following conjecture:

Conjecture 4.5. *The set* $\{\langle r_1, r_2\rangle :$ *where* r_i *is an index for an atomless r-maximal set* A_i *and* $\mathcal{L}(A_1) \approx \mathcal{L}(A_2)\}$ *is* Σ^1_1 *complete.*

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