# ISOMORPHISMS OF SPLITS OF COMPUTABLY ENUMERABLE SETS

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ABSTRACT. We show that if $A$ and $\widehat{A}$ are automorphic via $\Phi$ then the structures $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ are $\Delta^0_3$-isomorphic via an isomorphism $\Psi$ induced by $\Phi$. Then we use this result to classify completely the orbits of hhsimple sets.

## 1. INTRODUCTION

We will work in the structure of the computably enumerable sets. The language is just inclusion, $\subseteq$. This structure is called $\mathcal{E}$.

Our understanding of automorphisms of $\mathcal{E}$ is unique to $\mathcal{E}$. In most structures with nontrivial automorphisms we can construct automorphisms via the normal "back and forth" argument. But this is not the case with $\mathcal{E}$. To construct automorphisms we use the properties of being *well-visited* and *well-resided*. Well-visited is $\Pi^0_2$ and not being well-resided is $\Sigma^0_3$ (we use the negation). Since the complexity of these properties is at most $\Sigma^0_3$, the construction of the desired automorphism can be placed on a tree. (We will not discuss the details on this placement nor of the construction of an automorphism of $\mathcal{E}$ but direct the reader to Harrington and Soare [8] or Cholak [1].) If an automorphism $\Phi$ is constructed on a tree then $\Phi$ has a presentation computable in the true path (which is $\Delta^0_3$). Hence all automorphisms constructed in this way are $\Delta^0_3$-automorphisms. (In some cases we can improve this to make the automorphism effective.)

Our current goal is to prove that if $A$ and $\widehat{A}$ are automorphic by any (not necessarily $\Delta^0_3$) automorphism $\Phi$ (that is, $\Phi(A) = \widehat{A}$) then there are definable substructures of $\mathcal{E}$, $\mathcal{S}_{\mathcal{R}}(A)$, and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$—we will define these structures shortly—such that the isomorphism induced on these structures by $\Phi$ is $\Delta^0_3$. That is, whereas $\Phi$ need not be $\Delta^0_3$ it is $\Delta^0_3$ on a definable substructure of $\mathcal{E}$.

Before we can formally state this result we need a definition:

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**Definition 1.1.** Let $\mathcal{S}(A) = \{B : \exists C(B \sqcup C = A)\}$ ($\sqcup$ is disjoint union). So $\mathcal{S}(A)$ is the splits of $A$ and $\mathcal{S}(A)$ forms a Boolean algebra. Let $\mathcal{R}(A) = \{R : R \subseteq A$ and $R$ is computable$\}$. $\mathcal{R}(A)$ is the collection of all computable subsets of $A$ and is an ideal of $\mathcal{S}(A)$. Let $\mathcal{S}_{\mathcal{R}}(A)$ be the quotient structure $\mathcal{S}(A)$ modulo $\mathcal{R}(A)$.

If $W \in \mathcal{S}(A)$ then let $W^{R(A)}$ be the equivalence class of $W$ in $\mathcal{S}_{\mathcal{R}}(A)$. $\mathcal{S}_{\mathcal{R}}(A)$ is a Boolean algebra and is definable in $\mathcal{E}$ with a parameter, $A$. $\mathcal{S}_{\mathcal{R}}(A)$ is also definable (without parameters) in $\mathcal{E}(A)$ which is the computably enumerable sets contained in $A$ ordered by inclusion.

If $A$ and $B$ are automorphic then the structures $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(B)$ are isomorphic structures. In fact, if $A$ and $B$ are noncomputable then $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(B)$ are atomless Boolean algebras (see Lemma 2.2) and hence isomorphic. In fact, they can be seen to be $\Delta^0_3$-isomorphic; see Theorem 2.4.

But surprisingly, if $A$ and $\widehat{A}$ are automorphic in $\mathcal{E}$ via $\Phi$ then the automorphism $\Phi$ induces a $\Delta^0_3$-isomorphism on $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ even if $\Phi$ is not itself $\Delta^0_3$.

**Theorem 1.2.** *If $A$ and $\widehat{A}$ are automorphic via $\Phi$ then the structures $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ are $\Delta^0_3$-isomorphic via an isomorphism $\Psi$ induced by $\Phi$.*

In other words there is an isomorphism $\Psi$ between $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ such that $\Psi(W^{\mathcal{R}(A)})$ and $(\Phi(W))^{\mathcal{R}(\widehat{A})}$ are the same equivalence class (i.e., $\Psi$ is induced by $\Phi$) and there is a $\Delta^0_3$-function $f$ such that for $W_e \in \mathcal{S}(A)$, $W_{f(e)}$ is in $\Psi(W_e^{\mathcal{R}(A)})$. We will write this as $\mathcal{S}_{\mathcal{R}}(A) \simeq_{\Delta^0_3} \mathcal{S}_{\mathcal{R}}(\widehat{A})$.

We will not completely discuss here the full impact and potential of the above theorem since a partial discussion already appears in Section 3 of Cholak and Harrington [4] to which we direct the reader. However we will mention and prove one theorem which follows from the above theorem and some work in Maass [9]:

**Theorem 1.3.** *Let $H$ and $\widehat{H}$ be hhsimple. $H$ and $\widehat{H}$ are automorphic iff they are $\Delta^0_3$-automorphic iff $\mathcal{L}^*(H) \simeq_{\Delta^0_3} \mathcal{L}^*(\widehat{H})$.*

What is exciting about this result is that it completely characterizes when two hhsimple sets are automorphic. They are automorphic iff their $\mathcal{L}^*$ are $\Delta^0_3$-isomorphic. We can easily work with the $\mathcal{L}^*$ as they are just Boolean algebras. In Section 3.1 of Cholak and Harrington [4], we mentioned how this theorem easily implies some known results.

Theorem 1.2 and some other work of ours will be used in Cholak and Harrington [2] to show:

**Theorem 1.4.** *If $A$ and $\widehat{A}$ are automorphic via $\Psi$ then they are automorphic via $\Lambda$ where $\Lambda \upharpoonright \mathcal{L}^*(A) = \Psi \upharpoonright \mathcal{L}^*(A)$ and $\Lambda \upharpoonright \mathcal{E}^*(A)$ is $\Delta^0_3$.*

Hence if $A$ and $\widehat{A}$ are automorphic (via any automorphism) they are automorphic via an automorphism $\Lambda$ which is $\Delta^0_3$ on the inside of $A$. It has been observed that in any known construction of an automorphism taking a set $A$ to $\widehat{A}$ it is possible to use Soare's Extension Theorem. The above theorem (and its proof) imply that this observation is always the case; Soare's Extension Theorem can be used in every automorphism construction.

The rest of this paper consists of the proofs of Theorems 1.2 and 1.3. These proofs are very modular. The proof of Theorem 1.2 starts in Section 3 and continues through Section 10. The proof of Theorem 1.2 depends on the special $\mathcal{L}$-patterns which were introduced in Cholak and Harrington [4]. We will use several theorems about special $\mathcal{L}$-patterns which can be found in Cholak and Harrington [3]. As needed we will restate and rephrase these theorems; this can be found in Section 6. The proof of Theorem 1.3 appears in Section 11.3.

1.1. **Notation and definitions.** Our notation and definitions are standard and follow Cholak and Harrington [4] which follows Soare [13].

We think of $\Phi$ as a map from $\omega$ to another copy of $\omega$, $\widehat{\omega}$. All subsets of $\widehat{\omega}$ will wear hats. We refer to $\widehat{\omega}$ as the *hatted* side and sometimes we refer to $\omega$ as the *unhatted* side. Throughout the rest of the paper we will assume that $\Phi$ is an automorphism of $\mathcal{E}$ and that $\Phi(W) = \widehat{W}$, for all computably enumerable sets $W$. If $S$ is a split of a computably enumerable set $X$, we will use $\breve{S}$ to denote the computably enumerable set $X - S$ when the set $X$ is clear from context.

## 2. Some facts about $\mathcal{S}_{\mathcal{R}}(A)$

If $X, Y \in \mathcal{S}(A)$ then the symmetric difference of $X$ and $Y$, $X \triangle Y$, is a computably enumerable set, a split of $A$, and therefore in $\mathcal{S}(A)$ ($X \triangle Y = (X \cap (A - Y)) \cup (Y \cap (A - X)) = (X \cap \breve{Y}) \cup (Y \cap \breve{X})$). If $X \equiv_{\mathcal{R}(A)} Y$ then $X \triangle Y$ is contained in a computable subset $R$ of $A$. But if $X \triangle Y \subseteq R \subseteq A$ then $X \triangle Y$ is computable (given $x$ in $R$ it must enter $X \triangle Y$ or $A - (X \triangle Y)$). So for $X, Y \in \mathcal{S}(A)$, $X \equiv_{\mathcal{R}(A)} Y$ iff $X \triangle Y$ is a computable subset of $A$.

**Lemma 2.1.** *Given two splits $X$ and $Y$ (of $A$), whether $X \equiv_{\mathcal{R}} Y$ and $X \subseteq_{\mathcal{R}} Y$ is $\Sigma^0_3$.*

*Proof.* Given the index for $X$, it is possible to find in a $\Delta^0_3$ way an index for $\breve{X}$. Similarly for $Y$. Hence we can find an index for $X \triangle Y$ in a $\Delta^0_3$ fashion. Now $X \equiv_{\mathcal{R}} Y$ iff $X \triangle Y$ is computable iff there is an $l$ such that $W_l \sqcup (X \triangle Y) = \omega$. Since "$W_l \sqcup (X \triangle Y) = \omega$" is $\Pi^0_2$, the last clause in the above sentence is $\Sigma^0_3$. Now $X \subseteq_{\mathcal{R}} Y$ iff $X \cup (\breve{X} \cap Y) \equiv_{\mathcal{R}} Y$. ∎

**Lemma 2.2.** *If $C$ is noncomputable then $\mathcal{S}_{\mathcal{R}}(C)$ is the atomless Boolean algebra.*

*Proof.* Let $X$ and $Y$ be noncomputable splits of $A$ such that $X \subseteq Y$ but $X \not\equiv_{\mathcal{R}} Y$. Then by the Friedberg Splitting Theorem or the Owings Splitting Theorem we can split $Y$ over $X$ into two noncomputable halves each of which are splits of $C$. □

Since any two atomless Boolean algebras are isomorphic effectively in the order relation, we get the following corollary:

**Corollary 2.3.** *If $C$ and $\widehat{C}$ are noncomputable, then $\mathcal{S}_{\mathcal{R}}(C)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{C})$ are $\Delta^0_4$-isomorphic.*

However this can be improved.

**Theorem 2.4** (Nies)**.** *If $C$ and $\widehat{C}$ are noncomputable, then $\mathcal{S}_{\mathcal{R}}(C)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{C})$ are $\Delta^0_3$-isomorphic.*

*Proof.* Herrmann (unpublished) observed that if an isomorphism $\Phi$ between $\mathcal{E}^*(C)$ and $\mathcal{E}^*(\widehat{C})$ is constructed by first using Soare's Order-Preserving Enumeration Theorem (see Soare [13, XV.5.1]) followed by Soare's original Extension Theorem then the produced isomorphism preserves the computable subsets (the image and preimage of computable sets is computable) and is $\Delta^0_3$. (For more details we direct the reader to the sequel of this paper, Cholak and Harrington [2], where we discuss extension theorems.) Since $\Phi$ preserves computable sets, it clearly induces a $\Delta^0_3$-isomorphism between $\mathcal{S}_{\mathcal{R}}(C)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{C})$. (Nies's proof is different and is discussed below.) □

At some point we will need to consider a particular $\Sigma^0_3$-ideal, $\mathcal{I}$, of $\mathcal{S}_{\mathcal{R}}(C)$ where $C$ is a computably enumerable set which we construct. (Our particular $\mathcal{I}$ will be defined in Definition 5.6.) Ideals of $\mathcal{S}_{\mathcal{R}}(C)$ are subsets of $\mathcal{S}_{\mathcal{R}}(C)$ which are closed under finite union and inclusion. An ideal $\mathcal{I}$ of $\mathcal{S}_{\mathcal{R}}(C)$ is $\Sigma^0_n$ iff $W_e \in \mathcal{I}$ is $\Sigma^0_n$. Since whether $W$ is a split of $C$ is $\Sigma^0_3$, $\mathcal{S}_{\mathcal{R}}(C)$ is itself a $\Sigma^0_3$-ideal. Until we define our particular $\mathcal{I}$ we should think of $\mathcal{I}$ as just $\mathcal{S}_{\mathcal{R}}(C)$. Since $\Phi$ is an automorphism, there is an ideal, $\widehat{\mathcal{I}} = \{\widehat{X} : X \in \mathcal{I}\}$, of $\mathcal{S}_{\mathcal{R}}(\widehat{C})$.

We should point out that $\mathcal{S}_{\mathcal{R}}(A)$ plays a very important role in Nies [12]; it is the principal structure used to provide an interpretation of true arithmetic in intervals of $\mathcal{E}^*$ which are not Boolean algebras. In that paper, Nies considers effectively dense Boolean algebras (we refer the reader to Nies [12] for a definition). By using the Owings Splitting Theorem, Nies showed that $\mathcal{S}_{\mathcal{R}}(A)$ is a $\mathbf{0}''$-effectively dense Boolean algebra.

Recently Nies has extended some of his work concerning effectively dense Boolean algebras to effectively inseparable Boolean algebras which were introduced in Montagna and Sorbi [10] (see that paper or Nies's upcoming paper on this topic for a definition). Nies showed that effectively inseparable Boolean algebras are also effectively dense and that $\mathcal{S}_{\mathcal{R}}(A)$ is a $\mathbf{0}''$-effectively inseparable Boolean algebra. Montagna and Sorbi have a number of results about effectively inseparable Boolean algebras. One of these states that any two **a**-effectively inseparable Boolean algebras are **a**-isomorphic. This was the first proof of Theorem 2.4. Hence we have credited Theorem 2.4 to Nies.

## 3. Weeding out some splits $T$ of $A$

Our goal is to match in a $\Delta^0_3$ way each split $T$ of $A$ with a split $\widetilde{T}$ of $\widehat{A}$ such that $\widehat{T}$ and $\widetilde{T}$ are equivalent modulo $\mathcal{R}(\widehat{A})$; that is $\widehat{T} \triangle \widetilde{T}$ is computable. The isomorphism $\Psi$ will be defined as $\Psi(T^{\mathcal{R}(A)}) = (\widetilde{T})^{\mathcal{R}(\widehat{A})}$. Since $\Phi$ is an automorphism of $\mathcal{E}$ and $\mathcal{S}_{\mathcal{R}}(A)$ is a definable substructure of $\mathcal{E}$, $\Psi$ will be an isomorphism.

We start by building a subset $B$ of $A$ with certain definable and dynamic properties. We need to recall the following definition:

**Definition 3.1.** $B$ is a *small* subset of $A$ if for every pair of sets $X, Y$ if $X \cap (A - B) \subseteq Y$ then $Y \cup (X - A)$ is a computably enumerable set.

The following results about small subsets will prove useful. At least the first two are in Stob [14] (see Soare [13, X.4.11]).

**Lemma 3.2** (Stob)**.** *Assume $E$ is a small subset of $D$.*

1. *If $D \subseteq \widehat{D}$ then $E$ is a small subset of $\widehat{D}$.*
2. *If $\widehat{E} \subseteq E$ then $\widehat{E}$ is a small subset of $D$.*
3. *Then $E - R$ is small in $D - R$ for any computable set $R$.*

*Proof.* We are just chasing through the definitions to get these results.

(1) Assume that $X \cap (\widehat{D} - E) \subseteq Y$. Then $X \cap (D - E) \subseteq Y$. So $Y \cup (X - D)$ is a computably enumerable set. Now $X \cap (\widehat{D} - D) \subseteq Y$. So $Y \cup (X - D) = Y \cup (X - \widehat{D})$.

(2) Assume that $X \cap (D - \widehat{E}) \subseteq Y$. Then $X \cap (D - E) \subseteq Y$. So $Y \cup (X - D)$ is a computably enumerable set.

(3) Assume that $X \cap ((D-R)-(E-R)) \subseteq Y$. Then $X \cap (D-E) \subseteq Y \cup R$. So $Z = Y \cup R \cup (X - D)$ is a computably enumerable set. Then $Y \cup (X - (D - R)) = (Z \cap \overline{R}) \cup ((Y \cup X) \cap R)$ is also a computably enumerable set. ☐

**Theorem 3.3.** *There is a small subset $B$ of $A$ such that $B \neq^* A$ and for all splits $T$ of $A$, if $T \setminus B$ is infinite then $T \cap B$ is not computable.*

*Proof.* To show $B$ is a small subset of $A$ we will use the requirements and strategy presented in Soare [13, X.4.12]. Since these requirements and strategies will be used in the proof of Theorem 5.1 we will briefly review them below.

To show $B$ is small in $A$ we need to meet the negative requirements:

$$\mathcal{N}_i \qquad Y_i \supseteq X_i \cap (A - B) \Rightarrow (X_i - A) \cup Y_i \text{ is a c.e. set,}$$

where $\{X_i, Y_i\}_{i\in\omega}$ is a listing of all pairs of computably enumerable sets. To aid in meeting $N_i$ we define by induction on $s$, $g(i, s)$ and $Z_{i,s}$. For $s = 0$, let $g(i, 0) = 0$ and $Z_{i,s} = \emptyset$. For $s + 1$ define

$$g(i, s) = \begin{cases} (\mu x)[x \in (A_{s+1} - B_s) \cap (Z_{i,s} - Y_{i,s})] & \text{if such an } x \text{ exists,} \\ s + 1 & \text{otherwise,} \end{cases}$$

where $Z_{i,s+1} = \{x : x \in X_{i,s+1} \wedge x \leq g(i, s + 1)\} \cup Z_{i,s}$. Elements of $(A_{s+1} - B_s) \cap (Z_{i,s} - Y_{i,s})$ are restrained from entering $B$ with priority $\mathcal{N}_i$. So an element of $A$ and $X$ is restrained from entering $B$ until it enters $Y$.

Only finitely many integers are permanently restrained by $\mathcal{N}_i$ since if $Z_i$ is infinite then $\limsup_s g(i, s) = \infty$ and hence $Y_i \supseteq X_i \cap (A - B)$. However it might take some time before $g(i, s) > x$ and $x$ is not restrained by $\mathcal{N}_i$. The smallness requirements can cause a delay in the enumeration of $B$. So this requirement is not compatible with a requirement which forces integers to quickly enter $B$ such as a promptness requirement (or the requirements needed to make $B$ a true Friedberg split of $A$).

This is enough to show $B$ is small: Assume that $Y_i \supseteq X_i \cap (A - B)$ and that $\mathcal{N}_i$ is not injured at or after stage $t$. For $s < t$, let $U_{i,s} = \emptyset$. For $s \geq t$, let $U_{i,s} = \{x : x \in X_{i,s} - A_s \wedge x \leq g(i, s)\}$. We will show $U_i \cup Y_i =^* (X_i - A) \cup Y_i$. If $x \in U_{i,s}$ then $x$ is restrained from entering $B$ until $x$ enters $Y_i$. The other almost inclusion is clear. This strategy mixes well with finitary positive requirements (and some infinitary positive requirements).

For splits $T$ of $A$, to ensure that if $T \backslash B$ is infinite then $T \cap B$ is not computable, it is enough to meet the requirements:

$$\mathcal{P}_e \qquad T \backslash B \text{ is infinite} \Rightarrow \overline{T \cap B} \neq W_e.$$

(Actually we will need two more parameters for this requirement, the indices for $T$ and $\breve{T}$, but we will ignore them.) To meet this requirement, we will select a witness $x$ in $T - B$, hold $x$ out of $B$, and wait for $x$ to enter $W_e$. If $x$ is in $W_e$ by stage $s$ and is not restrained by any higher priority negative requirements at stage $s$, we will put $x$ into $B$ at stage $s$ for a win. If $x$ is in $W_e$ at stage $s$ but is restrained by some higher priority negative requirement at stage $s$, we will hold onto $x$ hoping that later we can put $x$ into $B$ for a win and in the meantime repeat the above process with a new larger witness.

If $T\backslash B$ is infinite then we will eventually find a usable witness $x$ in $T$ and be able to meet $\mathcal{P}_e$.

To get the strategies used for the $\mathcal{P}_e$ to work together we must put this construction on a tree, $2^{<\omega}$. The problem is getting usable balls in $T - B$. This depends on whether $T = W_i$ is a split of $A$ and if $W_i\backslash B$ is infinite.

We define the true path by induction as follows: Let $\alpha \subset f$ such that $|\alpha| = \langle i, j, e\rangle$. If $W_i \sqcup W_j = A$ and $W_i\backslash B$ is infinite then $\alpha\widehat{\ }0 \subset f$; otherwise $\alpha\widehat{\ }1 \subset f$. The approximation to the true path is also defined by induction. Let $\alpha \subseteq f_s$ such that $|\alpha| = \langle i, j, e\rangle$ and $|\alpha| \leq s$. We need a length of agreement function: $l_\alpha(s)$ is the greatest $z \leq s$ such that $(W_{i,s} \upharpoonright z) \cap (W_{j,s} \upharpoonright z) = \emptyset$, and $(W_{i,s} \cup W_{j,s}) \upharpoonright z = A_s \upharpoonright z$. Let $t < s$ be the last stage that $\alpha\widehat{\ }0 \subseteq f_s$ (if such a stage does not exist let $t = 0$). If $l_\alpha(t) < l_\alpha(s)$ and $|(W_i\backslash B)_s| > |(W_i\backslash B)_t|$ (an $\alpha$-expansionary stage) then $\alpha\widehat{\ }0 \subseteq f_s$; otherwise $\alpha\widehat{\ }1 \subseteq f_s$. It is not too hard to show that $f = \liminf_s f_s$.

Let $\beta = \alpha\widehat{\ }0$ where $|\alpha| = \langle i, j, e\rangle$. At $\beta$ we will try to meet $\mathcal{N}_i$ and $\mathcal{P}_e$. To meet $\mathcal{N}_i$ we will define a function $g(\beta, s)$ and a set $Z_{\beta,s}$ as above but we only allow these to change at stages when $\beta \subseteq f_s$. $\beta$ will also look for a witness $x \in W_i$ to meet $\mathcal{P}_e$. $\beta$ is given an interval from which it is allowed to use all balls. A witness is *usable* at stage $s$ if $x \in W_i$ and, for all $\beta' \subseteq \beta$, $x \notin (A_{s+1} - B_s) \cap (Z_{\beta',s} - Y_{i,s})$. It is not hard to show that only finitely many balls are permanently unusable. $\beta$'s goal is to find a usable ball (in its interval). If $\beta \subseteq f_s$ and $\beta$ has a usable ball $x$ such that $x \in W_{e,s}$ then $\beta$ will add $x$ to $B$, meeting $\mathcal{P}_e$.

To meet its goal $\beta$ will be given a number $d_\beta$ and a function $u_\beta(s)$. At the first stage when $\beta \subseteq f_s$ or the first such stage after being initialized we will set $d_\beta$ to be large (just some number not yet occurring in the construction), $u_\beta(s)$ to be even larger, and discontinue this stage of the construction. If $\beta \subseteq f_s$ and either $\mathcal{P}_e$ is not met or $\beta$ does not have any usable balls (in its interval) we will set $u_\beta(s)$ to be large; otherwise the value of $u_\beta(s)$ is $u_\beta(s-1)$. As always, the number $d_\beta$ and the function $u_\beta(s)$ will be initialized if $f_t <_L \beta$, for some later stage $t$. Balls in the interval $[d_\beta, u_\beta(s)]$ are assigned to $\beta$. □

From now on we will work with the set $B$ given by Theorem 3.3.

**Lemma 3.4.** *$B$ is simple in $A$ and hence $B$ is not computable.*

*Proof.* Assume that $W \subseteq A - B$ and $W$ is infinite. Then there is a computable set $R$ contained in $W$. $R \sqcup (\overline{R} \cap A) = A$. So $R$ is a split of $A$ such that $R\backslash B$ is infinite but $R \cap B = \emptyset$ is computable. Contradiction. □

**Lemma 3.5.** *Let $T$ be a split of $A$. If $T \subseteq^* B$ then $T$ is computable.*

*Proof.* $A - T$ is computably enumerable and almost contains $\omega \cap (A - B)$. By smallness, $(A - T) \cup (\omega - B)$ is a computably enumerable set. Since $T \subseteq^* B$, $(A - T) \cup (\omega - B) =^* \overline{T}$. □

**Lemma 3.6.** *Let $T$ be a split of $A$. If $T \cap B$ is computable then, by Theorem 3.3, $T \backslash B$ is finite, therefore $T \subseteq^* B$ and, by Lemma 3.5, $T =^* T \cap B$ is computable.*

We only need worry about those splits $T$ such that $T \backslash B$ is infinite. If $T \backslash B$ is finite then $T$ is computable and we can let $\widetilde{T} = \emptyset$. Notice that "$T \backslash B$ is infinite" is a $\Pi^0_2$ property.

**Lemma 3.7.** *For two splits $T$ and $\check{T}$ of $A$, if $T$ is not equivalent to $\check{T}$ modulo $\mathcal{R}(A)$ then $T \cap B$ is not equivalent to $\check{T} \cap B$ modulo $\mathcal{R}(B)$.*

*Proof.* Apply Lemma 3.6 to the split $(T \triangle \check{T})$ of $A$. If $(T \triangle \check{T}) \cap B$ is computable then $T \triangle \check{T}$ is computable. □

Notice that the above dynamic properties (i.e., properties of $T \backslash B$) about splits of $A$ do not transfer to the hatted side. With this in mind we will build $\widehat{C} \subset \widehat{B}$ with certain definable and dynamic properties. But first we must recall the following definition:

**Definition 3.8.** Given computably enumerable sets $C \subseteq B$, $C$ is *major* in $B$, $C \subseteq_M B$, if $B - C$ is infinite, and for every computably enumerable set $W$, $\overline{B} \subseteq^* W$ implies $\overline{C} \subseteq^* W$.

**Theorem 3.9.** *There are a set $\widehat{C} \subset \widehat{B}$ and a computable function $\widehat{p}$ such that $\widehat{C}$ is promptly simple in $\widehat{B}$ via $\widehat{p}$, $\widehat{C}$ is major in $\widehat{B}$, and for all splits $\widehat{T}$ of $\widehat{A}$, if $\widehat{T} \cap \widehat{B}$ is not computable then $\widehat{T} \cap \widehat{B}$ is not a subset of $\widehat{C}$.*

The dynamic property of being promptly simple does not transfer to $C$. But as we will later see the dynamic property will be useful in determining $\widetilde{T}$; it is used in Lemmas 9.1, 9.2, and 10.1. However, since $\Phi$ is an automorphism, the definable properties of $\widehat{C}$ do transfer to $C$. So $C$ is simple in $B$, $C$ is major in $B$, and for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap B$ is not a subset of $C$. $C$ is not computable and $C$ is simple in $A$. So if $T \backslash B$ is infinite then $T \cap B$ is not computable which in turn implies $T \cap B$ is not a subset of $C$.

To make life notationally easy we discuss and prove the nonhatted version of Theorem 3.9:

**Theorem 3.10.** *There are a set $C \subset B$ and a computable function $p$ such that $C$ is promptly simple in $B$ via $p$, $C$ is major in $B$, and for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap B$ is not a subset of $C$.*

*Proof.* Fix a computable function $p$. To make $C$ prompt in $B$ via the computable function $p$ it is enough to meet the requirements:

$$\mathcal{P}_e \qquad W_e \cap B \text{ is infinite } \Rightarrow \exists x \exists s[x \in (W_e \cap B)_{\text{at } s} \cap C_{p(s)}].$$

These are positive finitary requirements and will easily mix with the requirements below.

To make $C$ major in $B$ we use an $e$-state construction like the construction of a maximal set. We have a sequence of movable markers $\Gamma_n$ marking the elements of $B - C$. The markers must only move finitely often and they pull finitely often to maximize their $e$-states to meet the requirement:

$$\mathcal{Q}_e \qquad \overline{B} \subseteq W_e \Rightarrow \overline{C} \subseteq^* W_e.$$

The $e$-states are measured w.r.t. $\{V_e\}_{e\in\omega}$ where

$$V_{e,s} = \{x : x \in W_{e,s} \wedge (\forall y \leq x)[y \in W_{e,s} \cup B_s]\}.$$

There will be a $\sigma_e$ such that the final $e$-state of almost every ball in $B - C$ is $\sigma_e$. Let $R_{\sigma_e} = \{x : \exists s \exists y > x \exists n \geq e$[the $e$-state of $x$ at stage $s$ is smaller than $\sigma_e$, the $e$-state of $y$ at stage $s$ is larger than the $e$-state of $x$ at stage $s$, and $\Gamma_n$ marks $y$]$\}$. $R_{\sigma_e}$ is the computable subset of $B$ which $\mathcal{Q}_e$ dumps into $C$. The $\mathcal{Q}_e$ are positive infinitary requirements. A detailed construction of a major subset can be found in Soare [13, X.4.6].

Let $T_e$ be a split of $A$. Modulo higher priority positive requirements we will *try* to meet:

$$\mathcal{N}_e \qquad T_e \searrow B \text{ is infinite } \Rightarrow |(T_e \searrow B) - C| \geq 1.$$

If there is an element $x$ of $T_e$ entering $B$ at stage $s$ and there is no integer $y$ currently restrained by $\mathcal{N}_e$ in a higher $e$-state at stage $s$ then we allow $\mathcal{N}_e$ to restrain $x$ at stage $s$. If the final $e$-state of some restrained $x$ is $\sigma_e$ or larger (i.e., $x \in \overline{R_{\sigma_e}}$) then $\mathcal{N}_e$ is met and furthermore $T_e \cap B$ is not a subset of $C$.

Assume no such $x$ is restrained. Let $T = T_e$ and $R = R_{\sigma_e}$. Then $(T \cap \overline{R}) \searrow B = \emptyset$ (if $x \in (T \cap \overline{R}) \searrow B$ then $\mathcal{N}_e$ would restrain $x$ forever). Since $R \subseteq B$, $(T \cap \overline{R}) - B = T - B$. Now $(T \cap \overline{R}) \backslash B = ((T \cap \overline{R}) - B) \sqcup ((T \cap \overline{R}) \searrow B) = (T - B) \sqcup \emptyset = T - B$. Hence $T - B$ is a computably enumerable set. $T = (T - B) \sqcup (T \cap B)$. Therefore $T \cap B$ is a split of $A$. By Lemma 3.5, $T \cap B$ is computable. Hence if we fail to meet $\mathcal{N}_e$ then $T_e \cap B$ is computable.

These requirements and their strategies fit together. The major subset requirements ensure that $B - C$ is infinite. □

From this point on, when we discuss the set $C$ we mean the set given by Theorem 3.10.

## 4. Robust sets and robust splits

**Definition 4.1.** A (possibly non-c.e.) set $S$ is *robust* if for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap S$ is not computable.

By the definition, clearly $B$ is a robust set.

**Definition 4.2.** $S$ is a *robust split* if it is robust and a split of $C$ (possibly trivial).

If $S$ is a robust split then, since $S$ is a split of $C$, $S$ is computably enumerable.

**Lemma 4.3.** *$C$ is a robust split of itself.*

*Proof.* Assume $T \cap B$ is not computable but $T \cap C$ is computable. Then $X = (T \cap B) - (T \cap C)$ is an infinite computably enumerable subset of $B - C$. $X$ contains an infinite computable set $R$. $\overline{B} \subseteq \overline{R}$ but $\overline{C} \not\subseteq^* \overline{R}$ contradicting the fact that $C$ is major in $B$. □

**Lemma 4.4.** *No robust set is a computable subset of $A$.*

*Proof.* Let $R \subseteq A$ be computable. If $T$ is a noncomputable split of $A$ then, by Lemma 3.6, $T \cap B$ is not computable but $T \cap R$ is computable (given $x$ in $R$ wait for $x$ to enter $T$ or $A - T$ and then decide if $x$ is in $T \cap R$). So no robust set is computable. □

**Lemma 4.5.** *A robust set $S$ is never a split of $A$.*

*Proof.* Assume that $S$ is a robust split of $A$. Then, by the simplicity of $B$ in $A$, there is an enumeration of $A - S$ such that $(A - S) \backslash B$ is infinite. Hence, by Theorem 3.3, $(A - S) \cap B$ is not computable. But $(A - S) \cap S = \emptyset$. Contradiction. □

**Lemma 4.6.** *If $X \subseteq B$, $X \triangle S$ is a computable subset of $B$, and $S$ is a robust set, then $X$ is a robust set.*

*Proof.* For splits $T$ of $A$, if $T \cap X$ is computable then $T \cap S$ is computable. □

**Lemma 4.7.** *If $\check{S} \subseteq S$ is a robust split and $S$ is a split of $C$ then $S$ is a robust split.*

*Proof.* For splits $T$ of $A$, if $T \cap S$ is computable then so is $T \cap \check{S}$. □

**Lemma 4.8.** *Let $S$ be a robust set. Fix two splits $T$ and $\check{T}$ of $A$. If $T$ is not equivalent to $\check{T}$ modulo $\mathcal{R}(A)$ then $T \cap S$ is not equivalent to $\check{T} \cap S$ modulo $\mathcal{R}(C)$.*

*Proof.* $T \triangle \check{T}$ is a split of $A$. By Lemma 3.7, $(T \triangle \check{T}) \cap B$ is not computable and, since $S$ is a robust set, $(T \triangle \check{T}) \cap S$ is not computable. $\square$

Since the above properties of robust sets are definable, these properties transfer to the hatted side (even though their proofs may involve dynamic properties of $A$ and $B$).

True Friedberg splits provide a way to construct robust splits.

**Definition 4.9** (Downey and Stob [5]; also see Downey and Stob [6])**.** $S_1$ and $S_2$ form a *true Friedberg split* of $S$ if for all computably enumerable sets $W$, $W \searrow S$ is infinite implies $W \searrow S_i$ is infinite.

Being a true Friedberg split is a dynamic but not definable property. Both the Friedberg Splitting Theorem (see Soare [13, X.2.1]) and the Owings Splitting Theorem over $\emptyset$ (see Soare [13, X.2.5]) create a true Friedberg split.

**Lemma 4.10.** *If $S_1$ and $S_2$ form a true Friedberg split of a robust set (split) $S$ then the $S_i$ are robust sets (splits).*

*Proof.* Let $T$ be a split of $A$ such that $T$ witnesses that $S_1$ is not robust (WLOG we can assume $S_1$ is not robust). So $\overline{T \cap S_1} = X$ is a computably enumerable set. $\overline{S} \subseteq X \backslash S$ and $X \backslash S = (X - S) \sqcup (X \searrow S)$.

Assume $X \searrow S$ is finite. Then $X - S =^* \overline{S}$ is a computably enumerable set. But no robust set is computable.

Hence $X \searrow S$ is infinite. This implies that $X \searrow S_1$ is infinite and hence $X \cap S_1 \neq \emptyset$. But $X \cap S_1 = \emptyset$. Contradiction. $\square$

Lemma 4.10 is a dynamic lemma about robust sets and hence does not, on the face of it, transfer to the hatted side. The proof of Lemma 4.10 relied on one dynamic property: $\overline{T \cap S_1} \searrow S_1$ is infinite. But this was implied by a definable property: no robust set is computable. This property transfers to the hatted side. Hence the lemma transfers to the hatted side:

**Lemma 4.11.** *If $\widehat{S}_1$ and $\widehat{S}_2$ is a true Friedberg split of a robust set (split) $\widehat{S}$ then the $\widehat{S}_i$ are robust sets (splits).*

The requirements for a true Friedberg split require integers to quickly enter $S$. In some situations this might be a disadvantage. In Section 5.1.3, we discuss another option for constructing robust sets.

## 5. Small robust splits

The next theorem, proven using a tree argument, will allow us to associate, in a definable and $\Delta^0_3$ fashion, splits $T$ of $A$, where $T \backslash B$ is infinite, with certain robust splits $S$ of $C$. We will use this association to code splits $T$ of $A$, where $T \backslash B$ is infinite, with a split $\widetilde{T}$ of $\widehat{A}$.

On the hatted side we will decode this association/coding to compute $\widetilde{T}$. To help with this decoding we must define the ideal $\mathcal{I}$ and add some, at this point, seemingly unneeded features to the following theorem.

The next theorem will also be used to fix a listing of all splits of $A$. Fix a listing $\{X_i, Y_i\}_{i\in\omega}$ of all pairs of computably enumerable sets.

**Theorem 5.1.** *There are a computable tree $T = 2^{<\omega}$, a corresponding true path $f$, computable in $\mathbf{0}''$, and, for all $\alpha \in T$, splits $T_\alpha$ and $\breve{T}_\alpha$ of $A$ (note Condition 4a which implies if $\alpha \subset f$ then $T_\alpha$ and $\breve{T}_\alpha$ form almost the same* single *split of $A$ but this is not the case if $\alpha \not\subseteq f$), splits $S_\alpha$ and $\breve{D}_\alpha$ of $C$, and a subset $D_\alpha$ of $B$ such that if we define $T_j = T_\alpha$, $\breve{T}_j = \breve{T}_\alpha$, $S_j = S_\alpha$, $D_j = D_\alpha$, and $\breve{D}_j = \breve{D}_\alpha$, where $\alpha \subset f$ is of length $j = \langle i, k, l\rangle$ then*

1. *if $\alpha \not\subseteq f$ then $T_\alpha =^* \breve{T}_\alpha =^* S_\alpha =^* D_\alpha =^* \breve{D}_\alpha =^* \emptyset$;*
2. *if $\alpha \neq \beta$ then $S_\alpha \cap S_\beta = \emptyset$;*
3. *if $X_j$ and $Y_j$ do not form a split of $A$ or $X_j \backslash B$ is finite then all the sets constructed at $\alpha$ are finite; and*
4. *if $X_i \sqcup Y_i = A$ and $X_i \backslash B$ is infinite and $j' > j$ is the least $j' = \langle i', k', l'\rangle$ such that $X_{i'} \sqcup Y_{i'} = A$ and $X_{i'} \backslash B$ is infinite then*
   (a) *$T_j =^* X_i$ and $\breve{T}_j =^* Y_i$,*
   (b) *$(C \cap D_j) \sqcup \breve{D}_j = C$,*
   (c) *$S_j \equiv_{\mathcal{R}(C)} (D_j \cap \breve{D}_{j'}) \cap C$,*
   (d) *$D_0 = B$ is a small subset of $A$,*
   (e) *$D_{j'}$ is a small subset of $D_j$, and*
   (f) *$S_j$ is a robust split.*

Note that the $k$ and $l$ are not needed in this construction but in the coding. See Section 7.

5.1. **The proof of Theorem 5.1.** We will start out by discussing the tree $T = 2^{<\omega}$, the true path, $f$, and the approximation to the true path, $f_s$. We need to measure if $X_i \sqcup Y_i = A$ and $X_i \backslash B$ is infinite. This is $\Pi^0_2$.

We define the true path by induction as follows: Let $\alpha \subset f$ such that $|\alpha| = \langle i, k, l\rangle$. If $X_i \sqcup Y_i = A$ and $X_i \backslash B$ is infinite then $\alpha\widehat{\ }0 \subset f$; otherwise $\alpha\widehat{\ }1 \subset f$. The approximation to the true path is also defined by induction. Let $\alpha \subseteq f_s$ such that $|\alpha| = \langle i, k, l\rangle$ and $|\alpha| \leq s$. We need a length of agreement function: $l_\alpha(s)$ is the greatest $z \leq s$ such that $(X_{i,s} \upharpoonright z) \cap (Y_{i,s} \upharpoonright z) = \emptyset$, and $(X_{i,s} \cup Y_{i,s}) \upharpoonright z = A_s \upharpoonright z$. Let $t < s$ be the last stage that $\alpha\widehat{\ }0 \subseteq f_s$ (if such a stage does not exist let $t = 0$). If $l_\alpha(t) < l_\alpha(s)$ and $|(X_i \backslash B)_s| > |(X_i \backslash B)_t|$ (an $\alpha$-expansionary stage) then $\alpha\widehat{\ }0 \subseteq f_s$; otherwise $\alpha\widehat{\ }1 \subseteq f_s$. It is not too hard to show that $f = \liminf_s f_s$.

All sets constructed are initially empty and those sets constructed at $\alpha$ are initialized at stages $s$ when $f_s <_L \alpha$. We will only allow balls to enter any of the sets being constructed at $\alpha$ at stage $s$ if $\alpha\hat{}0 \subset f_s$. Hence a priori we realize Conditions 5.1.1 and 5.1.3. Meeting Condition 5.1.2 is also straightforward; once a ball enters $S_\alpha$ it will not be allowed to enter any other $S_\beta$.

5.1.1. *The first three parts of 5.1.4.* That leaves Condition 5.1.4. Assume $\alpha\hat{}0 \subset f$ and $\alpha$ is of length $j = \langle i, k, l\rangle$. Some of the parts of Condition 5.1.4 can be met by rather straightforward action.

We say $x$ is *$\alpha$-acceptable at stage $s$* if $|\alpha| \leq x$, for all stages $t$, such that $x \leq t \leq s$, $f_t \not<_L \alpha$, and there is a stage $s'$ such that $x \leq s' \leq s$ and $\alpha \subseteq f_{s'}$. If $x$ is $\alpha$-acceptable at stage $s$ and $\alpha <_L \beta$ then $x$ will never be $\beta$-acceptable at later stages. Assume $\alpha \subset f$. Then almost all balls $x$ will be $\alpha$-acceptable at almost all stages. Let $m_\alpha$ be the least stage such that for all $s \geq m_\alpha$, $f_s \not< \alpha$. Given $x \geq m_\alpha$, let $s_{\alpha,x}$ be the least stage $s$ such that $s \geq x$ and $\alpha \subseteq f_s$. $x$ is acceptable after this stage. Note that $s_{\alpha,x}$ can be found effectively in $x \geq m_\alpha$.

If $\alpha\hat{}0 \subseteq f_s$, $x$ is $\alpha$-acceptable at stage $s$, and $x \in X_i$ ($x \in Y_i$) at stage $s$ then we will add $x$ to $T_j$ ($\check{T}_j$) at stage $s$. This meets Condition 5.1.4a.

If $\alpha\hat{}0 \subseteq f_s$ and $x$ has entered $C$ since the last time $\alpha\hat{}0$ was on the approximation to the true path then if $x$ is not in $D_j$ at stage $s$ we will add $x$ to $\check{D}_j$ at stage $s$. This requires that we must add

$$C \searrow D_j = \emptyset$$

to conditions we must satisfy. If we meet the above new condition then Condition 5.1.4b is met.

If $\beta \subset \alpha$ and $\beta\hat{}0 \subseteq \alpha$ then before a ball can enter $D_\alpha$ it must first be in $D_\beta$. So if $\beta \subset \alpha$ then $D_\alpha \subseteq D_\beta$ and $D_\alpha \searrow D_\beta = \emptyset$.

Let $\alpha'$ be any node such that $\alpha\hat{}0 \subseteq \alpha'$ and if $\alpha\hat{}0 \subseteq \beta\hat{}0 \subseteq \alpha'\hat{}0$ then $\alpha' = \beta$. Hence if $|\alpha'| = j' = \langle i', k', l'\rangle$ then $j'$ is a candidate for the least $j' > j$ such that $X_{i'} \sqcup Y_{i'} = A$ and $X_{i'} \backslash B$ is infinite.

Assume some $\alpha'\hat{}0 \subseteq f_s$ and $x$ has entered $C$ since the last time $\alpha'\hat{}0$ was on the approximation to the true path then, as we discussed above, if $x$ is not in $D_{\alpha'}$ at stage $s$ we will add $x$ to $\check{D}_{\alpha'}$ at stage $s$. Furthermore if $x \in D_{\alpha,s}$ (we will construct this set later) and $x$ is $\alpha'$-acceptable at stage $s$ (and hence $\alpha$-acceptable at stage $s$) then we will add $x$ to $S_\alpha$ at stage $s$.

This action allows us to realize Condition 5.1.4c and show $S_\alpha$ is a split of $C$. Let $\alpha'$ be the unique node such that $\alpha\hat{}0 \subseteq \alpha' \subset f$ and if $\alpha\hat{}0 \subseteq \beta\hat{}0 \subseteq \alpha'\hat{}0$ then $\alpha' = \beta$. So if $|\alpha'| = j' = \langle i', k', l'\rangle$ then $j'$ is the least $j' > j$ such that $X_{i'} \sqcup Y_{i'} = A$ and $X_{i'} \backslash B$ is infinite. Let $R = \{x : x \in S_{\alpha, s_{\alpha',x}-1}\}$. $R$ is a computable subset of $S_\alpha \subseteq C$. Since $x$

cannot be $\beta$-acceptable for $\beta >_L \alpha'$ after stage $s_{x,\alpha'} - 1$, it is not hard to see that $S_\alpha - R =^* ((D_\alpha \cap \breve{D}_{\alpha'}) \cap C) - R$. By the conditions $C \searrow D_\alpha = \emptyset$ and $C \searrow D_{\alpha'} = \emptyset$ and the above construction of $\breve{D}_{\alpha'}$, $((D_\alpha \cap \breve{D}_{\alpha'}) \cap C) - R$ is a split of $C$ and so is $S_\alpha$.

5.1.2. *The last three conditions.* This leaves the last three parts of Condition 5.1.4 and satisfying $C \searrow D_\alpha = \emptyset$. By the above paragraph, we already have that $S_\alpha$ is a split of $C$. By Lemma 4.6, to show $S_\alpha$ is robust it is enough to show that $(D_\alpha \cap \breve{D}_{\alpha'}) \cap C = (D_\alpha - D_{\alpha'}) \cap C$ is robust. So it will be enough to satisfy the following:

(1) $C \searrow D_{\alpha'} = \emptyset$,
(2) $D_{\alpha'}$ is a small subset of $D_\alpha$ (this covers Conditions 5.1.4d and 5.1.4e),
(3) both $D_{\alpha'}$ and $(D_\alpha - D_{\alpha'}) \cap C$ are robust sets (the last clause covers Condition 5.1.4f), and
(4) furthermore, for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap D_{\alpha'} \nsubseteq C$,

for all nodes $\alpha'$, where $\alpha\hat{\ }0 \subseteq \alpha' \subset f$ and if $\alpha\hat{\ }0 \subseteq \beta\hat{\ }0 \subseteq \alpha'\hat{\ }0$ then $\alpha' = \beta$ (let $D_\lambda = B$, where $\lambda$ is the empty node). Call these 4 conditions collectively $P_{\alpha'}$.

Assume that $\alpha' \subset f$. Let $D_{\alpha \text{ at } \alpha'} = \{x : (\exists s \geq s_{\alpha',x})[x \in D_{\alpha, \text{at } s}]\}$, balls which enter $D_\alpha$ after they are $\alpha'$-acceptable. Now consider those $x$ which enter $D_\alpha$ before they are $\alpha'$-acceptable; let $R_{\alpha'} = \{x : (\exists s < s_{\alpha',x}) [x \in D_{\alpha,s}]\}$. $R_{\alpha'}$ is a computable subset of $B$ and $D_\alpha = D_{\alpha \text{ at } \alpha'} \sqcup R_{\alpha'}$.

$B$ is robust and, by Theorem 3.10, for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap B \nsubseteq C$. So let's inductively assume that $P_\alpha$ holds. Since $D_\alpha$ is robust, by Lemma 4.6, $D_{\alpha \text{ at } \alpha'}$ is robust. Let $T$ be a split of $A$ such that $T \cap B$ is not computable. Then $T - R_{\alpha'}$ is a split of $A$ such that $(T - R_{\alpha'}) \cap B$ is not computable. Therefore, by the inductive hypothesis, $(T - R_{\alpha'}) \cap D_\alpha = T \cap D_{\alpha \text{ at } \alpha'} \nsubseteq C$. We will apply the following technical theorem to $D = D_{\alpha \text{ at } \alpha'}$ to get $D_{\alpha'} = E$ and complete the induction.

**Theorem 5.2.** *Assume that $D$ is a robust subset of $B$ and for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap D \nsubseteq C$. Then effectively in an index for $D$ there is a subset $E$ of $D$ such that $C \searrow E = \emptyset$* (Condition 1)*, $E$ is a small subset of $D$* (almost Condition 2)*, both $E$ and $(D - E) \cap C$ are robust sets* (almost Condition 3) *and furthermore, for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap E \nsubseteq C$* (Condition 4)*.*

By Lemma 3.2, $E = D_{\alpha'}$ is small in $D \sqcup R_{\alpha'} = D_{\alpha \text{ at } \alpha'} \sqcup R_{\alpha'} = D_\alpha$ (completing Condition 2). Since $(D_{\alpha \text{ at } \alpha'} - D_{\alpha'}) \cap C$ is robust, so is $(D_\alpha - D_{\alpha'}) \cap C$ (completing Condition 3). Hence to complete the proof of Theorem 5.1, it will be enough to provide a proof of Theorem 5.2.

5.1.3. *The proof of Theorem 5.2.* We will provide a construction of $E$ in modular fashion.

**Lemma 5.3.** *Assume $D$ is a robust subset of $B$ and for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap D \nsubseteq C$. If $T \cap B$ is not computable then $(T \cap D) - C$ is infinite.*

*Proof.* Assume $(T \cap D) - C$ is finite. Then the split $X = T - ((T \cap D) - C)$ of $A$ has the properties that $X \cap B$ is not computable and $X \cap D \subseteq C$. Contradiction. $\square$

The next two lemmas use this lemma to provide two requirements which are needed in the construction of $E$.

**Lemma 5.4.** *Assume that $D$ is a robust subset of $B$ and for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap D \nsubseteq C$. Furthermore assume $E \subseteq D$ and for all computably enumerable sets $W$,*

$$\mathcal{R}_W: \qquad (W \cap D) - C \text{ is infinite} \Rightarrow W \cap E \nsubseteq C.$$

*Then $E$ is a robust set and furthermore, for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap E \nsubseteq C$.*

*Proof.* Let $T$ be a split of $A$ such that $T \cap B$ is not computable. Since $D$ is robust, $T \cap D$ is not computable. Assume $T \cap E$ is computable. Then $T - (T \cap E) = T - E$ is a split of $A$. Since $T \cap B = ((T - E) \cap B) \sqcup (T \cap E)$, $(T - E) \cap B$ is not computable. Hence, by Lemma 5.3, $((T - E) \cap D) - C$ is infinite and thus, by $\mathcal{R}_{T-E}$, $(T - E) \cap E \nsubseteq C$. Contradiction, since $((T - E) \cap E) = \emptyset \subset C$.

By Lemma 5.3, if $T \cap B$ is not computable then $(T \cap D) - C$ is infinite and hence, by $\mathcal{R}_T$, $T \cap E \nsubseteq C$. $\square$

The strategy to meet $\mathcal{R}_W$ is as follows: if $(W_s \cap E_s) - C_s = \emptyset$ then take the least element of $(W_s \cap D_s) - C_s$ which is not restrained by requirements of higher priority and enumerate it into $E$ at stage $s + 1$. If $(W \cap D) - C$ is infinite then this strategy will meet $\mathcal{R}_W$. Clearly this strategy can be successful even when mixed with negative requirements which only permanently restrain finitely many balls (such as the smallness requirements—see the proof of Theorem 3.3). This strategy is also compatible with ensuring $C \searrow E = \emptyset$.

**Lemma 5.5.** *Assume that $D$ is a robust subset of $B$ and for all splits $T$ of $A$, if $T \cap B$ is not computable then $T \cap D \nsubseteq C$. Assume that $C \searrow E = \emptyset$, $E \searrow D = \emptyset$, and $E \subseteq D$. Furthermore assume for all $W$,*

$$\mathcal{N}_W: \qquad (W \cap D) - C \text{ is infinite} \Rightarrow W \cap ((D - E) \cap C) \neq \emptyset.$$

*Then $(D - E) \cap C$ is a robust set.*

*Proof.* $(D - E) \cap C = (D \cap C) - E = (D \cap C) \backslash E$, since $C \searrow E = \emptyset$. Let $X = (D - E) \cap C$. $X$ is a computably enumerable set. Let $T$ be a split of $A$ such that $T \cap B$ is not computable. Since $D$ is robust, $T \cap D$ is not computable. Assume $T \cap X$ is computable. Then $T - (T \cap X) = T - X$ is a split of $A$. Since $T \cap B = ((T - X) \cap B) \sqcup (T \cap X)$, $(T - X) \cap B$ is not computable. Hence, by Lemma 5.3, $((T - X) \cap D) - C$ is infinite and thus, by $\mathcal{N}_{T-X}$, $(T - X) \cap X \neq \emptyset$. Contradiction. □

Assume that $W_s \cap ((D_s - E_s) \cap C_s) = \emptyset$. Then we will start restraining elements of $W \cap D$ from entering $E$. Since $C$ is simple in $B$, eventually one of these restrained balls must enter $C$ to meet $\mathcal{N}_W$. Hence the above strategy to meet $\mathcal{N}_W$ permanently restrains only finitely many balls.

Clearly the strategies for $R_W$, $\mathcal{N}_W$, the smallness requirements (see the proof of Theorem 3.3), $E \searrow D = \emptyset$, and $C \searrow E = \emptyset$ can be mixed together in a standard finite injury argument to construct a set $E$ satisfying Theorem 5.2 and hence completing the proof of Theorem 5.1. □

5.2. **Properties and ideals.** All properties that the sets $T_j$, $S_j$, and $D_j$ have in Theorem 5.1 are definable. Thus they transfer to the hatted side. For the rest of this paper, we will fix the ordering of splits of $A$, $B$, and $C$ given by the above theorem. For example, if $X_i$, $Y_i$ is a split of $A$ such that $X_i \backslash B$ is infinite, $\alpha\hat{\ }0 \subset f$ and $|\alpha| = j = \langle i, k, l\rangle$, then $T_j =^* X_i$, for any $k, l$.

**Definition 5.6.** Let $\mathcal{I}$ be the ideal of $\mathcal{S}_{\mathcal{R}}(C)$ generated by the nonfinite $\breve{D}_j$s. Let $\widehat{\mathcal{I}} = \{\widehat{X} : X \in \mathcal{I}\}$.

Since the nonfinite $\breve{D}_j$ are a $\Delta^0_3$ listing of splits of $C$, $\mathcal{I}$ is a $\Sigma^0_3$-ideal of $\mathcal{S}_{\mathcal{R}}(C)$. Since $\Phi$ is an automorphism, $\widehat{\mathcal{I}}$ is an ideal of $\mathcal{S}_{\mathcal{R}}(\widehat{C})$. By Theorem 5.1 Parts 4b and 4e, $\breve{D}_\alpha \subset \breve{D}_{\alpha'}$, where $\alpha \subset \alpha' \subset f$. Therefore if $X \in \mathcal{I}$, there is a least $j$ such that $X \subseteq_{\mathcal{R}(C)} \breve{D}_j$. Furthermore, by Theorem 5.1 Parts 4b and 4c, $X \subseteq_{\mathcal{R}(C)} \bigcup_{i \leq j} S_i$.

## 6. Special $\mathcal{L}$-patterns

Now it is necessary to know how the special $\mathcal{L}$-patterns $\mathcal{P}_i$ and the definable *and* $\Sigma^0_3$ property $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, Y)$ are defined. The reader is directed to Section 3 of Cholak and Harrington [3].

As one reads Section 3 of [3] there are some important differences to keep in mind. In Definition 3.13 of [3], the property $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C)$ is defined. Here we are interested in the property $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, Y)$ which is defined as $\varphi_{\mathcal{P}}(\emptyset, \vec{U}, \vec{B}, Y)$. Hence the roles of the $A$ are different. If one is reading Section 3 of [3] just to understand this paper, one should read the $A$ in [3] as $\emptyset$. The roles of $C$ in both papers are also different. In [3], $C$ is used as a variable; here $C$ is a fixed set and we are using $Y$ as a variable.

One important fact to notice is that if $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, Y)$ holds and $Z \subseteq Y$ then $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, Z)$ holds (If $\vec{D}$ witnesses that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, Y)$ holds then the $\vec{D}$ formed by intersecting each complement of original $\vec{D}$ with $Z$ witnesses that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, Z)$ holds).

The following theorem is both an extension and modification of Theorem 4.1 from Cholak and Harrington [3] into the current situation. It is about universal $\mathcal{B}$-interpretations.

**Theorem 6.1.** *Fix $\mathcal{L}$, $\vec{U}$ an $\mathcal{L}$-interpretation, $\mathcal{P} = (\mathcal{T}, \mathcal{R}, \mathcal{B}, l)$ a special $\mathcal{L}$-pattern, and $\mathcal{I}$ a $\Sigma^0_3$-ideal of $\mathcal{S}_{\mathcal{R}}(C)$. Uniformly in $\vec{U}$, $\mathcal{P}$, $A$, $C$, and $\mathcal{I}$ (think of all of these items as indexed by $\mathcal{P}$ and $\mathcal{I}$), there is a $\mathcal{B}$-interpretation, $\vec{B}_{\mathcal{P},\mathcal{I}}$, such that the following are equivalent:*

(1) *there is a robust split $S \in \mathcal{I}$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_{\mathcal{P},\mathcal{I}}, S)$;*
(2) *for all $\mathcal{B}$-interpretations, $\vec{B}$, there is a robust split $\check{S} \in \mathcal{I}$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, \check{S})$.*

*Proof.* (2) implies (1) is clear. The rest of the proof will focus on the other direction.

The main idea in the proof of Theorem 4.1 of [3] is that for a noncomputable modulo $A$ split $S$ of $C = W$ such that $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_{\mathcal{P}}, S)$, using the Owings Splitting Theorem, we split $S$ into infinitely many pairwise disjoint splits $S_i$ and on $S_i$ we copy the $i$th $\mathcal{B}$-interpretation, $\vec{B}_i$. Here we must first ensure that $S \in \mathcal{I}$ and then replace "noncomputable modulo $A$ split" with "robust split". (An Owings split over the empty set is a Friedberg split.)

It would be nice if we could just refer the proof of Theorem 4.1 of [3] but there are some subtle differences. We can, however, use the proof in [3] as a very good first approximation. The only difference between the proof there and here is the construction of the splits $S_\alpha$. There if $\widehat{S_i}$ is the first noncomputable modulo $A$ split so is $S_\alpha$, where $\alpha \subset f$ and $\alpha$ is of length $i+1$. Here $S_\alpha$ must be a Friedberg split of $\widehat{S_i}$ and hence, by Lemma 4.10 (or Lemma 4.11 in the dual case), if $\widehat{S_i}$ is a robust split so is $S_\alpha$. This is true for all $i$ and the corresponding $\alpha \subset f$ and hence the result is slightly stronger.

Changing the construction in [3] so that $S_\alpha$ is a Friedberg split of $\widehat{S_i}$ is not difficult but it is nontrivial so we provide a formal proof below. Since the construction of the $S_\alpha$ here gives a better result and it and its verification are shorter than what was used in [3], it should have been used there. But in the interest of brevity we will focus just on the above theorem.

We will use the tree method. We can safely assume, by the recursion theory, that we have an index for $\vec{B}_e$. If $S$ is a split of $C$ then there is an enumeration of $S$ such that if $x$ enters $C$ at stage $s$ then $x$ enters either $S$ or $C - S$ at stage $s$. Call such an enumeration of $S$ a *quick* split of $C$. $S \in \mathcal{I}$ is

$\Sigma^0_3$ so there is a $\Pi^0_2$ formula $\Xi(l)$ such that $S \in \mathcal{I}$ iff there is an $l$ such that $\Xi(l)$. We say such an $l$ witnesses that $S \in \mathcal{I}$.

Construct a sequence of 4-tuples $(\check{S}_i, \tilde{S}_i, \vec{D}_i, l_i)$. Now asking if $\tilde{S}_i$ witnesses $\check{S}_i$ is a quick split of $C$, $\vec{D}_i$ witnesses that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_e, \check{S}_i)$, and $l_i$ witnesses that $\check{S}_i \in \mathcal{I}$ is $\Pi^0_2$. We need to code this information on a $\Pi^0_2$ tree.

We define the true path as follows: Let $\alpha \in 2^{<\omega}$ be of length $i$ such that $\alpha \subset f$. We will let $\alpha\widehat{\ }0 \subset f$ iff $\tilde{S}_i$ witnesses $\check{S}_i$ is a quick split of $C$, $\vec{D}_i$ witnesses that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_e, \check{S}_i)$, and $l_i$ witnesses that $\check{S}_i \in \mathcal{I}$.

Now we need to define the approximation to the true path $f_s$. Let's assume that $\tilde{S}_i$ witnesses $\check{S}_i$ is a quick split of $C$, $\vec{D}_i$ witnesses $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_e, \check{S}_i)$, and $l_i$ witnesses that $\check{S}_i \in \mathcal{I}$ iff $(\forall x)(\exists s)[\Theta(x, s)]$, where $\Theta$ is computable ($\Theta$ can be found uniformly in the given parameters—Remark 3.14 of [3] might be useful here). Let $l_\alpha(s)$ be the greatest $x$ such that for all $y < x$ there is an $s_y \leq s$ such that $\Theta(y, s_y)$. Assume that $\alpha \subseteq f_s$ and $|\alpha| < s$. Then let $\alpha\widehat{\ }0 \subseteq f_s$ iff $l_\alpha(s) > l_\alpha(t)$, where $t$ is the last stage when $\alpha\widehat{\ }0 \subseteq f_t$ (if such a stage does not exist let $t = 0$). It is straightforward to show that $f = \liminf_s f_s$.

We need to discuss the action at $\beta = \alpha\widehat{\ }0$. At $\beta$ we will effectively build a Friedberg split $S_\beta$ of $\check{S}_i$. So we must meet the following requirements:

$$\mathcal{R}_{\beta,j} \qquad W_j \searrow S_i \text{ is infinite then } W_j \cap S_\beta \neq \emptyset,$$

for all $j$.

To this end we will maintain a list of requirements $\mathfrak{R}$. When $\beta \subseteq f_s$ we will place the requirements $\mathcal{R}_{\beta,j}$ on the list for all $j \leq s$. If $\beta$ is to right of $f_s$ we will cancel all the requirements $\beta$ placed on $\mathfrak{R}$.

As a ball $x$ enters $C$ at stage $s$ we will go through the uncanceled unmet requirements in order of placement on $\mathfrak{R}$ trying to meet them. For $\mathcal{R}_{\beta,j}$ if $x$ in $W_j \searrow S_i$ at stage $s$ and $x$ is not in any $S_\gamma$ at this (sub)stage of the construction then add $x$ to $S_\beta$ and we say $\mathcal{R}_{\beta,j}$ has been met.

If $\beta \subset f$ then $S_\beta$ is a Friedberg split of $\check{S}_i$. Eventually the requirement $\mathcal{R}_{\beta,j}$ is placed on $\mathfrak{R}$ and never canceled. Eventually all the requirements placed earlier on $\mathfrak{R}$ will be met or never receive any action as above. After such a stage, since $\check{S}_i$ is a quick split, if $W_j \searrow \check{S}_i$ then some $x$ will meet $\mathcal{R}_{\beta,j}$.

On $S_\beta$ we will use the module from Section 4.1.1 of [3]. Using the Friedberg Splitting Theorem, we will effectively split $S_\beta$ into $S_{\beta,j}$, for $j \in \omega$. On $S_{\beta,j}$, we can replicate the $j$th $\mathcal{B}$-interpretation, $\vec{B}_j$; that is, we can let $B_{e,p} \cap S_{\beta,j} = B_{j,p} \cap S_{\beta,j}$, for all $p \in \mathcal{T}$.

The construction of $S_\beta$ is uniformly effective in $\beta$. The $S_\beta$ are constructed to be pairwise disjoint. Therefore the construction of $\vec{B}_e$ is uniform in $\vec{U}$, $\mathcal{P}$, $A$, and $C$ and hence it is legal to use the Recursion Theorem as we did.

Let $S$ be a robust split in $\mathcal{I}$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_e, S)$ holds. Then there is an $i$ such that $S = \check{S}_i$, $\tilde{S}_i$ witnesses $\check{S}_i$ is a quick split of $C$, $\vec{D}_i$ witnesses that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_e, \check{S}_i)$, and $l_i$ witnesses that $\check{S}_i \in \mathcal{I}$. Let $\beta \subset f$ be of length $i+1$. Then, by Lemma 4.10 (or Lemma 4.11 for the dual), $S_\beta$ is a robust split in $\mathcal{I}$. Then, by the same reasoning, $S_{\beta,j}$ is a robust split in $\mathcal{I}$. Therefore $S_{\beta,j}$ is a robust split in $\mathcal{I}$ and $\vec{D}_i$ witnesses $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_j, S_{\beta,j})$. □

A $\mathcal{B}$-interpretation, $\vec{B}$, is $\mathcal{I}$-*universal* iff it satisfies the equivalence in Theorem 6.1. Whether a $\mathcal{B}$-interpretation, $\vec{B}$, is $\mathcal{I}$-universal is not a $\Sigma^0_3$ property; it is, however, definable and hence invariant under automorphisms. Since for our discussion $\mathcal{I}$ is a fixed (but still undefined) $\Sigma^0_3$-ideal, we can drop the $\mathcal{I}$ from $\vec{B}_{\mathcal{P},\mathcal{I}}$ to get $\vec{B}_{\mathcal{P}}$.

Let $T$ be a split of $A$ such that $T \cap B$ is not computable. If instead of considering the robust splits $S$ of $C$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_{\mathcal{P}}, S)$ we consider the robust splits $S$ of $C$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_{\mathcal{P}}, S \cap T)$ then we get the following:

**Theorem 6.2.** *Fix $\mathcal{L}$, $\vec{U}$ an $\mathcal{L}$-interpretation, $\mathcal{P} = (\mathcal{T}, \mathcal{R}, \mathcal{B}, l)$ a special $\mathcal{L}$-pattern, and $\mathcal{I}$ a $\Sigma^0_3$-ideal of $\mathcal{S}_{\mathcal{R}}(C)$. Let $T$ be a split of $A$ such that $T \cap B$ is not computable. Uniformly in $\vec{U}$, $\mathcal{P}$, $A$, $W$, $\mathcal{I}$, and $T$ (think of all of these items indexed by $\mathcal{P}$, $\mathcal{I}$ and $T$), there is a $\mathcal{B}$-interpretation, $\vec{B}_{\mathcal{P},T}$, such that the following are equivalent:*

1. *there is a robust split $S \in \mathcal{I}$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}_{\mathcal{P},T}, S \cap T)$;*
2. *for all $\mathcal{B}$-interpretations, $\vec{B}$, there is a robust split $\check{S} \in \mathcal{I}$ such that $\varphi_{\mathcal{P}}(\vec{U}, \vec{B}, \check{S} \cap T)$.*

Assuming that $\widehat{\mathcal{I}}$ is $\Sigma^0_3$, a hatted version of the above two theorems holds as well. Fix $\mathcal{P}$ and $\vec{U}$. Let $\vec{B}_{\mathcal{P},T}$ be the universal $\mathcal{B}$-interpretation effectively given to us by Theorem 6.2. $\Phi(\vec{B}_{\mathcal{P},T})$ is a universal $\widehat{\mathcal{B}}$-interpretation (since being universal is invariant under automorphisms), but not necessarily the one, $\vec{\widehat{B}}_{\widehat{\mathcal{P}},\widehat{T}}$, effectively given to us by the hatted version of Theorem 6.2.

We only put hats on the special $\mathcal{L}$-patterns $\mathcal{P}$ when we are using the hatted version of Theorem 6.1 to effectively get the universal pattern, $\vec{\widehat{B}}_{\widehat{\mathcal{P}}}$. This is a universal $\mathcal{B}$-interpretation which lives in $\widehat{\omega}$. Similarly if $\widehat{T}$ is a split of $\widehat{A}$ then $\vec{\widehat{\mathcal{B}}}_{\widehat{P},\widehat{T}}$ is the special $\mathcal{L}$-pattern given to us by the hatted version of Theorem 6.2.

The following theorem is a weaker version of Theorem 5.1 of [3]. It is about the special $\mathcal{L}$-patterns $\mathcal{P}_i$. The definitions of $\mathcal{P}_i$ can be found in Section 3.6 of [3].

**Theorem 6.3** (Theorem 5.1 of Cholak and Harrington [3])**.** *Effectively in $i$, a $\mathcal{B}$-interpretation $\vec{B}$, and $S$, a subset of $C$, there are $\vec{U}$ and $\vec{D}$ such that $\vec{D}$*

*witnesses that* $\varphi_{\mathcal{P}_i}(\vec{U}, \vec{B}, S)$ *holds (and all the components of* $\vec{U}$ *and* $\vec{D}$ *are contained in* $S$*).*

*Furthermore, for all* $j \neq i$*, effectively in* $j$ *there are* $\vec{B}_j$ *(each component being completely contained in* $S$*) such that for all splits* $\tilde{S}$ *of* $C$*, either* $\tilde{S}$ *is computable or, for all* $\vec{X}$*,* $\vec{X}$ *does not witness* $\varphi_{\mathcal{P}_j}(\vec{U}, \vec{B}_j, \tilde{S})$ *(for the same set* $\vec{U}$ *as above).*

The $\vec{B}_j$ is $\vec{B}_{j,C}$ from Theorem 5.1 of [3] which witnesses that $\vec{U}$, $\emptyset$, and $C$ do not realize $\mathcal{P}_j$ (we apply Theorem 5.1 of [3] to $S = W$ and $W$ and, moreover, in the second clause of Theorem 5.1 of [3], we only consider the one $C$ given to us by Theorem 6.3 not all subsets of $W$). In the notation we are using here, this means that, for all splits $\tilde{S}$ of $C$, either $\tilde{S}$ is computable or, for all $\vec{X}$, $\vec{X}$ does not witness $\varphi_{\mathcal{P}_j}(\vec{U}, \vec{B}_j, \tilde{S})$ (see Definition 3.17 of [3] and the two negative requirements in Section 5.1.2 of [3]).

This version is weaker than the original in the sense that the second paragraph of Theorem 5.3 of [3] actually holds for all subsets of $C$ in the original version. That is, for all $Z \subseteq C$ one can find a $\vec{B}_{j,Z}$ such that for all splits $\tilde{S}$ of $Z$, either $\tilde{S}$ is computable or, for all $\vec{X}$, $\vec{X}$ does not witness $\varphi_{\mathcal{P}_j}(\vec{U}, \vec{B}_{j,Z}, \tilde{S})$ (again see Definition 3.17 of [3] and the two negative requirements in Section 5.1.2 of [3] and Definition 3.17 of [3]).

## 7. Coding

We are ready to proceed with the coding: Let $\vec{B}_{i,k}$ be a computable list of all $\mathcal{B}_i$-interpretations, where $\mathcal{P}_i = \langle \mathcal{T}_i, \mathcal{R}_i, \mathcal{B}_i, l_i \rangle$. Again the definition of the $\mathcal{P}_i$ can be found in Section 3.6 of [3]. So there are a $k$ and $l$ such that $\vec{B}_{i,k} = \vec{B}_{\mathcal{P}_i,T_i}$ and $\vec{B}_{i,l} = \vec{B}_{\mathcal{P}_i,\breve{T}_i}$, where $\vec{B}_{\mathcal{P}_i,T_i}$, and $\vec{B}_{\mathcal{P}_i,\breve{T}_i}$ are from Theorem 6.2.

Let $\alpha$ be such that $|\alpha| = \langle i, k, l \rangle$. We will apply Theorem 6.3 to $2i$, $\vec{B}_{2i,k}$, and $S_\alpha \cap T_\alpha$. We also apply Theorem 6.3 to $2i+1$, $\vec{B}_{2i+1,l}$, and $S_\alpha \cap \breve{T}_\alpha$.

Each application of Theorem 6.3 uniformly creates a $\vec{U}_{\alpha,0}$ whose components are contained in $S_\alpha \cap T_\alpha$ and a $\vec{U}_{\alpha,1}$ whose components are contained in $S_\alpha \cap \breve{T}_\alpha$. Since all the $S_{\alpha,i}$ are componentwise pairwise disjoint, we can let $\vec{U}$ be the componentwise effective union (over the $i$s) of the $\vec{U}_{\alpha,i}$s. Hence

$$\varphi_{\mathcal{P}_{2i}}(\vec{U}, \vec{B}_{2i,k}, S_\alpha \cap T_\alpha) \text{ and } \varphi_{\mathcal{P}_{2i+1}}(\vec{U}, \vec{B}_{2i+1,l}, S_\alpha \cap \breve{T}_\alpha). \tag{7.1}$$

Furthermore, by Theorem 6.3, if $j \neq 2i$ then there is a $\vec{B}_{j,\alpha,0}$ such that for all splits $\tilde{S}$ of $S_\alpha \cap T_\alpha$ either $\tilde{S}$ is computable or $\varphi_{\mathcal{P}_j}(\vec{U}, \vec{B}_{j,\alpha,0}, \tilde{S})$ fails. Similarly for $j \neq 2i+1$ then there is a $\vec{B}_{j,\alpha,1}$ such that for all splits $\tilde{S}$ of $S_\alpha \cap \breve{T}_\alpha$ either $\tilde{S}$ is computable or $\varphi_{\mathcal{P}_j}(\vec{U}, \vec{B}_{j,\alpha,1}, \tilde{S})$ fails. Since the $\vec{B}_{j,\alpha,r}$ were uniformly constructed and their components are contained in $S_\alpha \cap T_\alpha$

or $S_\alpha \cap \breve{T}_\alpha$, we can let $\vec{B}_j$ be the componentwise effective union (over the $j$s) of the $\vec{B}_{j,\alpha,r}$s.

**Lemma 7.1.** *Assume $\check{S}_0$ and $\check{S}_1$ are robust splits in $\mathcal{I}$. Assume $T$ is a split of $A$, $\varphi_{\mathcal{P}_{2i}}(\vec{U}, \vec{B}_{2i}, \check{S}_0 \cap T)$ and $\varphi_{\mathcal{P}_{2i+1}}(\vec{U}, \vec{B}_{2i+1}, \check{S}_1 \cap \breve{T})$. Then $T \triangle X_i \cap B$ is computable and, by Lemma 3.7, $T \equiv_{\mathcal{R}(A)} X_i$. (The $X_i$ were defined just before Theorem 5.1 and used in that theorem.)*

*Proof.* Since $\check{S}_i$ is in $\mathcal{I}$, there is a least $m$ such that $\check{S}_i \subseteq_{\mathcal{R}(C)} \bigcup_{j \leq m} S_j$. $T - X_i$ is a split of $A$. $\check{S}_0 \cap (T - X_i) \subseteq \check{S}_0 \cap T$ and $\varphi_{\mathcal{P}_{2i}}(\vec{U}, \vec{B}_{2i}, \check{S}_0 \cap (T - X_i))$. So $j \leq m$, $\varphi_{\mathcal{P}_{2i}}(\vec{U}, \vec{B}_{2i}, S_j \cap \check{S}_0 \cap (T - X_i))$. By the above construction of $\vec{B}_{2i}$, for $2i$, Theorem 6.3 is applied inside $S_\alpha \cap X_i$. Therefore, for each $j \leq m$, $S_j \cap \check{S}_0 \cap (T - X_i)$ is computable. Therefore $\check{S}_0 \cap (T - X_i)$ is computable. Since $\check{S}_0$ is robust, $(T - X_i) \cap B$ is computable. Similarly, using $2i + 1$ and $\breve{T}$ rather than $2i$ and $T$, $(X_i - T) \cap B$ is computable. So $T \triangle X_i \cap B$ is computable. Now use Lemma 3.7. □

Since the properties involved in Lemma 7.1 are definable, there is a hatted version:

**Lemma 7.2.** *Assume $\widehat{\check{S}}_0$ and $\widehat{\check{S}}_1$ are robust splits in $\widehat{\mathcal{I}}$. Assume $\widehat{T}$ is a split of $\widehat{A}$, $\varphi_{\mathcal{P}_{2i}}(\Phi(\vec{U}), \Phi(\vec{B}_{2i}), \widehat{\check{S}}_0 \cap \widehat{T})$ and $\varphi_{\mathcal{P}_{2i+1}}(\Phi(\vec{U}), \Phi(\vec{B}_{2i+1}), \widehat{\check{S}}_1 - \widehat{T})$. Then $\widehat{T} \triangle \widehat{X}_i \cap \widehat{B}$ is computable and, by the hatted version of Lemma 3.7, $\widehat{T} \equiv_{\mathcal{R}(\widehat{A})} \widehat{X}_i$.*

## 8. Ideals

We want to use the hatted version of Theorem 6.2. To do this we must show that $\widehat{\mathcal{I}}$ is a $\Sigma^0_3$-ideal. In addition, it will be very useful to show that $\widehat{\mathcal{I}}$ is definable using a finite number of parameters. The following theorem of Harrington provides us with those tools.

**Harrington's Ideal Definability Theorem.** *For $n \geq 1$, there is a formula $\psi_n(X, C, \vec{Y})$, where $|\vec{Y}| = n$, such that, for each $\vec{Y}$, $\{X : \psi_n(X, C, \vec{Y})\}$ is a $\Sigma^0_{2n+1}$-ideal of $\mathcal{S}_{\mathcal{R}}(C)$ and, for each $\Sigma^0_{2n+1}$-ideal $\mathcal{I}$ of $\mathcal{S}_{\mathcal{R}}(C)$, there is a $\vec{Y}$ such that $\mathcal{I} = \{X : \psi_n(X, C, \vec{Y})\}$.*

The proof of Harrington's Ideal Definability Theorem can be found in Harrington and Nies [7]. A short proof for the case when $n = 1$ with an additional parameter can be found in Nies [11]. Here we only need the case when $n = 1$.

Since $\mathcal{I}$ is a $\Sigma^0_3$-ideal of $\mathcal{S}_{\mathcal{R}}(C)$, there are a $\psi$ and $\vec{Y}$ such that $\mathcal{I}$ is the collection of all $X$ such that $\psi(X, C, \vec{Y})$. Since $\Phi$ is an automorphism,

$\widehat{\mathcal{I}} = \{\widehat{X} : \psi(\widehat{X}, \widehat{C}, \widehat{\vec{Y}})\}$. By the hatted version of Harrington's Ideal Definability Theorem, $\widehat{\mathcal{I}}$ is a $\Sigma^0_3$-ideal of $\mathcal{S}_{\mathcal{R}}(\widehat{C})$. $\widehat{\mathcal{I}}$ is also definable using $\psi$ and $\widehat{\vec{Y}}$. From now on we will treat $\widehat{\mathcal{I}}$ as a parameter.

## 9. Promptness and smallness

Now it is time to use the dynamic properties we have built into $B$ and $\widehat{C}$ and various smallness conditions. We will use the hatted and nonhatted versions of these lemmas but for notational ease we will present the nonhatted versions.

**Lemma 9.1.** *Let $E \subseteq D$ be subsets of $B$ and $S = (D - E) \cap C$. Assume $C - D \in \mathcal{I}$, $C - E \in \mathcal{I}$, $\breve{S} = C \cap E$, $S \sqcup \breve{S} = D \cap C$, $D \cap C$ is promptly simple in $D$ via the computable function $p$, there is a computable function $g$ such that if $x \in (D \cap C)_{p(s)}$ then either $x \in S_{g(s)}$ or $x \in \breve{S}_{g(s)}$, and there is a computable set $R$ such that $\overline{R} \subseteq \breve{S}$ and for all $x \in R$ and all stages $s$, if $x \in D_{\text{at } s}$ then $x \notin \breve{S}_{g(s)}$.*

*Then $S$ is a robust split in $\mathcal{I}$.*

*Proof.* First we will show $D \cap C$ is robust. Since $C - D \in \mathcal{I}$, there is an $i$ such that $D_i \cap C \subseteq^* D \cap C$. So $S_i$ is almost a split of $D \cap C$. Since $S_i$ is robust, by Lemma 4.7, $D \cap C$ is robust.

Let $T$ be a split of $A$ such that $T \cap B$ is not computable. Since $D \cap C$ is robust, $T \cap D \cap C$ is not computable. Now $T \cap D \cap C = T \cap (S \sqcup \breve{S}) = T \cap (S \sqcup (R \cap \breve{S}) \sqcup \overline{R}) = (T \cap S) \sqcup (T \cap R \cap \breve{S}) \sqcup (T \cap \overline{R})$.

Assume, for a contradiction, that $W$ witnesses that $T \cap S$ is computable. So $(T \cap S) \sqcup W = \omega$. Since $T \cap \overline{R}$ is computable, $T \cap R \cap \breve{S}$ is not computable and hence infinite. Now $\breve{S} \subseteq W \cap D$. So $T \cap R \cap \breve{S} \subseteq T \cap R \cap W \cap D$. Hence $T \cap R \cap W \cap D$ is infinite.

There are infinitely many $x$ and stages $s$ such that $x \in (T \cap R \cap W)_s$ and $x \in D_{\text{at } s}$. (If necessary speed up the enumeration of $T \cap R \cap W$. When $x$ enters $D$ then $x$ is in $A$ so we can speed up the enumeration of $T$ and $\breve{T}$ such that $x$ is one of these sets at stage $s$. Since $R$ and $W$ are computable we can do the same for them.)

Therefore, since $C$ is promptly simple in $D$, there is an $x$ and a stage $s$ such that $x \in (T \cap R \cap W)_s$, $x \in D_{\text{at } s}$, and $x \in (D \cap C)_{p(s)}$. Thus $x \in S_{g(s)}$. So $x \in W \cap T \cap S$, a contradiction since $W \cap T \cap S$ is empty.

Hence $S$ is robust. It is clear that $S$ is a split of $C$ in $\mathcal{I}$. □

**Lemma 9.2.** *Let $E \subseteq D$ be subsets of $B$ such that $E \cap C$ and $D \cap C$ are splits of $C$. Let $S = (D - E) \cap C$ and $\breve{S} = C \cap E$ (so $S \sqcup \breve{S} = D \cap C$). Assume $D \cap C$ is promptly simple in $D$ via the computable function $p$. Furthermore assume that $E$ is a small subset of $D$.*

*Then there is a computable function $g$ such that if $x \in (D \cap C)_{p(s)}$ then either $x \in S_{g(s)}$ or $x \in \breve{S}_{g(s)}$, and there is a computable set $R$ such that $\overline{R} \subseteq \breve{S}$ and for all $x \in R$ and all stages $s$, if $x \in D_{at\ s}$ then $x \notin \breve{S}_{g(s)}$.*

*Proof.* Since $S \sqcup \breve{S} = D \cap C$, the computable function $g$ clearly exists. By Lemma 3.2, $\breve{S}$ is a small subset of $D$. Let $Y = \{x|\exists s[x \in D_{\text{at } s} \wedge x \notin \breve{S}_{g(s)}]\}$. $\omega \cap (D - \breve{S}) = D - \breve{S} \subseteq Y$. So, by smallness, $R = Y \cup (\omega - D)$ is a computably enumerable set. $\overline{R} = \{x|\exists s[x \in D_{\text{at } s} \wedge x \in \breve{S}_{g(s)}]\}$. So $R$ is computable. Clearly $R$ has the desired properties. □

## 10. The decoding

Using $\widehat{B}$, $\widehat{C}$, and $\widehat{\mathcal{I}}$ as parameters, we can tell in a $\Sigma^0_3$-fashion if a split $\widehat{T}$ of $\widehat{A}$ is equivalent modulo $\mathcal{R}(\widehat{A})$ to $\widehat{T}_j$, for some $T_j$.

**Theorem 10.1.** *Let $X_i$ be a split of $A$ such that $X_i \backslash B$ is infinite (here we are using the listings of splits fixed just before Theorem 5.1). Then a split $\widehat{T}$ of $\widehat{A}$ is equivalent modulo $\mathcal{R}(\widehat{A})$ to $\widehat{X}_i$ iff there are subsets $\widehat{E} \subset \widehat{D}$ of $\widehat{B}$ such that the following hold:*

(1) *$\widehat{C} - \widehat{D}$ and $\widehat{C} - \widehat{E}$ are in $\widehat{\mathcal{I}}$.*
(2) *Let $\widehat{S} = (\widehat{D} - \widehat{E}) \cap \widehat{C}$ and $\breve{S} = \widehat{C} \cap \widehat{E}$. There is a computable function $g$ such that if $x \in (\widehat{D} \cap \widehat{C})_{\widehat{p}(s)}$ then either $x \in \widehat{S}_{g(s)}$ or $x \in \breve{S}_{g(s)}$, and there is a set $\widehat{R}$ such that $\widehat{R}$ is computable, $\overline{\widehat{R}} \subseteq \breve{S}$, and for all $x \in R$ and all stages $s$, if $x \in \widehat{D}_{at\ s}$ then $x \notin \breve{S}_{g(s)}$. (Recall $\widehat{C}$ is promptly simple in $\widehat{B}$ via $\widehat{p}$; see Theorem 3.9.)*
(3) $\varphi_{\mathcal{P}_{2i}}(\Phi(\vec{U}), \vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i},\widehat{T}}, \widehat{S} \cap \widehat{T})$.
(4) $\varphi_{\mathcal{P}_{2i+1}}(\Phi(\vec{U}), \vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i+1},\widehat{A}-\widehat{T}}, \widehat{S} - \widehat{T})$.

*Proof.* ($\Rightarrow$) Recall that since $T \equiv_{\mathcal{R}(A)} X_i$, $T \triangle X_i$ is computable. First find the inverse of the hatted universal patterns $\vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i},\widehat{T}}$ and $\vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i+1},\widehat{A}-\widehat{T}}$ (given to us by the hatted version of Theorem 6.2). Let $\vec{B}_{2i,k} = \Phi^{-1}(\vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i},\widehat{T}})$ and $\vec{B}_{2i+1,l} = \Phi^{-1}(\vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i+1},\widehat{A}-\widehat{T}})$ (recall $\vec{B}_{n,m}$ is a computable list of all $\mathcal{B}_n$-interpretations).

Let $\alpha \subset f$ be such that $|\alpha| = \langle i, k, l\rangle$. Let $j = |\alpha|$ and $j' > j$ be the least $j' = \langle i', k', l'\rangle$ such that $X_{i'} \sqcup Y_{i'} = A$ and $X_{i'} \backslash B$ is infinite. By the coding in Section 7 (see Equation 7.1) and by applying $\Phi$, $\varphi_{\mathcal{P}_{2i}}(\Phi(\vec{U}), \vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i},\widehat{T}}, \widehat{S}_j \cap \widehat{X}_i)$ and $\varphi_{\mathcal{P}_{2i+1}}(\Phi(\vec{U}), \vec{\widehat{B}}_{\widehat{\mathcal{P}}_{2i+1},\widehat{A}-\widehat{T}}, \widehat{S}_j - \widehat{X}_i)$. By Lemma 4.8, $S_j \cap T$ and $S_j \cap X_i$ are equivalent modulo $\mathcal{R}(C)$. Similarly for $S_j - T$ and $S_j - X_i$. Let $S = S_j - (T \triangle X_i)$. Let $D = D_j - (T \triangle X_i)$ and $E = D_{j'} - (T \triangle X_i)$. $\widehat{C} - \widehat{D}$ and $\widehat{C} - \widehat{E}$ are in $\widehat{\mathcal{I}}$.

Since $\widehat{C}$ is promptly simple in $\widehat{B}$ via the computable function $\widehat{p}$ (see Theorem 3.9) and $\widehat{D} - \widehat{C}$ is infinite, $\widehat{D} \cap \widehat{C}$ is promptly simple in $\widehat{D}$ via the computable function $\widehat{p}$.

By Theorem 5.1 Part 4e and Lemma 3.2, $E$ is small in $D$. So $\widehat{E}$ is small in $\widehat{D}$. Hence we can apply the hatted version of Lemma 9.2 to get Condition 10.1.2. Using Equation 7.1 and our choice of $S$, the remaining two conditions are easy to verify.

($\Leftarrow$) $\widehat{D} \cap \widehat{C}$ is promptly simple in $\widehat{D}$ via the computable function $\widehat{p}$. Hence by the hatted version of Lemma 9.1, $\widehat{S}$ is a robust split. $\widehat{\vec{B}}_{\widehat{\mathcal{P}}_{2i}, \widehat{T}}$ is the universal $\mathcal{B}$-interpretation given by the hatted version of Theorem 6.2, similarly for $\widehat{\vec{B}}_{\widehat{\mathcal{P}}_{2i+1}, \widehat{A} - \widehat{T}}$. Hence, by the hatted version of Theorem 6.2, there are robust splits $\check{\widehat{S}}_0$ and $\check{\widehat{S}}_1$ contained in $\widehat{\mathcal{I}}$ such that $\varphi_{\mathcal{P}_{2i}}(\Phi(\vec{U}), \Phi(\vec{B}_{2i}), \check{\widehat{S}}_0 \cap \widehat{T})$ and $\varphi_{\mathcal{P}_{2i+1}}(\Phi(\vec{U}), \Phi(\vec{B}_{2i+1}), \check{\widehat{S}}_1 - \widehat{T})$. By Lemma 7.2, $\widehat{T} \equiv_{\mathcal{R}(\widehat{A})} \widehat{X}_i$. □

By counting the quantifiers one can verify that Condition 10.1.2 is $\Sigma^0_3$. Since $\varphi$ is $\Sigma^0_3$, Condition 10.1.3 is $\Sigma^0_3$. Condition 10.1.4 is $\Sigma^0_3$: there exists a $\check{\widehat{T}}$ such that $\widehat{T} \sqcup \check{\widehat{T}} = A$ and $\varphi_{\mathcal{P}_{2i+1}}(\Phi(\vec{U}), \widehat{\vec{B}}_{\widehat{\mathcal{P}}_{2i+1}}, \widehat{S} \cap \check{\widehat{T}})$. So the conditions of Theorem 10.1 are $\Sigma^0_3$.

Hence the map which takes a split $X_i$ of $A$ to a split $\widehat{T}$ of $\widehat{A}$ satisfying the conditions of Theorem 10.1 is an isomorphism between $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ and is $\Sigma^0_3$. But all $\Sigma^0_3$ (total) functions are $\Delta^0_3$ so we are done. This completes the proof of Theorem 1.2.

## 11. The hhsimple sets

11.1. **The substructure $\mathcal{C}_{\mathcal{R}}(A)$.** If $R$ is a computable set then $R \cap A$ is a split of $A$. Let $\mathcal{C}(A)$ be $\{R \cap A | R \text{ is computable}\}$. $\mathcal{C}(A)$ is the splits of $A$ which are formed by the computable sets, the computable splits. $\mathcal{C}(A)$ is a substructure of $\mathcal{S}(A)$. Let $\mathcal{C}_{\mathcal{R}}(A)$ be $\mathcal{C}(A)$ modulo $\mathcal{R}(A)$.

**Lemma 11.1.** *In $\mathcal{E}$, with a parameter for $A$, $\mathcal{C}_{\mathcal{R}}(A)$ is a definable substructure of $\mathcal{S}_{\mathcal{R}}(A)$.*

11.2. **$\mathcal{C}_{\mathcal{R}}(H)$ and $\mathcal{L}^*(H)$.** Let $H$ be a hhsimple set. Let $W$ be any computably enumerable set. By Lachlan, see Soare [13, Corollary X.2.7], $\mathcal{L}(H)$ is a Boolean algebra and hence there is a computably enumerable set $X$ such that $W \cup X \cup H = \omega$ and $W \cap X \subseteq H$. Let $R_W = (W \cup H) \backslash X$. Since $\overline{R_W} = X \backslash (W \cup H)$, $R_W$ is computable and $W \cap \overline{H} = R_W \cap \overline{H}$. This fact is nothing new and, in fact, can be found in Soare [13, Exercise X.2.12]. An index for $R_W$ can be found from an index of $X$ which can be found computably in $\mathbf{0}''$.

**Lemma 11.2.** *Let $H$ be hhsimple. Let $\Phi((W \cup H)^*) = (R_W \cap H)^{\mathcal{R}(A)}$. Then $\Phi$ is a $\Delta^0_3$-isomorphism between $\mathcal{L}^*(H)$ and $\mathcal{C}_{\mathcal{R}}(H)$.*

*Proof.* By the above, $\Phi$ is $\Delta^0_3$. To show $\Phi$ is an isomorphism we must show $\Phi$ is well defined, onto, one to one, and order preserving.

If $W_1 \cup H =^* W_2 \cup H$ then the symmetric difference of $R_{W_1}$ and $R_{W_2}$ is almost a computable subset of $H$ and hence $R_{W_1} \cap H \equiv_{\mathcal{R}(H)} R_{W_2} \cap H$. Assume $R_1 \cap H \equiv_{\mathcal{R}(H)} R_2 \cap H$. So $R = (R_1 \triangle R_2) \cap H$ is a computable subset of $H$. Then $(R_1 \triangle R_2) - R$ is a computable subset of $\overline{H}$. Since $H$ is simple, $(R_1 \triangle R_2) - R$ is finite. Therefore $R_1 \cup H =^* R_2 \cup H$. So $\Phi$ is well defined, onto, and one to one.

Assume $W_1 \subset W_2$ and $W_1 \neq^* W_2$. Then $(R_{W_1} \cup R_{W_2}) \cap \overline{H} =^* W_2 \cap \overline{H}$ and $R_{W_1} \cap R_{W_2} \cap \overline{H} =^* W_1 \cap \overline{H}$. Since $R_{W_1} \cap R_{W_2} \cap H \subset (R_{W_1} \cup R_{W_2}) \cap H$ and $\Phi$ is well defined, onto, and one to one, $R_{W_1} \subset_{\mathcal{R}(H)} R_{W_2}$ and $R_{W_1} \not\equiv_{\mathcal{R}(H)} R_{W_2}$. Therefore $\Phi$ is order preserving and hence an isomorphism. □

11.3. **The proof of Theorem 1.3.** The goal of this section is to prove Theorem 1.3.

**Theorem 1.3.** *Let $H$ and $\widehat{H}$ be hhsimple. $H$ and $\widehat{H}$ are automorphic iff they are $\Delta^0_3$-automorphic iff $\mathcal{L}^*(H) \simeq_{\Delta^0_3} \mathcal{L}^*(\widehat{H})$.*

This relies heavily on a result of Maass:

**Theorem 11.3** (Maass [9])**.** *Let $H$ and $\widehat{H}$ be hhsimple. $H$ and $\widehat{H}$ are $\Delta^0_3$-automorphic iff $\mathcal{L}^*(H) \simeq_{\Delta^0_3} \mathcal{L}^*(\widehat{H})$.*

*Proof of Theorem 1.3.* By Theorem 11.3, it is enough to show that if $H$ and $\widehat{H}$ are automorphic then $\mathcal{L}^*(H)$ and $\mathcal{L}^*(\widehat{H})$ are $\Delta^0_3$-isomorphic.

Assume $H$ and $\widehat{H}$ are automorphic via $\Phi$. By Theorem 1.2, $\mathcal{S}_{\mathcal{R}}(H)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{H})$ are $\Delta^0_3$-isomorphic via an isomorphism induced by $\Phi$. So, by Lemma 11.1, $\mathcal{C}_{\mathcal{R}}(H)$ and $\mathcal{C}_{\mathcal{R}}(\widehat{H})$ are $\Delta^0_3$-isomorphic via an isomorphism induced by $\Phi$. Hence, by Lemma 11.2 (applied twice), $\mathcal{L}^*(H)$ and $\mathcal{L}^*(\widehat{H})$ are $\Delta^0_3$-isomorphic. □

## References

[1] Peter Cholak. Automorphisms of the lattice of recursively enumerable sets. *Mem. Amer. Math. Soc.*, 113(541):viii+151, 1995. ISSN 0065-9266. 1

[2] Peter Cholak and Leo A. Harrington. Extension theorems, orbits and automorphisms of the computably enumerable sets. Submitted to Annals of Mathematics, 2002. 1, 2

[3] Peter Cholak and Leo A. Harrington. On the definability of the double jump in the computably enumerable sets. *Journal of Mathematical Logic*, 2(2):261–296, 2002. A copy can be found at http://www.nd.edu/~cholak. 1, 6, 6, 6, 6, 6.3, 6, 7

[4] Peter A. Cholak and Leo A. Harrington. Definable encodings in the computably enumerable sets. *Bull. Symbolic Logic*, 6(2):185–196, 2000. A copy can be found at http://www.nd.edu/~cholak. 1, 1, 1, 1.1

[5] Rod Downey and Michael Stob. Friedberg splittings of recursively enumerable sets. *Ann. Pure Appl. Logic*, 59(3):175–199, 1993. ISSN 0168-0072. Fourth Asian Logic Conference (Tokyo, 1990). 4.9

[6] Rod Downey and Michael Stob. Splitting theorems in recursion theory. *Ann. Pure Appl. Logic*, 65(1):106 pp, 1993. ISSN 0168-0072. 4.9

[7] Leo A. Harrington and André Nies. Coding in the partial order of enumerable sets. *Adv. Math.*, 133(1):133–162, 1998. ISSN 0001-8708. 8

[8] Leo A. Harrington and Robert I. Soare. The $\Delta^0_3$-automorphism method and noninvariant classes of degrees. *J. Amer. Math. Soc.*, 9(3):617–666, 1996. ISSN 0894-0347. 1

[9] W. Maass. On the orbit of hyperhypersimple sets. *J. Symbolic Logic*, 49:51–62, 1984. 1, 11.3

[10] Franco Montagna and Andrea Sorbi. Universal recursion-theoretic properties of r.e. preordered structures. *J. Symbolic Logic*, 50(2):397–406, 1985. ISSN 0022-4812. 2

[11] A. Nies. Intervals of the lattice of computably enumerable sets and effective Boolean algebras. *Bull. Lond. Math. Soc.*, 29:683–92, 1997. 8

[12] André Nies. Effectively dense Boolean algebras and their applications. *Trans. Amer. Math. Soc.*, 352(11):4989–5012 (electronic), 2000. ISSN 0002-9947. 2

[13] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987. 1.1, 2, 3, 3, 3, 4, 11.2

[14] M. Stob. *The Structure and Elementary Theory of the Recursive Enumerable Sets*. PhD thesis, University of Chicago, 1979. 3

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