On Ramsey's Theorem for Pairs

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Ramsey's theorem

- $[X]^n = \{Y \subseteq X : |Y| = n\}$.
- A k-coloring C of $[X]^n$ is a function from $[X]^n$ into a set of size k.
- H is *homogeneous* for C if C is constant on $[H]^n$, i.e. all n-element subsets of H are assigned the same color by C.
- RT^n_k is the statement that every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

Goals

- (Computability theory) Study the complexity (in terms of the arithmetical hierarchy or degrees) of infinite homogeneous sets for a coloring C relative to that of C. (For simplicity, assume that C is computable (recursive) and relativize.)
- (Reverse mathematics) Study the proof-theoretic strength of Ramsey's theorem (and its natural special cases) as a formal statement in second order arithmetic.

Language

Use the sorted language with the symbols: $=, \in, +, \times, 0, 1, <$; Number variables: $n, m, x, y, z \ldots$; Set Variables: $X, Y, Z \ldots$.

p is prime.

This just uses bounded quantification.

$$\forall \delta \exists \epsilon [|x - c| < \epsilon \Rightarrow |x^2 - c^2| < \delta]$$

This is an example of a Π^0_2 formula. The negation is Σ^0_2. A formula which is logically equivalent (over our base theory) to both a Π^0_n formula and a Σ^0_n formula is Δ^0_n.

Induction and Comprehension

$I\Sigma_n$ is the following statement: For every $\varphi(x)$, a Σ_n^0 formula, if $\varphi(0)$ and $\forall x[\varphi(x) \Rightarrow \varphi(x+1)]$ then $\forall x[\varphi(x)]$.

Over our base theory, $I\Sigma_n^0$ and $I\Pi_n^0$ are equivalent. $I\Sigma_n$ is also equivalent to every Π_n^0-definable set (Σ_n^0 set) has a least element.

Δ_1^0 comprehension is the statement: For every $\varphi(x)$, a Δ_1^0 formula, there is an X such that $X = \{x : \varphi(x)\}$.

For example, Δ_1^0 comprehension implies the set of all primes exists.

Arithmetic comprehension is the statement: For every $\varphi(x)$, a Δ_n^0 formula, there is an X such that $X = \{x : \varphi(x)\}$.

2^{nd}-order arithmetic

We work over models of 2^{nd}-order arithmetic.

The intended model: $(\mathbb{N}, \mathcal{P}(\mathbb{N}), +, \times, 0, 1, <)$.

P^{-} is the theory of finite sets. The base theory, RCA_0, is the logical closure of $P^{-} + I\Sigma_1^0$ and Δ_1^0 comprehension.

$$(\mathbb{N}, \{\text{all computable sets}\}, +, \times, 0, 1, <) \models RCA_0.$$

ACA_0 is RCA_0 plus arithmetic comprehension.

$$(\mathbb{N}, \{\text{all arithmetic sets}\}, +, \times, 0, 1, <) \models ACA_0.$$

Statements in 2nd-order arithmetic

Statement (*WKL*)

Every infinite tree of binary strings has an infinite branch.

$\forall T \exists P$[if T is an infinite binary branching tree then P is an infinite path through T].

Statement (RT^n_k)

For every infinite set X and for every k-coloring of $[X]^n$ there is an infinite homogeneous set H.

These are Π^1_2 sentences: look at the set quantifiers ignore the (inside) number quantifiers.

Conservation

Definition

If T_1 and T_2 are theories and Γ is a set of sentences then T_2 is Γ-conservative over T_1 if $\forall \varphi[(\varphi \in \Gamma \wedge T_2 \vdash \varphi) \Rightarrow T_1 \vdash \varphi]$.

Theorem

ACA_0 is arithmeticly conservative over PA.

Computability Theory

$A \leq_T B$ iff there is a computer which using an oracle for B can compute A. For all A, $\emptyset \leq_T A$.

A' (read A-jump) is all those programs e which using A as an oracle halt on input e. $A^{(n)}$ is the n^{th} jump of A. The jump operation is order preserving.

So for all A, $\emptyset^{(n)} \leq_T A^{(n)}$. A is low$_n$ iff $A^{(n)} \leq_T \emptyset^{(n)}$.

Key idea: if A is low$_n$ then sets which are Δ^0_{n+1} in A are Δ^0_{n+1} in $\emptyset$.

WKL

Theorem (Jockusch and Soare)

[The Low Basis Theorem] Every infinite computable tree of binary strings has a low path (working in the standard model).

Theorem (Harrington)

$RCA_0 + WKL$ is Π^1_1-conservative over RCA_0.

Ramsey's Theorem – Known Results

Theorem (Specker)

There is a computable 2*-coloring of* $[\mathbb{N}]^2$ *with no infinite computable homogeneous set.*

Corollary (Specker)

RT^2_2 *is* not *provable in* RCA_0.

Theorem

1. *(Jockusch) For each $n \geq 2$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that $0^{(n-2)} \leq_T A$ for each infinite homogeneous set A.*
2. *(Simpson) For each $n \geq 3$ and $k \geq 2$ (both n and k standard), the statements RT^n_k are equivalent to ACA_0 over RCA_0.*

Theorem (Seetapun)

For any computable 2-coloring C of $[\mathbb{N}]^2$ and any noncomputable sets $C_0, C_1, \ldots$, there is an infinite homogeneous set X such that $(\forall i)[C_i \not\leq_T X]$.

Corollary (Seetapun)

RT^2_2 does not imply ACA_0. Hence, over RCA_0, RT^2_2 is strictly weaker than RT^3_2.

Theorem (Hirst)

RT^2_2 is not Σ^0_3-conservative over RCA_0. RT^2_2 is stronger than RCA_0. RT^2_2 proves $B\Sigma_2$.

$B\Sigma_2$ is strictly between $I\Sigma_1$ and $I\Sigma_2$.

Our Work – Computability Theory

Theorem

For any computable 2*–coloring of* $[\mathbb{N}]^2$*, there is an infinite homogeneous set* X *which is low*$_2$*, i.e.* $X'' \leq_T 0''$*.*

Our work – Reverse mathematics

Theorem

$RCA_0 + RT^2_2$ is Π^1_1-conservative over $RCA_0 + I\Sigma_2$.

Corollary

RT^2_2 does not imply PA over RCA_0.

This improves Seetapun's result that RT^2_2 does not imply ACA_0 over RCA_0.

Question

Is RT^2_2 Π^0_2-conservative over RCA_0? In particular, does RT^2_2 prove the consistency of $P^- + I\Sigma_1$? Does RT^2_2 prove that Ackerman's function is total?