THIN SET THEOREMS AND CONE AVOIDANCE

PETER CHOLAK AND LUDOVIC PATEY

Abstract. The thin set theorem $\mathsf{RT}^n_{<\infty,\ell}$ asserts the existence, for every k-coloring of the subsets of natural numbers of size n, of an infinite set of natural numbers, all of whose subsets of size n use at most ℓ colors. Whenever $\ell = 1$, the statement corresponds to Ramsey's theorem. From a computational viewpoint, the thin set theorem admits a threshold phenomenon, in that whenever the number of colors ℓ is sufficiently large with respect to the size n of the tuples, then the thin set theorem admits strong cone avoidance.

Let $d_0, d_1, \dots$ be the sequence of Catalan numbers. For $n \geq 1$, $\mathsf{RT}^n_{<\infty,\ell}$ admits strong cone avoidance if and only if $\ell \geq d_n$ and cone avoidance if and only if $\ell \geq d_{n-1}$. We say that a set A is $\mathsf{RT}^n_{<\infty,\ell}$-*encodable* if there is an instance of $\mathsf{RT}^n_{<\infty,\ell}$ such that every solution computes A. The $\mathsf{RT}^n_{<\infty,\ell}$-encodable sets are precisely the hyperarithmetic sets if and only if $\ell < 2^{n-1}$, the arithmetic sets if and only if $2^{n-1} \leq \ell < d_n$, and the computable sets if and only if $d_n \leq \ell$.

1. Introduction

Ramsey's theorem asserts the existence, for every k-coloring of the subsets of natural numbers of size n, of an infinite set of natural numbers, all of whose subsets of size n are monochromatic. For notation ease, as in many papers on Ramsey's theorem, we will consider a set of n natural numbers as an increasing n-tuple. Ramsey's theorem plays a central role in reverse mathematics, as Ramsey's theorem for pairs historically provides an early example of a theorem which escapes the main observation of the early reverse mathematics, namely, the "Big Five" phenomenon [21]. From a computational viewpoint, Ramsey's theorem for n-tuples with $n \geq 3$ admits computable 2-colorings of the n-tuples such that every monochromatic set computes the halting set [9], while Ramsey's theorem for pairs does not [19].

More recently, Wang [25] considered a weakening of Ramsey's theorem now known as the *thin set theorem*, in which the constraint of monochromaticity of the resulting set is relaxed so that more colors are allowed. He proved that for every size n of the tuples, there exists a number of colors ℓ such that every computable coloring of the n-tuples into finitely many colors, there is an infinite set of natural numbers whose n-tuples have at most ℓ colors and which does not compute the halting set. On the other hand, Dorais et al. [5] proved that whenever the number of colors ℓ is not large enough with respect to the size of the tuples n, then this is not the case. However, the lower bound of Dorais et al. grows slower than the upper bound of Wang. Therefore, Wang [25] naturally asked where the threshold lies.

This work was partially supported by a grant from the Simons Foundation (#315283 to Peter Cholak) and NSF Conference grants (DMS-1640836 and DMS-1822193 to Peter Cholak).

In this paper, we address this question by exhibiting the exact bound at which this threshold phenomenon occurs, and reveal an intermediary computational behavior of the thin set theorem whenever the number ℓ of allowed colors in the outcome is exponential, but not large enough. We also provide some insights about the nature of the computational strength of Ramsey-type theorems.

1.1. **Reverse mathematics and Ramsey's theorem.** Reverse mathematics is a foundational program started by Harvey Friedman, which seeks to determine the optimal axioms to prove ordinary theorems. It uses the framework of second-order arithmetic, with a base theory, RCA_0, informally capturing *computable mathematics*. The early study of reverse mathematics revealed an empirical structural phenomenon. More precisely, there are four axioms, namely, WKL (weak König's lemma), ACA (arithmetical comprehension axiom), ATR and $\Pi^1_1\mathsf{CA}$, in increasing logical order, such that almost every theorem of ordinary mathematics is either provable in RCA_0 (hence computationally true), or equivalent over RCA_0 to one of those four systems. These systems, together with RCA_0, form the "Big Five" [12]. See Simpson [21] for a reference book on the early reverse mathematics.

Among the theorems studied in reverse mathematics, Ramsey's theorem received a special attention, as it was historically the first example of a theorem escaping this structural phenomenon. In what follows, $[X]^n$ denotes the set of all unordered n-tuples over X.

Definition 1.1 (Ramsey's theorem)**.** Given $n, k \geq 1$, RT^n_k is the statement "For every coloring $f : [\omega]^n \to k$, there is an infinite *f-homogeneous* set, that is, a set $H \subseteq \omega$ such that $|f[H]^n| = 1$."

Ramsey's theorem can be seen as a mathematical problem, expressed in terms of *instances* and *solutions*. Here, an instance is a coloring $f : [\omega]^n \to k$, and a solution to f is an infinite f-homogeneous set.

Specker [23] and Jockusch [9] studied Ramsey's theorem from a computational viewpoint. Specker [23] showed that there is a 2-coloring of pairs (of natural numbers) f with no infinite computable f-homogeneous set. When formalized in the framework of reverse mathematics, this shows that RT^2_2 does not hold in RCA_0. Jockusch constructed, for every $n \geq 3$, a computable coloring $f : [\omega]^n \to 2$ such that every infinite f-homogeneous set computes the halting set. When formalized in the framework of reverse mathematics, this shows that RT^n_2 is equivalent to ACA over RCA_0 whenever $n \geq 3$.

The case of Ramsey's theorem for pairs was a long-standing open problem, until solved by Seetapun (see [19]) using what is now known as *cone avoidance*.

Definition 1.2. A problem P admits *cone avoidance* if for every pair of sets $C \not\leq_T Z$ and every Z-computable instance of P, there is a solution Y such that $C \not\leq_T Y \oplus Z$.

Seetapun and Slaman proved that Ramsey's theorem for pairs (RT^2_k) admits cone avoidance. In particular, when taking Z to be a computable set and C to be the halting set, this shows that every computable instance of RT^2_k admits a solution which does not compute the halting set. Proving that a problem admits cone avoidance implies in particular that this problem does not imply ACA over RCA_0.

1.2. **The thin set theorem.** Friedman [7] first suggested studying a weakening of Ramsey's theorem, the thin set theorem, which asserts that every coloring of

$f : [\omega]^n \to \omega$ admits an infinite set $H \subseteq \omega$ such that $[H]^n$ avoids at least one color, that is, $f[H]^n \neq \omega$. We shall however consider stronger statements defined by Miller [12] and that we also refer to as thin set theorems.

Definition 1.3 (Thin set theorem)**.** Given $n, \ell \geq 1$, $\mathsf{RT}^n_{<\infty,\ell}$ is the statement "For every k and every coloring $f : [\omega]^n \to k$, there is an infinite set H such that $|f[H]^n| \leq \ell$."

H is called *f, ℓ-thin*. Most of the time we will drop the f and ℓ since they are understood by content. More or less, Friedman's thin set theorem restricted ℓ and k to when $k = \ell + 1$. But, in the setting of reverse mathematics and computability theory, the number of colors k of the instance $f : [\omega]^n \to k$ is not relevant, since we are allowed to apply the thin set theorem multiple times. (Apply the restricted thin set theorem $k - \ell$ times to get the desired thin set.) Whenever $\ell = 1$, we obtain Ramsey's theorem.

Wang [25] studied the thin set theorem (under the name Achromatic Ramsey Theorem), and proved that whenever ℓ is sufficiently large with respect to n, then $\mathsf{RT}^n_{<\infty,\ell}$ admits cone avoidance. His proof is an inductive interplay between the combinatorial and the computational weakness of the thin set theorems, which involves the notion of *strong cone avoidance*.

Definition 1.4. A problem P admits *strong cone avoidance* if for every pair of sets C, Z with $C \not\leq_T Z$ and every (arbitrary) instance of P, there is a solution Y such that $C \not\leq_T Y \oplus Z$.

Contrary to cone avoidance, strong cone avoidance does not consider only Z-computable instances of P, but arbitrary ones. Thus, while cone avoidance expresses a *computational weakness* of the problem P, strong cone avoidance reveals a *combinatorial weakness*, in the sense that even with an unlimited amount of power for defining the instance of P, one cannot code the set C in its solutions.

Note that in the case of the thin set theorems, there is a formal relationship between strong cone avoidance and cone avoidance.

Theorem 1.5. *For every $n, \ell \geq 1$, $\mathsf{RT}^n_{<\infty,\ell}$ admits strong cone avoidance if and only if $\mathsf{RT}^{n+1}_{<\infty,\ell}$ admits cone avoidance.*

Proof. $\Rightarrow$ (Wang [25]). Fix $C \not\leq_T Z$ and a Z-computable coloring $f : [\omega]^{n+1} \to k$. Let $G = \{x_0 < x_1 < \dots\}$ be a sufficiently generic set for computable Mathias genericity [1]. In particular, for every $\vec{x} \in [G]^n$, $\lim_{y \in G} f(\vec{x}, y)$ exists. Moreover, by Wang [25, Lemma 2.6], $C \not\leq_T G \oplus Z$. Let $g : [\omega]^n \to k$ be defined by $g(i_0, \dots, i_{n-1}) = \lim_{x_n \in G} f(x_{i_0}, \dots, x_{i_{n-1}}, x_n)$. By strong cone avoidance of $\mathsf{RT}^n_{<\infty,\ell}$ applied to g, there is an infinite set H such that $|g[H]^n| \leq \ell$ and such that $C \not\leq_T H \oplus G \oplus Z$. The set $H \oplus G \oplus Z$ computes an infinite set S such that $|f[S]^{n+1}| \leq \ell$. In particular, $C \not\leq_T S \oplus Z$.

$\Leftarrow$: Fix $C \not\leq_T Z$ and an arbitrary coloring $f : [\omega]^n \to k$. By Patey [13, Theorem 2.6], there is a set B such that f is $\Delta^0_2(B)$ and $C \not\leq_T B \oplus Z$. By Shoenfield's limit lemma [20], there is a B-computable coloring $g : [\omega]^{n+1} \to k$ such that for every $\vec{x} \in [\omega]^n$, $\lim_y g(\vec{x}, y) = f(\vec{x})$. By cone avoidance for $\mathsf{RT}^{n+1}_{<\infty,\ell}$, there is an infinite set H such that $|g[H]^{n+1}| \leq \ell$ and such that $C \not\leq_T B \oplus H \oplus Z$. In particular, $|f[H]^n| \leq \ell$. This completes the proof. $\square$

Strong cone avoidance has therefore two main interests: First, it gives some insight about the combinatorial nature of a problem P, by expressing the inability of the problem P to code some fixed set even with an arbitrary instance. Second, it can be used as a tool to prove that a problem does not imply ACA over RCA_0.

Wang proved that for every $n \geq 1$ and every sufficiently large ℓ, $\mathsf{RT}^n_{<\infty,\ell}$ admits strong cone avoidance, and, in particular, cone avoidance. On the other hand, Dorais et al. [5] proved that for every $n \geq 3$, $\mathsf{RT}^n_{<\infty,2^{n-2}-1}$ does not admit cone avoidance.

Remark 1.6. One of the goals of this paper is to explicitly determine these bounds. It turns that *Catalan* and *Schröder* numbers are involved. There are over 300 interpretations of Catalan numbers, and a few dozen of Schröder numbers. The easiest way to compare them is: The nth Catalan number is the number of paths from $(0,0)$ to (n,n) that take steps $(0,1)$ and $(1,0)$, and don't go above main diagonal; the nth Schröder number is the same, except the paths are also allowed to take $(1,1)$ steps. The best current references are http://oeis.org/A000108 and http://oeis.org/A006318. Another reference is Stanley [24].

The explicit bound given by Wang is the sequence of Schröder numbers, which starts with $1, 2, 6, 22, 90, 394, 1806, 8558, \dots$ and grows faster than the lower bound of Dorais et al. In particular, this left open whether $\mathsf{RT}^3_{<\infty,5}$ and $\mathsf{RT}^3_{<\infty,4}$ admit cone avoidance.

1.3. **Summary of our results.** We now give a summary on the known bounds on the threshold between strong cone avoidance of $\mathsf{RT}^n_{<\infty,\ell}$ and the existence of an instance of $\mathsf{RT}^n_{<\infty,\ell}$ all of whose solutions are above a fixed cone. We say that a set A is $\mathsf{RT}^n_{<\infty,\ell}$-*encodable* if there is an instance of $\mathsf{RT}^n_{<\infty,\ell}$ such that every solution computes A.

Let $d_0, d_1, \dots$ be the sequence of Catalan numbers inductively defined by $d_0 = 1$ and

$$d_{n+1} = \sum_{i=0}^{n} d_i d_{n-i}$$

In particular, $d_0 = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 5$, $d_4 = 14$, $d_5 = 42$, $d_6 = 132$, $d_7 = 429$.

Theorem 1.7. *The $\mathsf{RT}^n_{<\infty,\ell}$-encodable sets are precisely*

(a) the hyperarithmetic sets if and only if $\ell < 2^{n-1}$
(b) the arithmetic sets if and only if $2^{n-1} \leq \ell < d_n$
(c) the computable sets if and only if $d_n \leq \ell$

Proof. (a) By Solovay [22], any $\mathsf{RT}^n_{<\infty,\ell}$-encodable set must be hyperarithmetical. By Theorem 2.4, every hyperarithmetical set A must have a modulus μ_A (also see Definition 2.1). Apply Theorem 3.2 to μ_A to get a coloring $f : [\omega]^n \to 2^{n-1}$ where every infinite f-thin computes a function which dominates μ_A and hence computes A.

(b) In Theorem 4.15, we prove that any $\mathsf{RT}^n_{<\infty,\ell}$-encodable set must be arithmetical whenever $\ell \geq 2^{n-1}$. On the other hand, we prove in Theorem 4.15 that every arithmetical set is $\mathsf{RT}^n_{<\infty,\ell}$-encodable if $\ell < d_n$.

(c) In Theorem 4.18, we prove that any $\mathsf{RT}^n_{<\infty,\ell}$-encodable set must be computable whenever $\ell \geq d_n$. Of course, every computable set is trivially $\mathsf{RT}^n_{<\infty,\ell}$-encodable. □

In terms of (strong) cone avoidance, we obtained the following characterization.

Strongly computes	hyperarithmetic	arithmetic	computable
$\mathsf{RT}^1_{<\infty,\ell}$			$\ell \geq 1$
$\mathsf{RT}^2_{<\infty,\ell}$	$\ell = 1$		$\ell \geq 2$
$\mathsf{RT}^3_{<\infty,\ell}$	$\ell \leq 3$	$\ell = 4$	$\ell \geq 5$
$\mathsf{RT}^4_{<\infty,\ell}$	$\ell \leq 7$	$8 \leq \ell \leq 13$	$\ell \geq 14$
$\mathsf{RT}^5_{<\infty,\ell}$	$\ell \leq 15$	$16 \leq \ell \leq 41$	$\ell \geq 42$

FIGURE 1. This table gives a summary of the class of $\mathsf{RT}^n_{<\infty,\ell}$-encodable sets in function of the size n of the tuples. For example, the $\mathsf{RT}^2_{<\infty,4}$-encodable sets are exactly the arithmetical sets.

Theorem 1.8. *For $n \geq 1$, $\mathsf{RT}^n_{<\infty,\ell}$ admits*

(a) strong cone avoidance if and only if $\ell \geq d_n$
(b) cone avoidance if and only if $\ell \geq d_{n-1}$

Proof. By Theorem 4.18, Theorem 4.15 and Theorem 1.5. □

1.4. **A brief history of the Thin Set Theorems.** Basically the idea of a thin set for a coloring is a set where the number of colors used is less than the number of colors available to the coloring. This idea was first raised by Friedman [7]. The first paper about thin sets was in Cholak et al. [2] but in a different form than the one presented here. As we mentioned above, Wang [25] first studied the thin set theorem in the form we are considering now. He used the name Achromatic Ramsey Theorem. The lower bounds we exploit appeared in Dorais et al. [5]. Montalban [12] asked about the reverse mathematics of $\mathsf{RT}^n_{k,\ell}$. Patey [16] showed this hierarchy is strictly decreasing over RCA_0, by proving that the more colors you allow in the solution, the more non-c.e. definitions you can preserve simultaneously. This is the first known strictly decreasing hierarchy. Patey showed the same result in [14] with preservation of hyperimmunity.

There are two upcoming papers where our results play a role. Downey et al. [6] where they prove that (strong) preservation of 1-hyperimmunity is the same as (strong) cone avoidance. Patey [17] where the results here on the thin set theorem are used as a blackbox to decide (strong) cone avoidance for a whole class of Ramsey-like theorems. In some sense, this shows that the thin set theorem is "analysis-complete", in that its analysis contains the information to understand all the other Ramsey-like problems.

2. Thin set theorems and sparsity

Some degrees of unsolvability can be described by the ability to compute fast-growing functions. For example, a Turing degree is *hyperimmune* if it contains a function which is not computably dominated. By Martin's domination theorem, a Turing degree is *high*, that is, $\mathbf{d}' \geq \mathbf{0}''$ if it contains a function dominating every computable function. The notion of modulus establishes a bridge between the ability to compute fast-growing functions and the ability to compute sets.

Definition 2.1. A function g *dominates* a function f if $g(x) \geq f(x)$ for every x. A function $\mu_X : \omega \to \omega$ is a *modulus* for X if every function dominating μ_X computes X.

In the case of Ramsey's theorem and more generally Ramsey-type theorems, one usually proves lower bounds by constructing an instance such that every solution H will be sufficiently sparse, so that its *principal function* p_H is sufficiently fast-growing.

Definition 2.2. The *principal function* of an infinite set $X = \{x_0 < x_1 < \dots\}$ is the function p_X defined by $p_X(n) = x_n$.

Consider for example Ramsey theorem for pairs and two colors (RT^2_2). Given an arbitrary function $g : \omega \to \omega$, one can define a function $f : [\omega]^2 \to 2$ by $f(x, y) = 1$ if $g(x) \leq y$ and $f(x, y) = 0$ otherwise. Then every infinite f-homogeneous set H will be of color 1, and the principal function p_H will dominate g. By taking g to be a modulus for the halting set, this proves that RT^2_2 does not admit strong cone avoidance.

2.1. **Ramsey-type theorems and sparsity.** For many theorems in Ramsey Theory, the class of solutions to each instance always has certain structural features. In what follows, $[X]^\omega$ denotes the class of all infinite subsets of X.

Definition 2.3. A class $\mathcal{C} \subseteq [\omega]^\omega$ is *dense* if $(\forall X \in [\omega]^\omega)[X]^\omega \cap \mathcal{C} \neq \emptyset$, and is *downward-closed* if $(\forall X \in \mathcal{C})[X]^\omega \subseteq \mathcal{C}$. A class $\mathcal{C} \subseteq [\omega]^\omega$ is *Ramsey-like* if it is dense and downward-closed.

One can easily check that given an instance $f : [\omega]^n \to k$ of the thin set theorems, and some ℓ, the collection of all infinite sets H such that $|f[H]^n| \leq \ell$ is a Ramsey-like class. Since every Ramsey-like class is closed under subset, one can intuitively only compute with positive information, in that the absence of an integer in a solution H is not informative. It is therefore natural to conjecture that the only computational power of Ramsey-type theorems comes from the sparsity of their solutions, and thus from their ability to compute fast-growing functions.

The first result towards this intuition is the characterization of the sets which are computed by a downward-closed class, as those which admit a modulus function. We say that a set A is *computably encodable* if every infinite set has an infinite subset which computes A (so there is a dense class $\mathcal{C} \subseteq [\omega]^\omega$, all of whose elements compute A).

Theorem 2.4. *[Solovay [22], Groszek and Slaman [8]] Given a set A, the following are equivalent*

(a) A is computably encodable
(b) A is hyperarithmetic
(c) A admits a modulus

In particular, since the set of solutions of an instance f of $\mathsf{RT}^n_{<\infty,\ell}$ is Ramsey-like, if every solution to f computes a set A, then A is computably encodable, hence admits a modulus.

One can obtain a more precise result when the computation is uniform.[1] A function g is a *uniform modulus* for a set A if there is a Turing functional Φ such

[1]The authors thank Lu Liu for letting them include Theorem 2.5 and Theorem 2.6 in the paper.

that $\Phi^f = A$ for every function f dominating g. The following proposition shows that the uniform computation of a set by a Ramsey-type class can only be done by the sparsity of its members.

Theorem 2.5 (Liu and Patey)**.** *Fix a set A and a Ramsey-like class $\mathcal{C}$. If there is a Turing functional Φ such that $\Phi^H = A$ for every $H \in \mathcal{C}$, then p_H is a uniform modulus of A for every $H \in \mathcal{C}$.*

Proof. Let $H \in \mathcal{C}$ and f be a function dominating p_H. For every $x \in \omega$ and $v \in \{0, 1\}$, let $\mathcal{I}_{x,v}$ be the $\Pi^{0,f}_1$ class of all sets G such that p_G is dominated by f, and such that for every $E \subseteq G$, $\Phi^E(x) \downarrow \to \Phi^E(x) = v$. Note that $\mathcal{I}_{x,v}$ is $\Pi^{0,f}_1$ uniformly in f, x and v. It follows that the set $W = \{(x, v) : \mathcal{I}_{x,v} = \emptyset\}$ is f-c.e. uniformly in f. Since $\mathcal{C}$ is closed downward, $H \in \mathcal{I}_{x,A(x)}$, hence $(x, A(x)) \not\in W$ for every x. We have two cases.

- Case 1: There is some x such that $(x, 1 - (A(x))) \not\in W$. Let $G \in \mathcal{I}_{x,1-A(x)}$. Since $\mathcal{C}$ is dense, there is some $K \in [G]^\omega \cap \mathcal{C}$, and by definition of $\mathcal{I}_{x,1-A(x)}$, $\Phi^K(x) \downarrow \to \Phi^K(x) = 1 - A(x)$, but $\Phi^K(x) = A(x)$ by assumption. Contradiction.
- Case 2: For every x, $(x, 1 - A(x)) \in W$. Since $(x, A(x)) \not\in W$, we can W-compute A, and this uniformly in f.

This completes the proof. $\square$

However, the previous theorem cannot be generalized to non-uniform computations. In particular, there exists a set A and an instance f of RT^2_2 such that every solution to f computes A, but not through sparsity.

Theorem 2.6 (Liu and Patey)**.** *Let A be a hyperarithmetical set. There exists a function $f : [\omega]^2 \to 2$ such that*

(a) Every infinite f-homogeneous set computes A
(b) For each $i < 2$, there is an infinite f-homogeneous set H for color i such that p_H is dominated by a computable function.

Proof. Let $g : \omega \to \omega$ be a modulus for A. Let $h : 2^{<\omega} \to \omega$ be a computable bijection. Let $\tilde{f} : [\omega]^2 \to 2$ be defined by

$$\tilde{f}(x, y) = \begin{cases} 0 & \text{if } x = h(\sigma) \text{ with } \sigma \prec A \text{ or } y < g(x) \\ 1 & \text{otherwise} \end{cases}$$

We claim that every infinite $\tilde{f}$-homogeneous set computes A. If H is an infinite $\tilde{f}$-homogeneous set for color 0, then $H \subseteq \{h(\sigma) : \sigma \prec A\}$, in which case H computes A. If H is an infinite $\tilde{f}$-homogeneous set for color 1, then $p_H \geq g$, and again, H computes A.

$$f(x, y) = \begin{cases} \tilde{f}(x_1, y_1) & \text{if } x = 2x_1 \text{ and } y_1 = \lfloor y/2 \rfloor \\ 1 - \tilde{f}(x_1, y_1) & \text{if } x = 2x_1 + 1 \text{ and } y_1 = \lfloor y/2 \rfloor \end{cases}$$

We claim that every infinite f-homogeneous set computes an infinite $\tilde{f}$-homogeneous set. Let H be an infinite f-homogeneous set for some color $i < 2$. One of the two sets is infinite:

$$H_0 = \{x_1 : 2x_1 \in H\} \text{ and } H_1 = \{x_1 : 2x_1 + 1 \in H\}$$

Moreover, for every $x_1 < y_1 \in H_0$, $\tilde{f}(x_1, y_1) = f(2x_1, 2y_1) = i$ and for every $x_1, y_1 \in H_1$, $\tilde{f}(x_1, y_1) = 1 - f(2x_1 + 1, 2y_1 + 1) = 1 - i$. Therefore, both H_0 and H_1 are $\tilde{f}$-homogeneous.

Last, we prove (b). Let

$$G_0 = \{2h(\sigma) : \sigma \prec A\} \text{ and } G_1 = \{2h(\sigma) + 1 : \sigma \prec A\}$$

For each $i < 2$, G_i is an infinite f-homogeneous set for color i. Moreover, p_{G_0} and p_{G_1} are both dominated by the computable function which to n, associates $\max\{2h(\sigma) + 1 : |\sigma| = n\}$. This completes the proof. $\square$

Last, if we restrict ourselves to computable instances, one can prove that even non-uniform computation is done by sparsity. The following lemma tells us in some sense that whenever considering computable instances of $\mathsf{RT}^n_{<\infty,\ell}$, the analysis of the functions the thin sets dominate can be done, without loss of generality, by the study of the sparsity of these thin sets.

Definition 2.7. Given a function $g : \omega \to \omega$, a *domination modulus* is a function $\nu_g : \omega \to \omega$ such that every function dominating ν_g computes a function dominating g.

Theorem 2.8. *Fix a function $g : \omega \to \omega$. Let $f : [\omega]^n \to k$ be a computable instance of $\mathsf{RT}^n_{<\infty,\ell}$ such that every solution computes a function dominating g. Then for every solution H, p_H is a domination modulus for g.*

Proof. Fix g and f. Suppose for the sake of contradiction that there is an infinite set H such that $|f[H]^n| \leq \ell$ and such that p_H is not a domination modulus for g. By definition, there is a function h dominating p_H such that h does not computes a function dominating g. Let $T \subseteq \omega^{<\omega}$ be the h-computably bounded tree defined by

$$T = \left\{\sigma \in \omega^{<\omega} : (\forall x < |\sigma|)\sigma(x) < h(x)) \wedge |f[\sigma]^n| \leq \ell\right\}$$

In particular, $H \in [T]$, so the tree is infinite. Moreover, any infinite path through T is an $\mathsf{RT}^n_{<\infty,\ell}$-solution to f. By the hyperimmune-free basis theorem [10] relative to h, there is an infinite set $S \in [T]$ such that every S-computable function is dominated by an h-computable function. In particular, S does not compute a function dominating g. Contradiction. $\square$

3. The strength of the thin set theorems

In this section, we study the ability of thin set theorems to compute fast-growing functions. More precisely, given a fixed function $g : \omega \to \omega$ and some $n \geq 1$, we determine the largest number of colors ℓ such that there exists an instance of $\mathsf{RT}^n_{<\infty,\ell}$ such that every solution computes a function dominating g. Thanks to the notion of modulus, we will then apply this analysis to determine which sets can be computed by $\mathsf{RT}^n_{<\infty,\ell}$ whenever $n, \ell \geq 1$.

As explained, one approach to prove that $\mathsf{RT}^n_{<\infty,\ell}$ implies the existence of fast-growing functions is to define a coloring $f : [\omega]^n \to k$ such that every solution H is sparse, and then use the principal function of H.

Definition 3.1. Given a function $g : \omega \to \omega$, an interval $[a, b]$ is *g-large* if $b \geq g(a)$. Otherwise, it is *g-small*. By extension, we say that a finite set F is g-large (g-small) if $[\min F, \max F]$ is g-large (g-small).

Given a function $f : [\omega]^n \to k$, we say that a set $H \subseteq \omega$ is *f-thin* if $|f[H]^n| \leq k-1$.

Theorem 3.2. *For every function $g : \omega \to \omega$ and every $n \geq 1$, there is a g-computable function $f : [\omega]^n \to 2^{n-1}$ such that every infinite f-thin set computes a function dominating g.*

Proof. We prove this by induction over $n \geq 1$. Then case $n = 1$ vacuously holds, since $f_1 : \omega \to \{\langle\rangle\}$ has no infinite f_1-thin set. Furthermore, assume without loss of generality that g is increasing. Given $x_0 < x_1 < \dots < x_{n-1}$, let

$$f_n(x_0, \dots, x_{n-1}) = \langle gap(x_0, x_1), gap(x_1, x_2), \dots, gap(x_{n-2}, x_{n-1})\rangle$$

where $gap(a, b) = \ell$ if $[a, b]$ is g-large, and $gap(a, b) = s$ otherwise. Let H be an infinite f_n-thin set, say for color $\langle j_0, \dots, j_{n-2}\rangle$. We have several cases.

Case 1: H is f_n-thin for color $\langle s, s, \dots, s\rangle$. Then for every $x_0 < \dots < x_{n-1} \in H$, $[x_0, x_{n-1}]$ is g-large since g is increasing. Then the function which to i associates the $(i + n)$th element of H dominates g.

Case 2: H is f_n-thin for color $\langle j_0, \dots, j_{i-1}, \ell, s, s, s, \dots, s\rangle$. Case 2.1: H is f_{i+1}-thin for color $\langle j_0, \dots, j_{i-1}\rangle$ (where $\langle j_0, \dots, j_{i-1}\rangle = \langle\rangle$ if $i = 0$). Then by induction hypothesis, H computes a function dominating g. Case 2.2: There is some $x_0 < x_1 < \dots < x_i \in H$ such that $f_{i+1}(x_0, \dots, x_i) = \langle j_0, \dots, j_{i-1}\rangle$. Let $t > x_i$ be such that $gap(x_i, t) = \ell$. We claim that for every tuple $x_{i+1} < \dots < x_{n-1} \in H - \{0, \dots, t\}$, $[x_{i+1}, x_{n-1}]$ is g-large. Indeed, since $x_{i+1} > t$, then $gap(x_i, x_{i+1}) = \ell$, and by choice of i, H is f_n-thin for color $\langle j_0, \dots, j_{i-1}, \ell, s, \dots, s\rangle$. So all the intervals cannot be small after x_{i+1}, and since g is increasing, $[x_{i+1}, x_{n-1}]$ is g-large. The function which to u associates the $(u+n-i)$th element of $H - \{0, \dots, t\}$ dominates g. This completes the proof of Theorem 3.2. □

Corollary 3.3 (Dorais et al. [5])**.** *For every $n \geq 2$ and $k \geq 1$, there is a computable instance of $\mathsf{RT}^{n+k}_{<\infty, 2^{n-1}}$ such that every solution computes $\emptyset^{(k)}$.*

Proof. Let $g : \omega \to \omega$ be a $\emptyset^{(k)}$-computable modulus of $\emptyset^{(k)}$. By Theorem 3.2, there is a $\emptyset^{(k)}$-computable function $f = [\omega]^n \to 2^{n-1}$ such that every infinite f-thin set computes a function dominating g, hence computes $\emptyset^{(k)}$. By Schoenfield's limit lemma, there is a computable function $h : [\omega]^{n+k} \to 2^{n-1}$ such that for every $x_0 < \dots < x_{n-1} \in \omega$,

$$f(x_0, \dots, x_{n-1}) = \lim_{x_n} \dots \lim_{x_{n+k-1}} h(x_0, \dots, x_{n+k-1})$$

Every infinite h-thin set is f-thin, and therefore computes $\emptyset^{(k)}$. □

One can wonder about the optimality of Theorem 3.2. In particular, for $n = 3$, given a function $g : \omega \to \omega$, is there a function $f : [\omega]^3 \to 5$ such that every infinite f-thin set computes a function dominating g?

When considering the function constructed in the proof of Theorem 3.2, there seems at first sight to be some degree of freedom. In particular, given some $a < b < c$, whenever $[a, b]$ and $[b, c]$ are both g-small, $[a, c]$ can be either g-large or g-small. A candidate function would therefore be

$$f(a, b, c) = \langle gap(a, b), gap(b, c), gap(a, c)\rangle$$

However, as we shall see in Section 5, the last component $gap(a, c)$ is of no help to compute fast-growing functions, in the sense if g is not dominated by any computable function, then one can avoid the color $\langle s, s, \ell\rangle$ without dominating g.

Definition 3.4. A function f is *X-hyperimmune* if it is not dominated by any X-computable function.

Lemma 3.5. *If g is a hyperimmune function, then there is an infinite set H such that $\langle s, s, \ell\rangle \not\in f[H]^3$ and such that g is H-hyperimmune.*

Proof. See Theorem 5.12. $\square$

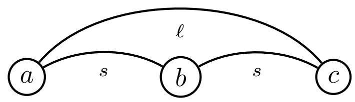

FIGURE 2. The following pattern can be avoided thanks to the GAP principle studied in Section 5.

Actually, we shall see that in Section 4 that Theorem 3.2 is optimal in the sense that for every function g which is hyperimmune relative to every arithmetical set, for every $n \geq 1$ and every instance of $\mathsf{RT}^n_{2^{n-1}+1,2^{n-1}}$, there is a solution which does not compute a function dominating g. We can however obtain better results in the case of left-c.e. functions.

Definition 3.6. A function $g : \omega \to \omega$ is *left-c.e.* if there is an uniformly computable sequence of functions $g_0, g_1, \dots$ such that for every x, $g_0(x), g_1(x), \dots$ is a non-decreasing sequence converging to $g(x)$.

We want our approximations $g_0, g_1, \dots$ to g to have some nice properties, as enumerated by the next lemma.

Lemma 3.7. *Let $g_0, g_1 \dots$ be an uniformly computable sequence of functions such that $\lim_s g_s(x) = g(x)$. Then there is a uniformly computable sequence of functions of $\hat{g}_0, \hat{g}_1, \dots$ such that*

(1) *For all x, $\lim_s \hat{g}_s(x)$ exists.*
(2) *$\lim_s \hat{g}_s(x) \geq g(x)$. So $\lim_s \hat{g}_s(x)$ dominates g.*
(3) *$\lim_s \hat{g}_s(x)$ is increasing.*
(4) *Each $\hat{g}_s$ is increasing.*
(5) *For all x, the sequence $\hat{g}_0(x), \hat{g}_1(x), \hat{g}_2(x), \dots$ is non-decreasing.*
(6) *If $a \leq x$, $\hat{g}_s(a) < x+1$, and $\hat{g}_{s+1}(a) \geq x+1$ then $\hat{g}_{s+1}(x+1) \geq s+1 \geq x+1$.*

Proof. Let $\hat{g}_0(0) = g_0(0)$; $\hat{g}_0(x+1)$ be the max of $g_0(x+1)$ and $\hat{g}_0(x)+1$ (so $\hat{g}_0$ is increasing); and $\hat{g}_{s+1}(0)$ be the max of $\hat{g}_s(0)$ and $g_{s+1}(0)$ (so the sequence $\hat{g}_0(0), \hat{g}_1(0), \hat{g}_2(0), \dots$ is non-decreasing and $\lim_s \hat{g}_s(0) \geq g(0)$)). Let $\hat{g}_{s+1}(x+1)$ be the max of $\hat{g}_{s+1}(x)+1$ (so $\hat{g}_{s+1}$ is increasing), $\hat{g}_s(x+1)$ (so the sequence $\hat{g}_0(x), \hat{g}_1(x), \hat{g}_2(x), \dots$ is non-decreasing), $g_{s+1}(x+1)$ (so $\lim_s \hat{g}_s(x) \geq g(x+1)$), and also the s's but only if there exists $a \leq x$ such that $\hat{g}_s(a) < x+1$ and $\hat{g}_{s+1}(a) \geq x+1$ (so last condition is meet). By induction on x, $\lim_s \hat{g}_s(x)$ exists. Since each g_s is increasing, so is $\lim_s \hat{g}_s(x)$. $\square$

Since we are only interested in functions which dominate g (see Theorem 3.17), the above lemma says that we can make the assumption all of our approximations have the properties enumerated in the above lemma and that g is increasing. Also note that the resulting function $\lim_s \hat{g}_s(x)$ is left-c.e. even in the case of an arbitrary Δ^0_2 function g. Here however we shall only consider only left-c.e. functions g.

Since g_s is computable, being g_s-large or g_s-small is decidable, contrary to g-largeness and g-smallness. If $[a, b]$ is g-large, then, for all s, since $g_s(a) \leq g(a)$, $[a, b]$ is g_s-large. The interesting feature of left-c.e. functions is that the set of their small intervals is c.e. Indeed, $[a, b]$ is g-small iff, there is some $s \in \omega$ such that $g_s(a) > b$, i.e. $[a, b]$ is g_s-small. Therefore, this notion is interesting to classify g-small sets. The last condition of the above lemma and the fact that g_s is increasing imply that if s is the first stage where $[a, b]$ is g_s-small then, for all $y \geq b$, $g_{g_s(y)}(a) > b$. So g grows sufficiently fast so that if $[a, b]$ is g-small and $[b, c]$ is g-large, then $[a, b]$ is g_c-small.

Before proving our main lower bound theorem, we need to introduce a combinatorial tool, namely, *largeness graphs*. They will be useful to count the number of colors needed for our theorem.

Given a tuple $x_0 < x_1 < \dots < x_{n-1}$, we study the properties of the graph whose vertices are $\{0, \dots, n-1\}$, and such that for every $i < n-1$, there is an edge between i and $i+1$ if $[x_i, x_{i+1}]$ is g-large, and for every $i+1 < j < n$, there is an edge between i and j if $[x_i, x_{i+1}]$ is g_{x_j}-small. This yields the notion of a largeness graph.

Definition 3.8. A *largeness graph* of size n is an undirected irreflexive graph (V, E), with $V = \{0, \dots, n-1\}$, such that

(a) If $\{i, i+1\} \in E$, then for every $j > i+1$, $\{i, j\} \notin E$
(b) If $i < j < n$, $\{i, i+1\} \notin E$ and $\{j, j+1\} \in E$, then $\{i, j+1\} \in E$
(c) If $i+1 < j < n-1$ and $\{i, j\} \in E$, then $\{i, j+1\} \in E$
(d) If $i+1 < j < k < n$ and $\{i, j\} \notin E$ but $\{i, k\} \in E$, then $\{j-1, k\} \in E$

Property (a) reflects the fact that if $[x_i, x_{i+1}]$ is g-large, then it is not g_{x_j}-small for any $j > i+1$. Property (b) says that if $[x_j, x_{j+1}]$ is g-large, then any value larger than x_{j+1} will already witness the smallness of all the g-small intervals before x_j. Property (c) says that if $[x_i, x_{i+1}]$ is g_{x_j}-small, then it will be g_{x_k}-small for every $k \geq j$. Last, Property (d) says that if $[x_i, x_{i+1}]$ is g_{x_k}-small, but g_{x_j}-large, then the interval $[x_{j-1}, x_j]$ is g_{x_k}-small. Actually, by (a), we already know that $[x_{j-1}, x_j]$ is g-small. What Property (d) adds is that this smallness is witnessed by time x_k. Since, for the first t such that $g_t(x_i) \geq x_{i+1}$, $x_j < t \leq x_k$, the conditions of Lemma 3.7 implies that $g_t(x_{i+1}) \geq t > x_j$. Since $t \leq x_k$ and the increasing properties of the approximations, $g_{x_k}(x_{j-1}) \geq t > x_j$.

Definition 3.9. A largeness graph $\mathcal{G} = (\{0, \dots, n-1\}, E)$ is *packed* if for every $i < n-2$, $\{i, i+1\} \notin E$.

Definition 3.10. Let $\mathcal{G}_0 = (\{0, \dots, n-1\}, E_0)$ and $\mathcal{G}_1 = (\{0, \dots, n-1\}, E_1)$ be two largeness graphs of size n. We define the equivalence relation $\mathcal{G}_0 \sim \mathcal{G}_1$ to hold if for every $i+1 < j < n$, $\{i, j\} \in E_0$ if and only if $\{i, j\} \in E_1$.

In other words, $\mathcal{G}_0 \sim \mathcal{G}_1$ if and only if only the edges of the form $\{i, i+1\}$ can vary.

Lemma 3.11. *Every largeness graph of size $n \geq 1$ is equivalent to a packed largeness graph.*

Proof. Let $\mathcal{G}_0 = (\{0, \dots, n-1\}, E_0)$ be a largeness graph of size n. Let $E_1 = E_0 \setminus \{\{i, i+1\} : i < n-1\}$ and $\mathcal{G}_1 = (\{0, \dots, n-1\}, E_1)$. We claim that $\mathcal{G}_1$ is a largeness graph by checking properties (a-d). Properties (a) and (b) are vacuously

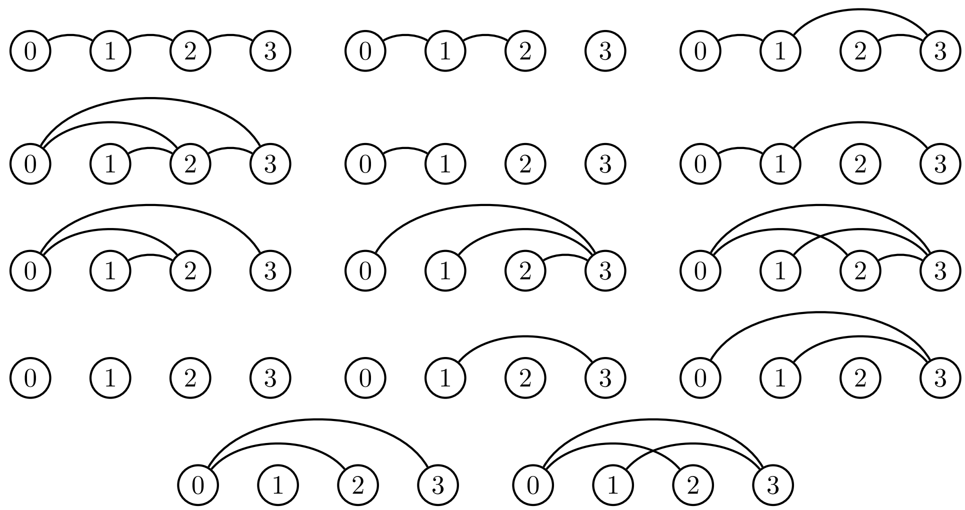

Figure 3. List of the 14 largeness graphs of size 4. The last 5 graphs are the packed largeness graphs of size 4.

true. Properties (c) and (d) are inherited from $\mathcal{G}_0$. By construction, $\mathcal{G}_0 \sim \mathcal{G}_1$ and $\mathcal{G}_1$ is packed. $\square$

Lemma 3.12. *Let $\mathcal{G}_0 = (\{0, \dots, n-1\}, E_0)$ be a largeness graph of size n. Let $\ell < n-2$ be minimal (if it exists) such that $\{\ell, \ell+1\} \not\in E_0$ and $\{\ell, n-1\} \not\in E_0$. Then the graph $\mathcal{G}_1 = (\{0, \dots, n-1\}, E_0 \cup \{\ell, \ell+1\})$ is a largeness graph such that $\mathcal{G}_0 \sim \mathcal{G}_1$.*

Proof. We check that Property (a-d) are satisfied for $\mathcal{G}_1$.

(a) We need to check that for every $j > \ell+1$, $\{\ell, j\} \not\in E_0 \cup \{\ell, \ell+1\}$). If $\{\ell, j\} \in E_0$ for some $j > \ell+1$, then by property (c) of $\mathcal{G}_0$, $\{\ell, n-1\} \in E_0$, contradicting our hypothesis.

(b) We need to check that if $i < \ell$ and $\{i, i+1\} \not\in E_0 \cup \{\ell, \ell+1\}$), then $\{i, \ell+1\} \in E_0 \cup \{\ell, \ell+1\}$). By minimality of ℓ, $\{i, n-1\} \in E_0$. If $\{i, \ell+1\} \not\in E_0 \cup \{\ell, \ell+1\}$), then by property (d) of $\mathcal{G}_0$, $\{\ell, n-1\} \in E_0$, contradicting our hypothesis.

(c-d) are inherited from properties (c-d) of $\mathcal{G}_0$. $\square$

Definition 3.13. A largeness graph $\mathcal{G} = (\{0, \dots, n-1\}, E)$ of size $n \geq 2$ is *normal* if $\{n-2, n-1\} \in E$.

Lemma 3.14. *Every largeness graph of size $n \geq 2$ is equivalent to a normal largeness graph.*

Proof. Fix a largeness graph $\mathcal{G}_0 = (\{0, \dots, n-1\}, E_0)$. By iterating Lemma 3.12, there is a graph $\mathcal{G}_1 = (\{0, \dots, n-1\}, E_1)$ equivalent to $\mathcal{G}_0$ such that for every $\ell < n-2$ such that $\{\ell, \ell+1\} \not\in E_1$, then $\{\ell, n-1\} \in E_1$. The graph $\mathcal{G}_2 = (\{0, \dots, n-1\}, E_1 \cup \{n-2, n-1\})$ is a normal largeness graph equivalent to $\mathcal{G}_0$. $\square$

The following lemma will be very useful for counting purposes.

Lemma 3.15. *The following are in one-to-one correspondance for every $n \geq 2$:*

(i) packed largeness graphs of size n

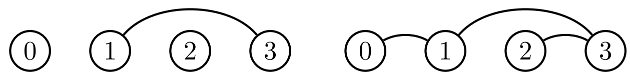

FIGURE 4. Example of a non-normal largeness graph equivalent to a normal one.

(ii) normal largeness graphs of size n
(iii) largeness graphs of size $n-1$

Proof. $(i) \Leftrightarrow (ii)$: By Lemma 3.11 and Lemma 3.14, every equivalence class from $\sim$ contains a packed largeness graph of size n and a normal largeness graph of size n. Moreover, there cannot be two packed largeness graphs in the same equivalence class. We now prove that two normal largeness graphs cannot belong to the same equivalence class. Indeed, let $\mathcal{G}_0 = (\{0,\dots,n-1\}, E_0)$ and $\mathcal{G}_1 = (\{0,\dots,n-1\}, E_1)$ be two normal largeness graphs of size n such that $\mathcal{G}_0 \sim \mathcal{G}_1$ but $\mathcal{G}_0 \neq \mathcal{G}_1$. Then without loss of generality, we can assume there is some $i < n-1$ such that $\{i, i+1\} \in E_0$ and $\{i, i+1\} \notin E_1$. By definition of normality, $\{n-2, n-1\} \in E_0 \cap E_1$, so by property (b) of the definition of a largeness graph, $\{i, n-1\} \in E_1$, and by property (a) of the definition of a largeness graph, $\{i, n-1\} \notin E_0$, contradicting $\mathcal{G}_0 \sim \mathcal{G}_1$.

$(ii) \Leftrightarrow (iii)$: Every largeness graph $\mathcal{G}_0 = (\{0,\dots,n-2\}, E_0)$ of size $n-1$ can be augmented into a normal largeness graph $\mathcal{G}_1 = (\{0,\dots,n-1\}, E_1)$ of size n by adding a node $n-1$ with $\{n-2, n-1\} \in E_1$. The edges with endpoint $n-1$ are all uniquely determined by the definition of a largeness graph, so the graph $\mathcal{G}_1$ is unique. Conversely, given a normal largeness graph $\mathcal{G}_1$, the subgraph induced by removing the last node is a largeness graph. □

We will now count the number of packed and general largeness graphs of size n. Let $d_0, d_1, \dots$ be the sequence of Catalan numbers inductively defined by $d_0 = 1$ and

$$d_{n+1} = \sum_{i=0}^{n} d_i d_{n-i}$$

Lemma 3.16. *For every* $n \geq 0$*, there exists exactly* d_n *many largeness graphs of size* n*.*

Proof. By induction over n. By convention, there exists a unique largeness graph with no nodes, and we let $d_0 = 1$. Assume the property holds for every $j \leq n$. Consider an arbitrary largeness graph of size $n+1$. Let $i < n$ be the least index such that $\{i, i+1\}$ has an edge, if it exists. If there is no such index, then set $i = n$. We have two cases.

Case 1: $i = 0$. Then all the edges with the endpoint 0 are already specified, and the graph with nodes $\{1, \dots, n\}$ is unspecified. Therefore, by induction hypothesis, there are d_n many possibilities. Since $d_0 = 1$, there are $d_i \cdot d_{n-1}$ many possibilities.

Case 2: $0 < i \leq n$. Then by minimality of i, there is no edge $\{j, j+1\}$ for any $j < i$, and an edge $\{i, i+1\}$. By definition of a largeness coloring, all the edges between nodes on the left of i and on the right of $i+1$ are fully specified. However, the subgraph on $\{0, \dots, i\}$ can be an arbitrary packed largeness graph of size $i+1$, and the subgraph on $\{i+1, \dots, n\}$ can be an arbitrary largeness graph of size $n-i$. By the one-to-one correspondence between packed largeness graphs of size $i+1$ and largeness graphs of size i, and by induction hypothesis, there are $d_i \cdot d_{n-i}$ many possibilities. See Figure 5.

Summing up the possibilities, we get $d_{n+1} = \sum_{i=0}^{n} d_i d_{n-i}$. This completes the proof. □

It follows that d_n is also the number of packed largeness graphs of size $n+1$, and the number of normal largeness graphs of size $n+1$.

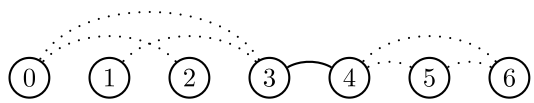

FIGURE 5. Assuming that $\{3,4\}$ is the left-most adjacent pair with an edge. Then the packed largeness coloring $\{0,1,2,3\}$ and the largeness coloring $\{4,5,6\}$ remain to be determined. Therefore there are $d_3 \cdot d_3$ many possibilities.

We are ready to prove our main lower bound theorem.

Theorem 3.17. *Let $g : \omega \to \omega$ be a left-c.e. increasing function. For every $n \geq 1$, there is a Δ^0_2 coloring $f : [\omega]^n \to d_n$ such that every infinite f-thin set computes a function dominating g.*

Proof. Let $\mathcal{L}_n$ be the collection of all largeness graphs of size n. For every $n \geq 1$, we construct a function $f_n : [\omega]^n \to \mathcal{L}_n$ such that every infinite f_n-thin set computes a function dominating g. By Lemma 3.16, the range of f_n has size at most d_n.

The case $n = 1$ is vacuously true, since $d_1 = 1$, and there is no infinite thin set for a 1-coloring of ω.

Assume that the property holds up to $n-1$. Let $f_n(x_0, \dots, x_{n-1})$ be the largeness graph $\mathcal{G} = (\{0, \dots, n-1\}, E)$ such that for $i < n-1$, $\{i, i+1\} \in E$ iff $[x_i, x_{i+1}]$ is g-large, and for $i+1 < j < n$, $\{i, j\} \in E$ iff $[x_i, x_{i+1}]$ is g_{x_j}-small.

We now prove that any infinite f-thin set computes a function dominating g. Let H be an infinite f-thin set for some largeness graph $\mathcal{G} = (\{0, \dots, n-1\}, E)$. We have two cases.

Case 1: $\mathcal{G}$ is not a packed largeness graph. There is some $i < n-1$ such that $\{i, i+1\} \in E$. There must be some $x_0 < \dots < x_i \in H$ such that $f_{i+1}(x_0, \dots, x_i)$ is the largeness subgraph of $\mathcal{G}$ of size $i+1$ induced by the vertices $\{0, 1, \dots, i\}$, otherwise H is f_{i+1}-thin, and by induction hypothesis, H computes a function dominating g and we are done. Let t be large enough such that $[x_i, t]$ is g-large. There must be some $x_{i+1} < \dots < x_{n-1} \in H - \{0, \dots, t\}$ such that $f_{n-i-1}(x_{i+1}, \dots, x_{n-1})$ is the largeness subgraph of $\mathcal{G}$ induced by the vertices $\{i+1, \dots, n-1\}$, otherwise $H - \{0, \dots, t\}$ is f_{n-i-1}-thin, and by induction hypothesis, H computes a function dominating g. In particular, $[x_i, x_{i+1}]$ is g-large, so $f(x_0, \dots, x_{n-1}) = \mathcal{G}$. This contradicts the fact that H is f_n-thin for color $\mathcal{G}$.

Case 2: $\mathcal{G}$ is a packed largeness graph. Note that the set

$$W = \{\{x_0, x_1, \dots, x_{n-1}\} \in [H]^n : f_n(x_0, \dots, x_{n-1}) \sim \mathcal{G}\}$$

is H-c.e. Indeed, this requires only to check that for $i+1 < j < n$, $\{i, j\} \in E$ iff $[x_i, x_{i+1}]$ is g_{x_j}-small, which is decidable. Since H is f_n-thin for color $\mathcal{G}$, for every $\{x_0, \dots, x_{n-1}\} \in W$, one of $[x_i, x_{i+1}]$ is g-large. Therefore, it suffices to prove that for every $t \in \omega$, there is such a tuple with $t < \min\{x_0, \dots, x_{n-1}\}$. Indeed, if so, the function $h : \omega \to \omega$ which on t H-computably searches for such a tuple $\{x_0, \dots, x_{n-1}\} \in W$ and outputs x_{n-1} is a function dominating g.

By Lemma 3.14, there is a normal largeness graph $\mathcal{G}_1 = (\{0, \dots, n-1\}, E_1) \sim \mathcal{G}$. For every $t \in \omega$, there is some $x_0 < \dots < x_{n-2} \in H - \{0, \dots, t\}$ such that $f_{n-1}(x_0, \dots, x_{n-2})$ is the largeness subgraph of $\mathcal{G}_1$ of size $n-1$ induced by the vertices $\{0, \dots, n-2\}$, otherwise $H - \{0, \dots, t\}$ is f_{n-1}-thin, and by induction hypothesis, H computes a function dominating g. Let $x_{n-1} \in H$ be sufficiently large so that $[x_{n-2}, x_{n-1}]$ is g-large. Then $f_n(x_0, \dots, x_{n-1}) = \mathcal{G}_1$, and therefore $\{x_0, \dots, x_{n-1}\} \in W$. This completes the proof of Theorem 3.17. □

Corollary 3.18. *For every $n \geq 2$ and $k \geq 1$, there is a computable instance of $\mathsf{RT}^{n+k}_{<\infty, d_n - 1}$ such that every solution computes $\emptyset^{(k)}$.*

Proof. $\emptyset^{(k)}$ is c.e. in $\emptyset^{(k-1)}$. Let $g : \omega \to \omega$ be a $\emptyset^{(k-1)}$-computable modulus of $\emptyset^{(k)}$. g is left c.e. over $\emptyset^{(k-1)}$. By Theorem 3.17, there is a $\emptyset^{(k)}$-computable function $f : [\omega]^n \to d_n$ such that every infinite f-thin set computes a function dominating g, hence computes $\emptyset^{(k)}$. By Schoenfield's limit lemma, there is a computable function $h : [\omega]^{n+k} \to d_n$ such that for every $x_0 < \dots < x_{n-1} \in \omega$,

$$f(x_0, \dots, x_{n-1}) = \lim_{x_n} \dots \lim_{x_{n+k-1}} h(x_0, \dots, x_{n+k-1})$$

Every infinite h-thin set is f-thin, and therefore computes $\emptyset^{(k)}$. □

By a relativization of the proof of Theorem 3.17 and using more colors, one can code every arithmetical set.

Theorem 3.19. *Let A be an arithmetical set. For every $n \geq 1$, there is a coloring $f : [\omega]^n \to k \cdot d_n$ such that every infinite set H such that $|f[H]^n| < d_n$ computes A.*

Proof. Fix n and A. Since A is arithmetical, it is Δ^0_{k+1} for some $k \in \omega$. Let Γ be a functional such that for every set X, Γ^X is a left-c.e. modulus of X' relative to X, that is, for every function g dominating Γ^X, $g \oplus X \geq_T X'$.

Let Ψ_n be the functional such that $\Psi^X_n(x_0, \dots, x_{n-1})$ is the largeness coloring of $(x_0, \dots, x_{n-1})$ defined by setting $\{x_i, x_{i+1}\}$ to be ℓ if $[x_i, x_{i+1}]$ is Γ^X-large, and $\{x_i, x_{i+1}\}$ is s otherwise. Moreover, $\{x_i, x_{i+2}\}$ has color ℓ if either $[x_i, x_{i+1}]$ is Γ^X-large, or $[x_i, x_{i+1}]$ is x_{i+2}-Γ^X-small, and $\{x_i, x_{i+2}\}$ has color s otherwise.

By a relativization of the proof of Theorem 3.17, for every Ψ^X_n-thin set H, $H \oplus X$ computes a function dominating Γ^X, hence computes X'. Let

$$f_n(x_0, \dots, x_{n-1}) = \langle \Psi^{\emptyset}_n(x_0, \dots, x_{n-1}), \dots, \Psi^{\emptyset^{(k-1)}}_n(x_0, \dots, x_{n-1}) \rangle$$

See f_n as an instance of $\mathsf{RT}^n_{<\infty, d_n - 1}$. Let H be an infinite set such that $|f_n[H]^n| \leq d_n - 1$. In particular, H is $\Psi^{\emptyset}_n$-thin, so $H \geq_T \emptyset'$. Moreover, H is $\Psi^{\emptyset'}_n$-thin, so $H \oplus \emptyset' \geq_T \emptyset''$, hence $H \geq_T \emptyset''$. By iterating the argument, $H \geq_T \emptyset^{(k)}$, hence $H \geq_T A$. This completes the proof. □

4. The weakness of the thin set theorems

Wang [25] proved that $\mathsf{RT}^n_{<\infty, \ell}$ admits strong cone avoidance whenever ℓ is at least the nth Schröder number, and asked whether this bound is optimal. In this section, we answer negatively this question and prove that the exact bound corresponds to Catalan numbers. We also prove the tightness of Dorais et al. [5]

by proving that $\mathsf{RT}^n_{<\infty,\ell}$ admits strong cone avoidance for non-arithmetical cones whenever $\ell \geq 2^{n-1}$.

Definition 4.1. Let Dec_n be the set of all strictly decreasing non-empty sequences over $\{1, \dots, n-1\}$. Given some $\sigma \in \mathrm{Dec}_n$, we let σ^+ be its last (smallest) element, and σ^- be the sequence truncated by its last element. If $|\sigma| = 1$, then σ^- is the empty sequence ϵ. By convention, we also define $\varepsilon^+ = n$.

By a simple counting argument, $|\mathrm{Dec}_n| = 2^{n-1} - 1$. Indeed, the strictly decreasing non-empty sequences over $\{1, \dots, n-1\}$ are in one-to-one correspondance with the non-empty subsets of $\{1, \dots, n-1\}$.

We will use the set of decreasing sequences in the proof of Theorem 4.3. See the explanations before Definition 4.4. We also need the following technical definition which will be used in Lemma 4.10.

Definition 4.2. Fix $n \in \omega$ and a vector $\vec{\ell} = \langle \ell_1, \ell_2, \dots, \ell_n \rangle$ of natural numbers. Given some $\sigma = n_0 n_1 \dots n_s \in \mathrm{Dec}_n$, let $\vec{\ell}(n, \sigma) = \ell_{n-n_0} \cdot \prod_{i=1}^{s} \ell_{n_{i-1}-n_i}$ By convention, $\vec{\ell}(n, \varepsilon) = 1$.

Theorem 4.3. *Fix $n \geq 1$, and let $\mathcal{M}$ be a countable Scott set such that*

$$(\forall s \in \{1, \dots, n\})(\exists \ell_s)\mathcal{M} \models \mathsf{RT}^s_{<\infty,\ell_s}$$

Let

$$\ell = \ell_n + \sum_{\sigma \in \mathrm{Dec}_n} \vec{\ell}(n, \sigma) \cdot \ell_{\sigma^+}$$

For every $B \notin \mathcal{M}$ and every instance $f : [\omega]^n \to k$ of $\mathsf{RT}^n_{<\infty,\ell}$, there is a solution G such that for every $C \in \mathcal{M}$, $B \not\leq_T G \oplus C$.

Proof. Fix n, $\mathcal{M}$, B, and f.

Given two sets A and B, we write $A \subseteq_n B$ for $A \subseteq B$ and $|A| \leq n$. We identify an integer $k \in \omega$ with the set $\{0, \dots, k-1\}$. We want to construct an infinite set G such that $A \not\leq_T G \oplus C$ for every $C \in \mathcal{M}$, and $f[G]^n \subseteq_\ell k$. Suppose there is no such set, otherwise we are done. We are going to build our set G by forcing.

Let us illustrate the general idea in the case $n = 3$ with a function $f : [\omega]^3 \to k$. We write $\mathcal{P}_s(X)$ for the collection of all finite subsets of X of size s. Our goal is to build an infinite set G such that $f[G]^3$ will use at most ℓ colors. For this, we will use a variant of Mathias forcing with conditions of the form $(F^K, X : K \in \mathcal{P}_\ell(k))$. Here, we build simultaneously $|\mathcal{P}_\ell(k)| = \binom{k}{\ell}$ many solutions. For each $K \in \mathcal{P}_\ell(k)$, F^K represents a finite stem of a solution G^K such that $f[G^K]^3 \subseteq K$. We will ensure that at least one of $G^K : K \in \mathcal{P}_\ell(k)$ will be infinite and be cone avoiding. The set X is a shared reservoir from which all the future elements of F^K will come. During the construction, the reservoir X will become more and more restrictive, so that Π^0_1 facts about the constructed solution can be forced.

We therefore require a condition $c = (F^K, X : K \in \mathcal{P}_\ell(k))$ to satisfy the following property:

(1): For every $K \in \mathcal{P}_\ell(k)$, $f[F^K]^3 \subseteq K$.

However, this property is not enough to ensure that the stems are extendible. Indeed, given a finite set $E \subseteq X$ satisfying again $f[E]^3 \subseteq K$, it may not be the case that $f[F^K \cup E]^3 \subseteq K$. A bad case is when there is some $x \in F^K$ such that for every $y, z \in X$, $f(x, y, z) \notin K$. Another bad case is when for some $x, y \in F^K$, for every $z \in X$, $f(x,y,z) \notin K$. We therefore need to strengthen the property (1). We would therefore want add the following properties:

(2.1): For every $K \in \mathcal{P}_\ell(k)$, every $x, y \in F^K$, $\lim_z f(x,y,z) \in K$.
(2.2): For every $K \in \mathcal{P}_\ell(k)$, every $x \in F^K$, $\lim_y \lim_z f(x,y,z) \in K$.

Properties (2.1) and (2.2) are in charge of propagating property (1), in that if $E \subseteq X$ is such that $f[E]^3 \subseteq K$, then $f[F^K \cup E]^3 \subseteq K$. There is however an issue: there is no reason to believe the function f admits a limit. By Ramsey's theorem, we know there is a restriction Y of the reservoir X over which f admits a limit, but we cannot ensure that $Y \in \mathcal{M}$. We will therefore have to "guess" every possibility of limiting behavior of the function f. The limiting behavior of the function f can be specified by two functions $g_2 : [\omega]^2 \to k$ and $g_1 : \omega \to k$, which informally should satisfy the equations $g_2(x,y) = \lim_z f(x,y,z)$ and $g_1(x) = \lim_y \lim_z f(x,y,z)$. The equations become:

(2.1): For every $K \in \mathcal{P}_\ell(k)$, every $x, y \in F^K$, $g_2(x,y) \in K$.
(2.2): For every $K \in \mathcal{P}_\ell(k)$, every $x \in F^K$, $g_1(x) \in K$.

Since we don't know the functions g_2 and g_1 ahead of time, we will need to try every possibility. Therefore, the notion of forcing becomes $(F^K_{g_2,g_1}, X : K \in \mathcal{P}_\ell(k), g_2 : [\omega]^2 \to k, g_1 : \omega \to k)$. A condition must satisfy the following properties for every $K \in \mathcal{P}_\ell(k)$, every $g_2 : [\omega]^2 \to k$, and every $g_1 : \omega \to k$:

(1): $f[F^K_{g_2,g_1}]^3 \subseteq K$
(2.1): For every $x, y \in F^K_{g_2,g_1}$, $g_2(x,y) \in K$.
(2.2): For every $x \in F^K_{g_2,g_1}$, $g_1(x) \in K$.

As explained, properties (2.1) and (2.2) are useful to propagate Property (1). However, we will encounter a similar issue of the propagation of Property (2.1): Suppose that there is some $x \in F^K_{g_2,g_1}$ such that for every $y \in X$, $g_2(x,y) \notin K$. Then the property (2.1) will never be satisfied for $F^K_{g_2,g_1}$ for any set $E \subseteq X$. We need to also consider the limiting behavior of the function g_2. It is specified by a function $g_{2,1} : \omega \to k$. The notion of forcing becomes $(F^K_{g_2,g_{2,1},g_1}, X : K \in \mathcal{P}_\ell(k), g_2 : [\omega]^2 \to k, g_{2,1} : \omega \to k, g_1 : \omega \to k)$. A condition must satisfy the following properties for every $K \in \mathcal{P}_\ell(k)$, every $g_2 : [\omega]^2 \to k$, every $g_{2,1} : [\omega]^2 \to k$ and every $g_1 : \omega \to k$:

(1): $f[F^K_{g_2,g_{2,1},g_1}]^3 \subseteq K$
(2.1): For every $x, y \in F^K_{g_2,g_{2,1},g_1}$, $g_2(x,y) \in K$.
(2.1.1): For every $x \in F^K_{g_2,g_{2,1},g_1}$, $g_{2,1}(x) \in K$.
(2.2): For every $x \in F^K_{g_2,g_{2,1},g_1}$, $g_1(x) \in K$.

Thus, Property (2.1.1) is necessary to propagate Property (2.1), and Properties (2.1) and (2.2) are necessary for Property (1).

Actually, for technical reasons appearing in Lemma 4.10, given a function $f : [\omega]^3 \to k$, we will define its limit behavior g_2 on $a < b$ by applying $\mathsf{RT}^{3-2}_{<\infty,\ell_{3-2}}$ to the function $c \mapsto f(a,b,c)$ and let $g_2(a,b)$ be the resulting set of colors. Therefore, $g_2(a,b)$ will not be a single limit color, but a set of colors of size ℓ_{3-2}. Thus g_2 has type $[\omega]^2 \to \mathcal{P}_{\ell_{3-2}}(k)$. Similarly, g_1 will be defined on a by applying $\mathsf{RT}^{3-1}_{<\infty,\ell_{3-1}}$ to the function $(b,c) \mapsto f(a,b,c)$ and let $g_1(a)$ be the limit set of colors of size ℓ_{3-1}. So g_1 has type $\omega \to \mathcal{P}_{\ell_{3-1}}$. Last, $g_{2,1}$ is now defining the limit behavior of the function $g_2 : [\omega] \to \mathcal{P}_{\ell_{3-2}}(k)$. It will be defined on input a by applying $\mathsf{RT}^{2-1}_{<\infty,\ell_{2-1}}$ to the function $b \mapsto g_2(a,b)$. Then, we get ℓ_{2-1} many values of g_2. However, the values of g_2 are also sets of colors of size ℓ_{3-2}. Therefore, $g_{2,1}(a)$ will collect a set of colors of size $\ell_{3-2} \times \ell_{2-1}$. Thus $g_{2,1}$ is of type $\omega \to \mathcal{P}_{\ell_{3-2}\times\ell_{2-1}}(k)$. Notice that the number of colors corresponds to Definition 4.2. The notion of forcing becomes $(F^K_{g_2,g_{2,1},g_1}, X : K \in \mathcal{P}_\ell(k), g_2 : [\omega]^2 \to \mathcal{P}_{\ell_1}(k), g_{2,1} : \omega \to \mathcal{P}_{\ell_1}(k), g_1 : \omega \to \mathcal{P}_{\ell_1}(k))$. A condition must satisfy the following properties for every $K \in \mathcal{P}_\ell(k)$, every $g_2 : [\omega]^2 \to \mathcal{P}_{\ell_1}(k)$, every $g_{2,1} : [\omega]^2 \to \mathcal{P}_{\ell_1}(k)$ and every $g_1 : \omega \to \mathcal{P}_{\ell_1}(k)$:

(1): $f[F^K_{g_2,g_{2,1},g_1}]^3 \subseteq K$

(2.1): For every $x, y \in F^K_{g_2,g_{2,1},g_1}$, $g_2(x, y) \subseteq K$.

(2.1.1): For every $x \in F^K_{g_2,g_{2,1},g_1}$, $g_{2,1}(x) \subseteq K$.

(2.2): For every $x \in F^K_{g_2,g_{2,1},g_1}$, $g_1(x) \subseteq K$.

Last, we can use compactness to the definition of a condition. Indeed, if for every $g_2 : [\omega]^2 \to \mathcal{P}_{\ell_1}(k)$, every $g_{2,1} : [\omega]^2 \to \mathcal{P}_{\ell_1}(k)$ and every $g_1 : \omega \to \mathcal{P}_{\ell_1}(k)$, we can find a tuple $\langle F^K_{g_2,g_{2,1},g_1} : K \in \mathcal{P}_\ell(k)\rangle$ satisfying properties (1), (2.1), (2.1.1) and (2.2), then we can find finitely many such tuples covering all the possible functions $g_2, g_{2,1}$ and g_1. Therefore, a condition becomes a tuple $(F^K_{g_2,g_{2,1},g_1}, X : K \in \mathcal{P}_\ell(k), g_2 : [\{0, \dots, p-1\}]^2 \to \mathcal{P}_{\ell_1}(k), g_{2,1} : \{0, \dots, p-1\} \to \mathcal{P}_{\ell_1}(k), g_1 : \{0, \dots, p-1\} \to \mathcal{P}_{\ell_1}(k))$ satisfying properties (1), (2.1), (2.1.1) and (2.2).

The exact computation of the size ℓ of the set K appears in Case 2 of Lemma 4.13, in order to force Π^0_1 facts.

Definition 4.4. Let $\mathbb{I}^{<\omega}$ be the set of all tuples $\vec{g} = \langle g_\sigma : \sigma \in \mathrm{Dec}_n\rangle$ such that for every $\sigma \in \mathrm{Dec}_n$, g_σ is a function of type $[\{0, \dots, p-1\}]^{\sigma^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma)}(k)$ for some $p \in \omega$. We then let the *height* of $\vec{g}$ be $\mathrm{ht}(\vec{g}) = p$. Let $\mathbb{I}^\omega$ be the set of all tuples $\vec{h} = \langle h_\sigma : \sigma \in \mathrm{Dec}_n\rangle$ such that for every $\sigma \in \mathrm{Dec}_n$, h_σ is a function of type $[\omega]^{\sigma^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma)}(k)$.

Given some $\vec{h} = \langle h_\sigma : \sigma \in \mathrm{Dec}_n\rangle \in \mathbb{I}^{<\omega} \cup \mathbb{I}^\omega$ and $\vec{g} = \langle g_\sigma : \sigma \in \mathrm{Dec}_n\rangle \in \mathbb{I}^{<\omega}$, we write $\vec{h} \le \vec{g}$ if $g_\sigma \subseteq h_\sigma$ for each $\sigma \in \mathrm{Dec}_n$.

An *index set* is a finite set $I \subseteq \mathbb{I}^{<\omega}$ such that for every $\vec{h} \in \mathbb{I}^\omega$, there is a tuple $\vec{g} \in I$, such that $\vec{h} \le \vec{g}$. Given two index sets I, J, we write $J \le I$ if for every $\vec{h} \in J$, there is some $\vec{g} \in I$ such that $\vec{h} \le \vec{g}$). The *height* of an index I is $\mathrm{ht}(I) = \max\{\mathrm{ht}(\vec{g}) : \vec{g} \in I\}$.

Definition 4.5. A *condition* is a tuple $(F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I)$ such that, letting $g_\epsilon = \vec{x} \mapsto \{f(\vec{x})\}$,

(a) I is an index set
(b) $g_\sigma(\vec{x}) \subseteq K$ for each $K \in \mathcal{P}_\ell(k)$, $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$, $\vec{g} \in I$ and $\vec{x} \in [F^K_{\vec{g}}]^{\sigma^+}$
(c) $X \in \mathcal{M}$ is an infinite set with $\min X > h(I)$

Note that the reservoir X is shared with all the stems $F^K_{\vec{g}}$. We refer to $\vec{g}$ as a *branch* of the condition c. Each branch can be seen as specifying $\binom{k}{\ell}$ simultaneous Mathias conditions $(F^K_{\vec{g}}, X)$ for each $K \in \mathcal{P}_\ell(k)$. Also note that, letting $\sigma = \epsilon$, we require that $f[F^K_{\vec{g}}]^n \subseteq K$ for each $\vec{g} \in I$.

Definition 4.6. A condition $d = (E^K_{\vec{h}}, Y : K \in \mathcal{P}_\ell(k), \vec{h} \in J)$ *extends* a condition $c = (F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I)$ (written $d \le c$) if $J \le I$, $Y \subseteq X$, and for each $K \in \mathcal{P}_\ell(k)$ and $\vec{h} \in J$ and $\vec{g} \in I$ such that $\vec{h} \le \vec{g}$, $F^K_{\vec{g}} \subseteq E^K_{\vec{h}}$ and $E^K_{\vec{h}} \smallsetminus F^K_{\vec{g}} \subseteq X$.

Definition 4.7. Let $\vec{h} \in \mathbb{I}^\omega$ and $\vec{g} \in \mathbb{I}^{<\omega}$ be such that $\vec{h} \leq \vec{g}$. Let $h_\epsilon : [\omega]^n \to \mathcal{P}_1(k)$. A set $F > \mathrm{ht}(\vec{g})$ is $(h_\epsilon, \vec{h})$*-compatible with* $\vec{g}$ if for every $\sigma \in \mathrm{Dec}_n$, letting $\tau = \sigma^-$, every $\vec{x} \in \mathrm{dom}\, g_\sigma$ and every $\vec{y} \in [F]^{\tau^+ - \sigma^+}$, then $g_\sigma(\vec{x}) \supseteq h_\tau(\vec{x}, \vec{y})$.

The notion of $(h_\epsilon, \vec{h})$-compatibility has been designed so that one can join two sets F and E satisfying property (b) of a forcing condition, and obtain a set $F \cup E$ still satisfying property (b), as proven in Lemma 4.8.

Lemma 4.8. *Let $\vec{h} \in \mathbb{I}^\omega$ and $\vec{g} \in \mathbb{I}^{<\omega}$ be such that $\vec{h} \leq \vec{g}$. Let $h_\epsilon = g_\epsilon$ be a function of type $[\omega]^n \to \mathcal{P}_1(k)$. Let F and E be two sets such that $F < \mathrm{ht}(\vec{g}) < E$ and*

(a) $g_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F]^{\sigma^+}$
(b) $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [E]^{\sigma^+}$
(c) E is $(h_\epsilon, \vec{h})$-compatible with $\vec{g}$

Then $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F \cup E]^{\sigma^+}$.

Proof. Let $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $p = \sigma^+$. Recall that by convention, $\epsilon^+ = n$. We show that $h_\sigma(\vec{x}) \subseteq K$ for each $\vec{x} \in [F \cup E]^p$. Let $\vec{x} = \{x_0, \dots, x_{p-1}\} \in [F \cup E]^p$, with $x_0 < \dots < x_{p-1}$. We have three cases. Case 1: $x_{p-1} \in F$. Then $\{x_0, \dots, x_{p-1}\} \in [F]^p$, and by (a), $h_\sigma(x_0, \dots, x_{p-1}) = g_\sigma(x_0, \dots, x_{p-1}) \subseteq K$. Case 2: $x_0 \in F$. Then $\{x_0, \dots, x_{p-1}\} \in [E]^p$, and by (b), $h_\sigma(x_0, \dots, x_{p-1}) \subseteq K$. Case 3: there is some $i \in \{1, \dots, p-1\}$ such that $x_{i-1} \in F$ and $x_i \in E$. Let $\tau = \sigma^\frown i$. Since $0 < i < p$, then $\tau \in \mathrm{Dec}_n$. By (a), $g_\tau(x_0, \dots, x_{i-1}) \subseteq K$. Since $\sigma = \tau^-$, then by (c), $g_\tau(x_0, \dots, x_{i-1}) \supseteq h_\sigma(x_0, \dots, x_{p-1})$. Hence $h_\sigma(x_0, \dots, x_{p-1}) \subseteq K$. □

Definition 4.9. Let $\prec_L$ be a linearization of the prefix order $\prec$ on Dec_n. We have $\sigma_0 \prec_L \sigma_1 \prec_L \dots \prec_L \sigma_{2^{n-1}-2}$. Given $p \in \omega$, let $\mathbb{T}_p$ be the set of all sequences $\langle g_{\sigma_0}, g_{\sigma_1}, \dots, g_{\sigma_s} \rangle$ for some $s < 2^{n-1} - 1$, such that $g_{\sigma_i} : [p]^{\sigma_i^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_i)}(k)$. The empty sequence $\langle\rangle$ also belongs to $\mathbb{T}_p$. The set $\mathbb{T}_p$ is naturally equipped with a partial order $\leq_{\mathbb{T}_p}$ corresponding to the prefix relation.

A $\mathbb{T}_p$*-tree* is a function S whose domain is $\mathbb{T}_p$, and such that $S(\langle g_{\sigma_0}, g_{\sigma_1}, \dots, g_{\sigma_s} \rangle)$ is a function $h_{\sigma_s} : [\omega]^{\sigma_s^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_i)}(k)$. By convention, $S(\langle\rangle)$ is a function $h_\epsilon : [\omega]^n \to \mathcal{P}_1(k)$.

Note that the maximal sequences in $\mathbb{T}_p$ are precisely the $\vec{g} \in \mathbb{I}^{<\omega}$ such that $\mathrm{ht}(\vec{g}) = p$. In some sense, $\mathbb{T}_p$ is the downward closure of such $\vec{g}$ under the $\prec_L$ relation. The following lemma justifies the combinatorial design of the notion of forcing.

Lemma 4.10. *Let $S \in \mathcal{M}$ be a $\mathbb{T}_p$-tree and $X \in \mathcal{M}$ be an infinite set with $X > p$. Then there is an infinite subset $Y \subseteq X$ in $\mathcal{M}$ and some $\vec{g} \in \mathbb{I}^{<\infty}$ with $\mathrm{ht}(\vec{g}) = p$, such that, letting $h_\epsilon = S(\langle\rangle)$ and $\vec{h} = \langle S(\xi) : \xi \leq_{\mathbb{T}_p} \vec{g} \rangle$, Y is $(h_\epsilon, \vec{h})$-compatible with $\vec{g}$.*

Proof. Fix $\sigma_0 \prec_L \sigma_1 \prec_L \dots \prec_L \sigma_{2^{n-1}-2}$. We define inductively two sequences

(a) $X = X_0 \supseteq X_1 \supseteq \dots \supseteq X_{2^{n-1}-1}$ where $X_i \in \mathcal{M}$ is an infinite set
(b) $g_{\sigma_0}, g_{\sigma_1}, \dots, g_{\sigma_{2^{n-1}-2}}$ where $g_{\sigma_i} : [p]^{\sigma_i^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_i)}(k)$

This induces a sequence $h_\epsilon, h_{\sigma_0}, h_{\sigma_1}, \dots, h_{\sigma_{2^{n}-2}}$ defined by $h_\epsilon = S(\langle\rangle)$ and $h_{\sigma_i} = S(\langle g_{\sigma_0}, \dots, g_{\sigma_i} \rangle)$. Note that $h_{\sigma_i} \in \mathcal{M}$ is a function of type $[\omega]^{\sigma_i^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_i)}(k)$.

At step $i < 2^{n-1} - 1$, we have already defined an infinite set $X_i \in \mathcal{M}$ and the functions g_{σ_j} and h_{σ_j} for every $j < i$. Let $\tau = \sigma_i^-$, $a = \sigma_i^+$ and $b = \tau^+$. Since $\prec_L$ is a linearization of $\prec$, the functions $g_\tau : [p]^b \to \mathcal{P}_{\vec{\ell}(n,\tau)}(k)$ and $h_\tau : [\omega]^b \to \mathcal{P}_{\vec{\ell}(n,\tau)}(k)$ are already defined.

For each tuple $\{x_0, \dots, x_{a-1}\} \in [p]^a$, we can apply $\mathsf{RT}^{b-a}_{<\infty,\ell_{b-a}}$ to $x_a, \dots, x_{b-1} \mapsto h_\tau(x_0, \dots, x_{b-1})$ on the domain X_i to obtain a set of colors $C \in \mathcal{P}_{\ell_{b-a}}(\mathcal{P}_{\vec{\ell}(n,\tau)}(k))$ and an infinite set $Y \subseteq X_i$ with $Y \in \mathcal{M}$ such that

$$(\forall \{x_a, \dots, x_{b-1}\} \in [Y]^{b-a}) h_\tau(x_0, \dots, x_{b-1}) \in C$$

Let $g_{\sigma_i}(x_0, \dots, x_{a-1}) = \bigcup C$. Note that $|C| = \ell_{b-a}$ and that each element of C has size $\vec{\ell}(n, \tau)$, so $|\bigcup C| = \vec{\ell}(n, \tau) \cdot \ell_{b-a} = \vec{\ell}(n, \sigma_i)$. By applying the operation iteratively for each tuple in $[p]^a$, we obtain an infinite set $X_{i+1} \subseteq X_i$ in $\mathcal{M}$ and a function $g_{\sigma_i} : [p]^a \to \mathcal{P}_{\vec{\ell}(n,\sigma_i)}(k)$ such that for every $\{x_0, \dots, x_{a-1}\} \in [p]^a$ and $\{x_a, \dots, x_{b-1}\} \in [X_{i+1}]^{b-a}$,

$$g_{\sigma_i}(x_0, \dots, x_{a-1}) \supseteq h_\tau(x_0, \dots, x_{b-1})$$

We then go to the next step. At the end of the construction, we obtain an infinite set $X_{2^{n-1}-1}$ and some $\vec{g} \in \mathbb{I}^{<\omega}$ satisfying the statement of our lemma. $\square$

Definition 4.11. Let $c = (F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I)$ be a condition and $\varphi(G, x)$ be a Δ^0_0 formula.

(a) $c \Vdash^K_{\vec{g}} (\exists x)\varphi(G, x)$ if there is some $x \in \omega$ such that $\varphi(F^K_{\vec{g}}, x)$ holds
(b) $c \Vdash^K_{\vec{g}} (\forall x)\varphi(G, x)$ if for every $x \in \omega$, every $E \subseteq X$, $\varphi(F^K_{\vec{g}} \cup E, x)$ holds.

Note that the forcing relation for Π^0_1 formulas seems too strong since no color restrain is imposed on the tuples over E.

Definition 4.12. Let $c = (F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I)$ and $d = (E^K_{\vec{h}}, Y : K \in \mathcal{P}_\ell(k), \vec{h} \in J)$ be conditions. We say that d is an *R-extension* of c for some $R \subseteq J$ if $R \neq \emptyset$ and $J - I \subseteq R$.

Lemma 4.13. *Let $c = (F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I)$ be a condition. For every $\vec{g} \in I$, let $\langle e^K_{\vec{g}} : K \in \mathcal{P}_\ell(k), \vec{g} \in I\rangle$ be Turing indices. Then there is a branch $\vec{g} \in I$ and an R-extension d such that for every branch $\vec{h} \in R$ of d refining some branch $\vec{g}$ in c,*

$$d \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_e(x) \uparrow \quad \text{or} \quad d \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_e(x) \downarrow \neq A(x)$$

for some $K \in \mathcal{P}_\ell(k)$, some $x \in \omega$ and $e = e^K_{\vec{g}}$.

Proof. Fix $\sigma_0 \prec_L \sigma_1 \prec_L \dots \prec_L \sigma_{2^{n-1}-2}$. Fix $p = \mathrm{ht}(I)$, $x \in \omega$ and $v < 2$.

Given some $\rho_{\sigma_{2^{n-1}-2}} : [p]^{\sigma^+_{2^{n-1}-2}} \to \mathcal{P}_{\vec{\ell}(n,\sigma_{2^{n-1}-2})}(k)$ and some $h_\epsilon, h_{\sigma_0}, \dots, h_{\sigma_{2^{n-1}-3}}$ such that $h_\epsilon : [\omega]^n \to \mathcal{P}_1(k)$ and $h_{\sigma_i} : [\omega]^{\sigma^+_i} \to \mathcal{P}_{\vec{\ell}(n,\sigma_i)}(k)$, let

$$\mathcal{C}_{x,v}(\rho_{\sigma_{2^{n-1}-2}}, h_\epsilon, h_{\sigma_0}, \dots, h_{\sigma_{2^{n-1}-3}})$$

be the $\Pi^0_1(X \oplus C \oplus \vec{h})$ class of all $h_{\sigma_{2^{n-1}-2}} : [\omega]^{\sigma^+_{2^{n-1}-2}} \to \mathcal{P}_{\vec{\ell}(n,\sigma_{2^{n-1}-2})}(k)$ such that $\rho_{\sigma_{2^{n-1}-2}} \subseteq h_{\sigma_{2^{n-1}-2}}$ and for every $K \in \mathcal{P}_\ell(k)$ and for every finite set $E \subseteq X$ such that $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F^K_{\vec{\rho}} \cup E]^{\sigma^+}$,

$$\Phi^{(F^K_{\vec{g}} \cup E) \oplus C}_{e^K_{\vec{g}}}(x) \uparrow \text{ or } \Phi^{(F^K_{\vec{g}} \cup E) \oplus C}_{e^K_{\vec{g}}}(x) \downarrow \neq v$$

where $\vec{g} \in I$ is such that $\vec{h} \leq \vec{g}$.

Given some $\rho_{\sigma_{2^{n-1}-3}} : [p]^{\sigma^+_{2^{n-1}-3}} \to \mathcal{P}_{\vec{\ell}(n,\sigma_{2^{n-1}-3})}(k)$ and some $h_\epsilon, h_{\sigma_0}, \dots, h_{\sigma_{2^{n-1}-4}}$, let $\mathcal{C}_{x,v}(\rho_{\sigma_{2^{n-1}-3}}, h_\epsilon, h_{\sigma_0}, \dots, h_{\sigma_{2^{n-1}-4}})$ be the $\Pi^0_1(X \oplus C \oplus \vec{h})$ class of all $h_{\sigma_{2^{n-1}-3}} : [\omega]^{\sigma^+_{2^{n-1}-3}} \to \mathcal{P}_{\vec{\ell}(n,\sigma_{2^{n-1}-3})}(k)$ such that $\rho_{\sigma_{2^{n-1}-3}} \subseteq h_{\sigma_{2^{n-1}-3}}$ and for every $\rho_{\sigma_{2^{n-1}-2}} : [p]^{\sigma^+_{2^{n-1}-2}} \to \mathcal{P}_{\vec{\ell}(n,\sigma_{2^{n-1}-2})}(k)$,

$$\mathcal{C}_{x,v}(\rho_{\sigma_{2^{n-1}-2}}, h_\epsilon, h_{\sigma_0}, \dots, h_{\sigma_{2^{n-1}-4}}, h_{\sigma_{2^{n-1}-3}}) \neq \emptyset$$

And so on. Then we let $\mathcal{C}_{x,v}$ be the $\Pi^0_1(X \oplus C)$ class of all $h_\epsilon : [\omega]^n \to \mathcal{P}_1(k)$ such that for every $\rho_{\sigma_0} : [p]^{\sigma_0^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_0)}(k)$, $\mathcal{C}_{x,v}(\rho_{\sigma_0}, h_\epsilon) \neq \emptyset$. Finally, let $W = \{(x,v) : \mathcal{C}_{x,v} = \emptyset\}$. Note that W is an $X \oplus C$-c.e. set. We have three cases.

- Case 1: There is some $x \in \omega$ such that $(x, 1 - A(x)) \in W$. By definition, $\mathcal{C}_{x,1-A(x)} = \emptyset$. In particular, the function $h_\epsilon = \vec{x} \mapsto \{f(\vec{x})\}$ is not in $\mathcal{C}_{x,1-A(x)}$, so there is a $\rho_{\sigma_0} : [p]^{\sigma_0^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_0)}(k)$ such that $\mathcal{C}_{x,1-A(x)}(\rho_{\sigma_0}, h_\epsilon) = \emptyset$. By compactness, there is some $p_0 \in \omega$ such that for every $h_{\sigma_0} : [p_0]^{\sigma_0^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_0)}(k)$, there is a $\rho_{\sigma_1} : [p]^{\sigma_1^+} \to \mathcal{P}_{\vec{\ell}(n,\sigma_1)}(k)$ such that $\mathcal{C}_{x,1-A(x)}(\rho_{\sigma_1}, h_\epsilon, h_{\sigma_0}) = \emptyset$. By iterating the reasoning and assuming that p_0 is large enough to be the same witness of compactness, we obtain a non-empty collection R of $\vec{h} \in \mathbb{I}^{<\omega}$ with $\mathrm{ht}(\vec{h}) = p_0$ satisfying the following two properties: First, letting $J = R \cup \{\vec{g} \in I : (\forall \vec{h} \in R) \vec{h} \not\leq \vec{g}\}$, the set J is an index set. Second, for each $\vec{h} \in R$, there is some $K \in \mathcal{P}_\ell(k)$ and some finite set $E_{\vec{h}} \subseteq X$ such that $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F^K_{\vec{\rho}} \cup E_{\vec{h}}]^{\sigma^+}$, and
$$\Phi^{(F^K_{\vec{g}} \cup E) \oplus C}_{e^K_{\vec{g}}}(x) \downarrow = 1 - A(x)$$
where $\vec{g} \in I$ is such that $\vec{h} \leq \vec{g}$. Define the R-extension $d = (H^K_{\vec{h}}, X - \{0, \dots, p_0\} : K \in \mathcal{P}_\ell(k), \vec{h} \in J)$ of c by setting $H^K_{\vec{h}} = F^K_{\vec{g}} \cup E_{\vec{h}}$ if $\vec{h} \in R$ and $\vec{g} \in I$ is such that $\vec{h} \leq \vec{g}$ and $K_{\vec{g}} = K$. Otherwise, set $H^K_{\vec{h}} = F^K_{\vec{g}}$ where $\vec{g} \in I$ is such that $\vec{h} \leq \vec{g}$. For every branch $\vec{h} \in R$ of d refining $\vec{g}$, letting $K = K_{\vec{h}}$ and $e = e^K_{\vec{g}}$, $d \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e^K_{\vec{g}}}(x) \downarrow \neq A(x)$.
- Case 2: There is some $x \in \omega$ such that $(x,0), (x,1) \notin W$. In particular $(x, A(x)) \notin W$, so $\mathcal{C}_{x,A(x)} \neq \emptyset$. Since $\mathcal{M} \models \mathsf{WKL}$, there is some $\mathbb{T}_p$-tree $S \in \mathcal{M}$ such that for every maximal sequence $\vec{\rho} = \langle \rho_{\sigma_0}, \dots, \rho_{\sigma_{2^{n-1}-2}} \rangle \in \mathbb{T}_p$, letting $h_\epsilon = S(\langle\rangle)$ and, for each $i < 2^{n-1}-1$, letting $h_{\sigma_i} = S(\langle \rho_{\sigma_0}, \dots, \rho_{\sigma_i} \rangle)$, for every $K \in \mathcal{P}_\ell(k)$ and for every finite set $E \subseteq X$ such that $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F^K_{\vec{\rho}} \cup E]^{\sigma^+}$,
$$\Phi^{(F^K_{\vec{g}} \cup E) \oplus C}_{e^K_{\vec{g}}}(x) \uparrow \text{ or } \Phi^{(F^K_{\vec{g}} \cup E) \oplus C}_{e^K_{\vec{g}}}(x) \downarrow \neq A(x)$$
where $\vec{g} \in I$ is such that $\vec{h} \leq \vec{g}$. By Lemma 4.10, there is an infinite set $Y \subseteq X$ in $\mathcal{M}$, and some $\vec{\rho} \in \mathbb{I}^{<\infty}$ with $\mathrm{ht}(\vec{g}) = p$, such that, letting $h_\epsilon = S(\langle\rangle)$ and $\vec{h} = \langle S(\xi) : \xi \leq_{\mathbb{T}_p} \vec{g} \rangle$, Y is $(h_\epsilon, \vec{h})$-compatible with $\vec{\rho}$. Let $\vec{g} \in I$ be such that $\vec{\rho} \leq \vec{g}$. In particular, Y is $(h_\epsilon, \vec{h})$-compatible with $\vec{g}$.

Since for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$, $\mathcal{M} \models \mathsf{RT}^{\sigma^+}_{<\infty,\ell_{\sigma^+}}$, then by an iterative process, we obtain an infinite set $Y_1 \subseteq Y$ in $\mathcal{M}$ and for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ some set of colors $C_\sigma \in \mathcal{P}_{\ell_{\sigma^+}}(\mathcal{P}_{\vec{\ell}(n,\sigma)}(k))$ such that $h_\sigma[Y_1]^{\sigma^+} \subseteq C_\sigma$. In particular, for $\sigma \in \mathrm{Dec}_n$, $|C_\sigma| = \ell_{\sigma^+}$ and each element of C_σ has size $\vec{\ell}(n,\sigma)$, so $|\bigcup C_\sigma| = \vec{\ell}(n,\sigma) \times \ell_{\sigma^+}$. Moreover, $C_\epsilon \in \mathcal{P}_{\ell_n}(\mathcal{P}_{\vec{\ell}(n,\epsilon)}(k))$ with $\vec{\ell}(n,\epsilon) = 1$, so $|C_\epsilon| = \ell_n$ and $|\bigcup C_\epsilon| = \ell_n$. It follows that

$$\begin{aligned} |\bigcup_{\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}} \bigcup C_\sigma| &\leq |\bigcup C_\epsilon| + \sum_{\sigma \in \mathrm{Dec}_n} |\bigcup C_\sigma|| \\ &\leq \ell_n + \sum_{\sigma \in \mathrm{Dec}_n} \vec{\ell}(n,\sigma) \times \ell_{\sigma^+} \end{aligned}$$

Therefore, there is some $K \in \mathcal{P}_\ell(k)$ such that $K \supseteq \bigcup_{\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}} \bigcup C_\sigma$. By definition of a condition, $g_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F^K_{\vec{g}}]^{\sigma^+}$. By choice of K, $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [Y]^{\sigma^+}$. By choice of Y, Y is $(h_\epsilon, \vec{h})$-compatible with $\vec{g}$. Therefore, by Lemma 4.8, $h_\sigma(\vec{x}) \subseteq K$ for each $\sigma \in \mathrm{Dec}_n \cup \{\epsilon\}$ and $\vec{x} \in [F^K_{\vec{g}} \cup Y]^{\sigma^+}$. The condition $d = (F^K_{\vec{g}}, Y : K \in \mathcal{P}_\ell(k), \vec{g} \in I)$ is a $\{\vec{g}\}$-extension of c such that $d^K_{\vec{g}} \Vdash \Phi^{G \oplus C}_e(x)\uparrow$, where $e = e^K_{\vec{g}}$.

- Case 3: None of the above cases hold. In this case, we can $X \oplus C$-compute the set A, contradicting our assumption.

□

For the simplicity of notation, given some $\vec{h} \in \mathbb{I}^\omega$ and a condition c with index set I, we might write $\Vdash^K_{\vec{h}}$ for $\Vdash^K_{\vec{g}}$ where $\vec{g}$ is the unique branch in I such that $\vec{h} \leq \vec{g}$.

Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. By Lemma 4.13, there is some $\vec{h} \in \mathbb{I}^\omega$ such that for every tuple of indices $\langle e_K \in \omega : K \in \mathcal{P}_\ell(k)\rangle$,

$$c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e_K}(x)\uparrow \quad \text{or} \quad c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e_K}(x)\downarrow \neq A(x) \tag{1}$$

for some $c \in \mathcal{F}$, $K \in \mathcal{P}_\ell(k)$ and some $x \in \omega$. We claim that there is some $K \in \mathcal{P}_\ell(k)$ such that for every index $e \in \omega$,

$$c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e}(x)\uparrow \quad \text{or} \quad c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e}(x)\downarrow \neq A(x) \tag{2}$$

for some $c \in \mathcal{F}$ and some $x \in \omega$. Indeed, suppose not. Then for every $K \in \mathcal{P}_\ell(k)$, there is some e_K such that for every $c \in \mathcal{F}$ and $x \in \omega$, the equation (2) does not hold. Then this contradicts the equation (1) for the tuple $\langle e_K \in \omega : K \in \mathcal{P}_\ell(k)\rangle$.

In what follows, we fix $\mathcal{F}$, $\vec{h}$ and K such that the equation (2) holds. Let

$$G = \bigcup\{F^K_{\vec{g}} : (F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I) \in \mathcal{F}, \vec{h} \leq \vec{g}\}$$

Lemma 4.14. *The set G is infinite.*

Proof. Let $t \in \omega$. Let $\Phi^{G \oplus C}_e$ be the Turing functional which on input x searches for some $y \in G$ such that $y > t$. If found, the program halts and output 1. Otherwise it diverges. Let $c = (F^K_{\vec{g}}, X : K \in \mathcal{P}_\ell(k), \vec{g} \in I) \in \mathcal{F}$ and x be such that

$$c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e}(x)\uparrow \quad \text{or} \quad c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_{e}(x)\downarrow \neq A(x)$$

Note that $c \nVdash^K_{\vec{h}} \Phi^{G \oplus C}_e(x)\uparrow$ since the reservoir X is infinite. It follows that $c \Vdash^K_{\vec{h}} \Phi^{G \oplus C}_e(x)\downarrow \neq A(x)$. Unfolding the definition of the forcing relation, $\Phi^{F^K_{\vec{g}} \oplus C}_e(x)\downarrow$

where $\vec{g} \in I$ is such that $\vec{h} \leq \vec{g}$. In other words, $\max F^K_{\vec{g}} > t$. Since $F^K_{\vec{g}} \subseteq G$, there is some $y \in G$ with $y > t$. ☐

By construction, $f[G]^n \subseteq K$, and by choice of $\vec{h}$, $G \oplus C$ does not compute A. This completes the proof of Theorem 4.3. ☐

Theorem 4.15. *For every $n \geq 1$, $\mathsf{RT}^n_{<\infty,2^{n-1}}$ admits strong cone avoidance for non-arithmetical cones.*

Proof. Fix a set C, a set A which is not arithmetical in C, and a coloring $f : [\omega]^n \to k$. By Jockusch and Soare [10], every computable instance of WKL has a low solution, and by Jockusch [9], every computable instance of RT has an arithmetical solution. Therefore, there is a countable ω-model $\mathcal{M}$ of $\mathsf{WKL} + \mathsf{RT}$ such that $C \in \mathcal{M}$ and containing only sets arithmetical in C. In particular, A is not arithmetical in any element of $\mathcal{M}$. By Theorem 4.3, letting

$$\ell = 1 + \sum_{\sigma \in \mathrm{Dec}_n} 1 = 2^{n-1}$$

there is an infinite set $G \subseteq \omega$ such that $f[G]^n \subseteq_\ell k$, and such that for every $C \in \mathcal{M}$, $A \not\leq_T G \oplus C$. This completes the proof of Theorem 4.15. ☐

Let $\ell_1, \ell_2, \dots$ be the sequence inductively defined by $\ell_1 = 1$, and

$$\ell_{n+1} = \ell_n + \sum_{\sigma \in \mathrm{Dec}_n} \vec{\ell}(n, \sigma) \cdot \ell_{\sigma^+}$$

Lemma 4.16. *For every $n \geq 1$,*

$$\ell_{n+1} = \sum_{i=0}^{n-1} \ell_{i+1}\ell_{n-i}$$

Proof. By induction over n. Assume $\ell_{i+1} = \sum_{i=0}^{i-1} \ell_{i+1}\ell_{i-1}$ for every $i \in \{1, \dots, n-1\}$. By definition, $\ell_{n+1} = \ell_n + \sum_{\sigma \in \mathrm{Dec}_n} \vec{\ell}(n, \sigma) \cdot \ell_{\sigma^+}$.

Fix $i \in \{1, \dots, n-1\}$. The strings $\sigma \in \mathrm{Dec}_n$ such that $\sigma(0) = i$ are precisely the strings of the form $i^\frown\tau$ for some $\tau \in \mathrm{Dec}_i \cup \{\epsilon\}$. Therefore,

$$\begin{aligned}
\textstyle\sum_{\sigma \in \mathrm{Dec}_n, \sigma(0)=i} \vec{\ell}(n, \sigma) \cdot \ell_{\sigma^+} &= \textstyle\sum_{\tau \in \mathrm{Dec}_i \cup \{\epsilon\}} \vec{\ell}(n, i^\frown\tau) \cdot \ell_{(i^\frown\tau)^+} \\
&= \vec{\ell}(n, i)\ell_i + \textstyle\sum_{\tau \in \mathrm{Dec}_i} \ell_{n-i} \cdot \vec{\ell}(i, \tau) \cdot \ell_{\tau^+} \\
&= \ell_{n-i}(\ell_i + \textstyle\sum_{\tau \in \mathrm{Dec}_i} \vec{\ell}(i, \tau) \cdot \ell_{\tau^+}) \\
&= \ell_{n-i} \cdot \ell_{i+1}
\end{aligned}$$

Therefore,

$$\ell_{n+1} = \ell_n + \sum_{\sigma \in \mathrm{Dec}_n} \vec{\ell}(n, \sigma) \cdot \ell_{\sigma^+} = \ell_n + \sum_{i=1}^{n-1} \ell_{n-i}\ell_{i+1} = \sum_{i=0}^{n-1} \ell_{i+1}\ell_{n-i}$$

This completes the proof of Lemma 4.16. ☐

Recall that $d_0, d_1, \dots$ denotes the Catalan sequence, inductively defined by $d_0 = 1$ and

$$d_{n+1} = \sum_{i=0}^{n} d_i d_{n-i}$$

Corollary 4.17. *For every $n \geq 0$, $d_n = \ell_{n+1}$.*

Proof. Immediate by Lemma 4.16. □

Theorem 4.18. *For every $n \geq 1$, $\mathsf{RT}^n_{<\infty,d_n}$ admits strong cone avoidance.*

Proof. By induction over $n \geq 1$. Fix a set C, a set $A \not\leq_T C$, and a coloring $f : [\omega]^n \to k$. By Jocksuch and Soare [10], every C-computable instance of WKL has a solution P such that $A \not\leq_T P \oplus C$. By induction hypothesis, we can build a countable ω-model $\mathcal{M}$ of WKL $\bigwedge_{s \in \{1,\dots,n\}} \mathsf{RT}^s_{<\infty,d_{s-1}}$ such that $C \in \mathcal{M}$ and $A \notin \mathcal{M}$. By Corollary 4.17, $\mathcal{M} \models \bigwedge_{s \in \{1,\dots,n\}} \mathsf{RT}^s_{<\infty,\ell_s}$. By Theorem 4.3, there is an infinite set $G \subseteq \omega$ such that $f[G]^n \subseteq_{\ell_{n+1}} k$, and such that for every $C \in \mathcal{M}$, $A \not\leq_T G \oplus C$. By Corollary 4.17, $\ell_{n+1} = d_n$. This completes the proof of Theorem 4.18. □

Corollary 4.19. *For every $n \geq 1$, $\mathsf{RT}^{n+1}_{<\infty,d_n}$ admits cone avoidance.*

Proof. Immediate by Theorem 4.18 and Theorem 1.5. □

5. The GAP principle

As explained in Section 3, a candidate function to improve the lower bound on the strength of the thin set theorem for 3-tuples was

$$f(a,b,c) = \langle gap(a,b), gap(b,c), gap(a,c)\rangle$$

where $gap(a,b) = \ell$ if $[a,b]$ is g-large, and $gap(a,b) = s$ otherwise. In this section, we prove that there always exists an infinite set $H \subseteq \omega$ which avoids the color $\langle s,s,\ell\rangle$ and which does not compute the halting set. We define the corresponding problem GAP, and study its reverse mathematical strength.

Definition 5.1. A set H is *g-transitive* if for every $x < y < z \in H$ such that $[x,y]$ and $[y,z]$ are g-small, then $[x,z]$ is g-small.

The notion of g-transitivity exactly says that the color $\langle s,s,\ell\rangle$ is avoided for the previously defined function f.

Statement 5.2. GAP is the statement "For every increasing function $g : \omega \to \omega$, there is an infinite g-transitive set H." DGAP is the statement "For every Δ^0_2 increasing function $g : \omega \to \omega$, there is an infinite g-transitive set H."

The main motivation of the GAP principle is the study of strong cone avoidance of $\mathsf{RT}^3_{5,4}$. We start by proving that GAP follows from a stable version of the Erdős-Moser theorem, which is already known to admit strong cone avoidance.

Definition 5.3 (Erdős-Moser theorem)**.** A tournament T on a domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation on D such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x,y)$ or $T(y,x)$ holds. A tournament T is *transitive* if the corresponding relation T is transitive in the usual sense. A tournament T is *stable* if $(\forall x \in D)[(\forall^\infty s)T(x,s) \vee (\forall^\infty s)T(s,x)]$. EM is the statement "Every infinite tournament T has an infinite transitive subtournament." SEM is the restriction of EM to stable tournaments.

Definition 5.4. $\mathsf{P} \leq_{sc} \mathsf{Q}$ iff, for every instance of P, I_{P}, there is an instance of Q, I_{Q}, such that for every solution of I_{Q}, S_{Q}, computes a solution of I_{P}, S_{P}.

Theorem 5.5. $\mathsf{GAP} \leq_{sc} \mathsf{SEM}$.

Proof. Let $g : \omega \to \omega$ be a function. Set $T(x, y)$ to hold if $x < y$ and $[x, y]$ is g-small, or $y \leq x$ and $[x, y]$ is g-large. Note that T is stable. Let H be an infinite T-transitive subtournament. Then for every $x < y < z$ such that $[x, y]$ and $[y, z]$ are g-small, $T(x, y)$ and $T(y, z)$ holds. By T-transitivity of H, $T(x, z)$ holds, hence $[x, z]$ is g-small. □

Corollary 5.6. GAP *admits strong cone avoidance.*

Proof. EM admits strong cone avoidance [13] and $\mathsf{GAP} \leq_{sc} \mathsf{EM}$. □

Corollary 5.7. $\mathsf{RCA}_0 + \mathsf{DGAP} \nvdash \mathsf{ACA}$.

Proof. Build an ω-model of $\mathsf{RCA}_0 + \mathsf{DGAP}$ which does not contain the halting set. □

Theorem 5.8. $\mathsf{GAP} \leq_{sc} \mathsf{RT}^3_{5,4}$.

Proof. Let $g : \omega \to \omega$ be a an increasing function. Given $x < y \in \omega$, let $i(x, y) = 1$ if $[x, y]$ is g-large, and $i(x, y) = 0$ otherwise. Let $f : [\omega]^3 \to 5$ be defined on $x < y < z$ by $f(x, y, z) = \langle i(x, y), i(y, zs), i(x, z)\rangle$. Note that f is a 5-coloring, since the colors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, $\langle 1, 1, 0\rangle$ cannot occur. Let H be an infinite set such that $f[H]^3$ avoids one color c. We have several cases.

- Case 1: $c = \langle 1, 1, 1\rangle$. This case is impossible, since we can always pick three elements $x < y < z \in H$ sufficiently sparse so that $[x, y]$ and $[y, z]$ is g-large.
- Case 2: $c = \langle 0, 1, 1\rangle$. In this case, the set H is only made of g-large intervals, and therefore is g-transitive. Indeed, suppose there is a g-small interval $[x, y]$ with $x < y \in H$. Then picking z sufficiently far, $i(y, z) = 1$ and $i(x, z) = 1$, in which case $f(x, y, z) = \langle 0, 1, 1\rangle$.
- Case 3: $c = \langle 1, 0, 1\rangle$. Let $x < y \in H$ be such that $[x, y]$ is g-large, and let $G = \{z \in H : z > y\}$. We claim that G is made only of g-large intervals, hence is g-transitive. Indeed, suppose that $[u, v]$ is g-small for some $u < v \in G$. Then, since $[x, u]$ is g-large, $f(x, u, v) = \langle 1, 0, 1\rangle$, contradiction.
- Case 4: $c = \langle 0, 0, 1\rangle$. This means exactly that H is g-transitive.
- Case 5: $c = \langle 0, 0, 0\rangle$. Suppose $H = \{x_0 < x_1 < \dots\}$, and let $G = \{x_{2n} : n \in \omega\}$. We claim that G is made only of g-large intervals, hence is g-transitive. Indeed, if $[x_{2n}, x_{2n+2}]$ were g-small, then so would be $[x_{2n}, x_{2n+1}]$ and $[x_{2n+1}, x_{2n+2}]$, in which case $f(x_{2n}, x_{2n+1}, x_{2n+2}) = \langle 0, 0, 0\rangle$.

Note that case 3 is the only one which prevents the reduction from being uniform, and case 4 is the only one which prevents the reduction from constructing an infinite set on which all the intervals are g-large. In particular, $\mathsf{GAP} \leq_{sW} \mathsf{RT}^3_{5,3}$. □

Definition 5.9. A function g is *hyperimmune* if it is not dominated by any computable function. An infinite set $H = \{x_0 < x_1 < \dots\}$ is *hyperimmune* if the function p_H defined by $p_H(n) = x_n$ is hyperimmune.

Lemma 5.10. *Let $g : \omega \to \omega$ be an increasing function. Every function dominating g computes an infinite g-transitive set.*

Proof. Let f be a function dominating g. Let $H = \{x_0 < x_1 < \dots\}$ be defined by $x_0 = 0$, and $x_{n+1} = f(x_n)$. Then every interval in H is f-large, hence g-large, and in particular is g-transitive. □

Definition 5.11. A function $f : \omega \to \omega$ is *diagonally non-computable* relative to X if $f(e) \neq \Phi_e^X(e)$ for every $e \in \omega$. DNC is the statement "For every set X, there is a diagonally non-computable function relative to X."

We then say that a Turing degree is DNC if it contains a diagonally non-computable function. The notion of DNC degree is very weak and not being able to bound such a degree is a good measure of the computability-theoretic weakness of a problem.

Theorem 5.12. *Let $g : \omega \to \omega$ be an increasing function and $f_0, f_1, \dots$ be a countable sequence of hyperimmune functions. Then there is an infinite g-transitive set H of non-DNC degree such that f_i is H-hyperimmune for every i.*

Proof. Suppose there is no computable infinite g-transitive set, otherwise we are done. We will build the set H using a variant of computable Mathias forcing. A condition is a pair (F, X) where F is a finite g-transitive set, X is an infinite, computable set such that $\max F < \min X$, and $[x, y]$ is g-large for every $x \in F$ and $y \in X$. A condition (E, Y) extends (F, X) if $F \subseteq E$, $Y \subseteq X$ and $E \setminus F \subseteq X$.

Every sufficiently generic filter for this notion of forcing yields an infinite g-transitive set G. Indeed, given a condition (F, X), one can pick any $x \in X$, add it to F, and remove finitely many elements from X to obtain an infinite set Y such that $[x, y]$ is g-large for every $y \in Y$.

Lemma 5.13. *Given a Turing functional Φ_e, some $i \in \omega$ and a condition c, there is an extension d forcing either Φ_e^G to be partial, or Φ_e^G not to dominate f_i.*

Proof. Fix a condition (F, X). Define the partial computable function $h : \omega \to \omega$ which on input n, searches for some finite sets $E_0 < E_1 < \dots < E_{k-1} \subseteq X$ such that $\Phi_e^{F \cup E_j}(n) \downarrow$ for each $j < k$, and $\Phi_e^{F \cup H}(n) \downarrow$, where $H = \{\max E_j : j < k\}$. If found, $h(n) = \max_j \{\Phi_e^{F \cup E_j}(n), \Phi_e^{F \cup H}(n)\}$. Otherwise $h(n) \uparrow$. We have two cases.

Case 1: h is total. Since f_i is hyperimmune, there is some n such that $h(n) < f_i(n)$. Let $E_0 < E_1 < \dots < E_{k-1} \subseteq X$ witness that $h(n) \downarrow$, and let Y be obtained from X by removing finitely many elements so that $[x, y]$ is g-large for every $x \in E_i$ and $y \in Y$. Either there is some E_j which is g-small, in which case, $(F \cup E_i, Y)$ is an extension forcing $\Phi_e^G(n) \downarrow < f_i(n)$, or for every $j < k$, E_j is g-large. Then H is made only of large intervals, so is g-transitive. $(F \cup H, Y)$ is then an extension forcing again $\Phi_e^G(n) \downarrow < f_i(n)$.

Case 2: h is partial, say $h(n) \uparrow$. Let $E_0 < E_1 < \dots$ be a maximal computable sequence of finite sets such that $\Phi_e^{F \cup E_j}(n) \downarrow$ for each j. If the sequence is finite, then letting $Y = X \setminus \{0, \dots, \max E_j\}$, (F, Y) forces $\Phi_e^G(n) \uparrow$. If the sequence is infinite, then letting $Y = \{\max E_j : j \in \omega\}$, the condition (F, Y) is an extension forcing again $\Phi_e^G(n) \uparrow$. □

By the previous lemma, f_i is G-hyperimmune for every sufficiently generic for this notion of forcing and every $i \in \omega$.

Lemma 5.14. *Given a Turing functional Φ_e, and a condition c, there is an extension d forcing either Φ_e^G to be partial, or $\Phi_e^G(n) = \Phi_n(n)$ for some $n \in \omega$.*

Proof. Fix a condition (F, X). Define a maximal computable sequence $E_0 < E_1 < \dots \subseteq X$ such that for every n, $\Phi_e^{F \cup E_n}(n) \downarrow$. Suppose first that this sequence is finite, with maximum element E_n, then $(F, X - \{0, \dots, \max E_n\})$ is an extension forcing $\Phi_e^G(n+1)\uparrow$. Suppose now that this sequence is infinite. Then, there must be infinitely many n such that $\Phi_e^{F\cup E_n}(n)\downarrow=\Phi_n(n)$, otherwise we would compute a DNC function. Let $W=\{n : \Phi_e^{F\cup E_n}(n)\downarrow=\Phi_n(n)\}$. If E_n is g-small for some $n\in W$, then in particular $F\cup E_n$ is g-transitive. Let $d=(F\cup E_n, Y)$ where Y is obtained from X by removing finitely many elements so that $[x,y]$ is g-large for every $x\in E_n$. The condition d is an extension of c forcing $\Phi_e^G(n)=\Phi_n(n)$. If E_n is g-large for every $n\in W$, then we can compute a function dominating g, and by Lemma 5.10, compute an infinite g-transitive set, contradicting our assumption. □

By the previous lemma, G is not of DNC degree for every sufficiently generic for this notion of forcing. This completes the proof. □

Definition 5.15 (Ascending descending sequence)**.** Given a linear order $(L,<_L)$, an *ascending* (*descending*) sequence is a set S such that for every $x<_{\mathbb{N}} y\in S$, $x<_L y$ ($x>_L y$). ADS is the statement "Every infinite linear order admits an infinite ascending or descending sequence". SADS is the restriction of ADS to orders of type $\omega^*+\omega$.

Corollary 5.16. $\mathsf{RCA}_0+\mathsf{DGAP}\nvdash\mathsf{SADS}$.

Proof. By Tennenbaum (see Rosenstein [18]), there is a computable linear ordering $\mathcal{L}$ of order type $\omega+\omega^*$ with no computable infinite ascending or descending sequence. Let U and V the the ω and the ω^* part, respectively. In particular, both U and V must be hyperimmune. By a relativization of Theorem 5.12, there is a Turing ideal $\mathcal{M}\models\mathsf{DGAP}$ such that U and V are hyperimmune relative to every element of this model. However, U is not hyperimmune relative to any infinite ascending sequence for $\mathcal{L}$, and the V part is not hyperimmune relative to any infinite descending sequence for $\mathcal{L}$. Therefore, $\mathcal{M}\not\models\mathsf{SADS}$. □

Corollary 5.17. $\mathsf{RCA}_0+\mathsf{DGAP}\nvdash\mathsf{DNC}$.

Proof. By a relativized version of Theorem 5.12, there is a Turing ideal $\mathcal{M}\models\mathsf{DGAP}$ with no DNC function. In particular, $\mathcal{M}\not\models\mathsf{DNC}$. □

Theorem 5.18. *For every Δ^0_2 function f, there is a Δ^0_2 increasing function g such that f does not dominate p_H for any infinite g-transitive set H.*

Proof. Let $g:\omega\to\omega$ be the Δ^0_2 function which on input x, returns $\max\{f(y) : y\le x+1\}$. Let $H=\{x_0<x_1<\dots\}$ be an infinite g-transitive set. We claim that f does not dominate p_H. Since H is g-transitive, there must be some n such that $[x_n,x_{n+1}]$ is g-large, otherwise $[x_i,x_j]$ would be g-small for every $i<j$. In particular, $p_H(n+1)=x_{n+1}\ge g(x_n)\ge f(n+1)$ since $n\le x_n$. This completes the proof. □

Corollary 5.19. *Every ω-model of* DGAP *is a model of* AMT*.*

Proof. Csima et al. [4] and Conidis [3] proved that AMT is computably equivalent to the statement "For every Δ^0_2 function $f:\omega\to\omega$, there is a function not dominated by f." Apply Theorem 5.18. □

Theorem 5.20. *For every Δ^0_2 increasing function g, there is an infinite g-transitive set H of low degree.*

Proof. Suppose there is no computable infinite g-transitive set, otherwise we are done. We construct the set H by the finite extension method. For this, we define a Δ^0_2 sequence $F_0 \subsetneq F_1 \subsetneq \dots$ of finite g-transitive sets such that $F_{n+1} \setminus F_n > F_n$, and then set $H = \bigcup_n F_n$. Start with $F_0 = \{0\}$. At stage e, search $\emptyset'$-computably for one of the following:

(i) Some non-empty finite set $E > F_e$ such that $F_e \cup E$ is g-transitive, and $\Phi_e^{F_e \cup E}(e) \downarrow$
(ii) Some n such that for every non-empty finite set $E > n$, $\Phi_e^{F_e \cup E}(e) \uparrow$.

We claim that one of the two must be found. Indeed, suppose (ii) is not found. Then there is an infinite computable sequence of non-empty finite sets $E_0 < E_1 < \dots$ such that $\Phi_e^{F_e \cup E_i}(e) \downarrow$. If E_i is g-large (meaning that $[\min E_i, \max E_i]$ is g-large) for all but finitely many i, then we can computably dominate the function g. Any function f dominating g computes a g-transitive set by constructing an infinite set consisting only of f-large (hence g-large) intervals. This contradicts the assumption that there is no computable infinite g-transitive set. If E_i is g-small for infinitely many i, then pick some i such that $[\max F_e, \min E_i]$ is g-large. Then $F_e \cup E_i$ is g-transitive, and we are in the case (i).

If we are in case (i), set $F_{e+1} = F_e \cup E$, and if we are in case (ii), pick some $x > n$ such that $[\max F_e, x]$ is g-large, and set $F_{e+1} = F_e \cup \{x\}$. This completes the construction. □

Corollary 5.21. $\mathsf{RCA_0} + \mathsf{DGAP} \nvdash \mathsf{SEM}$.

Proof. Using a relativized version of Theorem 5.20, we can build a Turing ideal $\mathcal{M} \models \mathsf{DGAP}$ with only low sets. By Kreuzer [11], $\mathcal{M} \not\models \mathsf{SEM}$. □

Definition 5.22. An infinite set H is *immune* if it has no computable infinite subset. A infinite set H is *k-immune* if there is no computable sequence of non-empty sets $F_0, F_1, \dots$ such that $F_i > i$, F_i has at most k elements, and $F_i \cap H \neq \emptyset$. An infinite set H is *constant-bound immune* if it is k-immune for every k.

Definition 5.23. A function g is *left-c.e.* if $\{(x, v) : g(x) \geq v\}$ is c.e.

We are actually interested in the case when the function g is a modulus for the halting set. In particular, $\emptyset'$ admits a left-c.e. modulus $\mu_{\emptyset'}$. With the extra assumption that the function g is left-c.e., we can obtain a stronger preservation property, namely, preservation of countably many immune sets. The second theorem can be used to construct models of CAC together with the statement "For every X, there is an infinite transitive set for the modulus of X' " which is not a model of DNC, since CAC admits preservation of one constant-bound immunity which DNC does not (see Patey [15]).

Theorem 5.24. *Let g be an increasing left-c.e. function, $B_0, B_1, \dots$ be a countable sequence of immune sets. Then there is an infinite g-transitive set H such that B_i is H-immune for every i.*

Proof. We construct the set H by the finite extension method. For this, we define a sequence $F_0 \subsetneq F_1 \subsetneq \dots$ of finite g-transitive sets such that $F_{n+1} \setminus F_n > F_n$, and then set $H = \bigcup_n F_n$. At stage $s = \langle e, i \rangle$, we want to satisfy the following requirement:

$$\mathcal{R}_{e,i} : W_e^H \text{ is finite or } W_e^H \not\subseteq B_i$$

Assume F_s is already defined, and fix some threshold t such that $[\max F_s, t]$ is g-large. For every n, computably search for a g-small finite set $E_n \geq \max(t,n)$ such that $W_e^{F_s \cup E_n} \cap (n, \infty) \neq \emptyset$. If not found, E_n is undefined. Note that this search can be made computably since the function g is left-c.e., so being g-small is a c.e. event. If there is some n such that E_n is undefined, then set $F_{s+1} = F_s \cup \{t\}$. We have ensured that $\max W_e^H \leq n$. If E_n is defined for every n, then there must be some n such that $W_e^{F_s \cup E_n} \not\subseteq B_i$, otherwise $\bigcup_n W_e^{F_s \cup E_n}$ would be an infinite c.e. subset of B_i, contradicting immunity of B_i. Then letting $F_{s+1} = F_s \cup E_n$, we ensured that $W_e^H \not\subseteq B_i$. We then go to the next stage. This completes the construction. □

Theorem 5.25. *Let g be an increasing Δ_2^0 function dominating $\mu_{\emptyset'}$, and let $B_0, B_1, \dots$ be a countable sequence of constant-bound immune sets. Then there is an infinite g-transitive set G such that B_i is constant-bound G-immune for every i.*

Proof. We construct the set G by the finite extension method. For this, we define a sequence $F_0 \subsetneq F_1 \subsetneq \dots$ of finite g-transitive sets such that $F_{n+1} \setminus F_n > F_n$, and then set $G = \bigcup_n F_n$. At stage $s = \langle e, k, i \rangle$, we want to satisfy the following requirement:

$$\mathcal{R}_{e,k,i} : \Phi_{e,k}^G \text{ is finite or } (\exists n)\Phi_e e, k^G(n) \cap B_i = \emptyset$$

where $\Phi_{0,k}, \Phi_{1,k}, \dots$ is an effective enumeration of all k-enumeration functionals, that is, whenever $\Phi_{e,k}^G(n) \downarrow$, then $\Phi_{e,k}^G(n)$ is interpreted as a k-set of elements greater than n. Assume F_s is already defined, and fix some threshold t such that $[\max F_s, t]$ is g-large.

Define a partial computable $2k$-enumeration $U_0, U_1, \dots$ as follows. For every n, computably search for two finite sets $E, H \geq \max(t,n)$ such that $\Phi_{e,k}^{F_s \cup E}(n) \downarrow$, $\Phi_{e,k}^{F_s \cup H}(n) \downarrow$, and H is g-transitive according to $\emptyset'_{\max E} \upharpoonright \min E$, where $\emptyset'_t$ is the approximation of $\emptyset'$ at stage t. If not found, U_n is undefined. Otherwise, $U_n = \Phi_{e,k}^{F_s \cup E}(n) \cup \Phi_{e,k}^{F_s \cup H}(n)$. Note that since g is Δ_2^0, $\emptyset'$ can decide whether a set is g-small or not. We have two cases.

Case 1: there is some n such that U_n is undefined. Suppose first there is some g-transitive finite set $E \geq \max(t,n)$ such that $\Phi_{e,k}^{F_s \cup E}(n) \downarrow$. Let u be sufficiently large so that E is g-transitive according to $\emptyset'_u \upharpoonright u$. Letting $F_{s+1} = F_s \cup \{u\}$, we have forced $\Phi_{e,k}^G(n) \uparrow$. Suppose now that whenever $E \geq \max(t,n)$ is g-transitive, then $\Phi_{e,k}^{F_s \cup E}(n) \uparrow$. Then letting $F_{s+1} = F_s \cup \{t\}$, we have forced $\Phi_{e,k}^G(n) \uparrow$.

Case 2: U_n is defined for every n. Then by constant-bound immunity of B_i, there must be some n such that $U_n \cap B_i = \emptyset$. Let E and H witness that U_n is defined. Suppose first that E is g-small. Then letting $F_{s+1} = F_s \cup E$, we have forced $\Phi_{e,k}^G(n) \downarrow \cap B_i = \emptyset$. Suppose now that E is g-large. Since g dominates $\mu_{\emptyset'}$, $\emptyset'_{\max E} \upharpoonright \min E$ agrees with $\emptyset'$ and H is truly g-transitive. Then letting $F_{s+1} = F_s \cup H$, we have forced $\Phi_{e,k}^G(n) \downarrow \cap B_i = \emptyset$. We then go to the next stage. This completes the construction. □

References

[1] Peter A. Cholak, Damir D. Dzhafarov, Jeffry L. Hirst, and Theodore A. Slaman. Generics for computable Mathias forcing. *Annals of Pure and Applied Logic*, 165(9):1418–1428, 2014.

[2] Peter A. Cholak, Mariagnese Giusto, Jeffry L. Hirst, and Carl G. Jockusch Jr. Free sets and reverse mathematics. *Reverse mathematics*, 21:104–119, 2001.

[3] Chris J. Conidis. Classifying model-theoretic properties. *Journal of Symbolic Logic*, 73(03):885–905, 2008.

[4] Barbara F. Csima, Denis R. Hirschfeldt, Julia F. Knight, and Robert I. Soare. Bounding prime models. *Journal of Symbolic Logic*, pages 1117–1142, 2004.

[5] François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer. On uniform relationships between combinatorial problems. *Trans. Amer. Math. Soc.*, 368(2):1321–1359, 2016.

[6] Rod Downey, Noam Greenberg, Matthew Harrison-Trainor, Ludovic Patey, and Dan Turetsky. Relationships between computability-theoretic properties of problems. *arXiv.org*, March 2019.

[7] Harvey M. Friedman. Fom:53:free sets and reverse math and fom:54:recursion theory and dynamics. Available at https://www.cs.nyu.edu/pipermail/fom/.

[8] Marcia J Groszek and Theodore A Slaman. Moduli of computation (talk). *Buenos Aires, Argentina*, 2007.

[9] Carl G. Jockusch. Ramsey's theorem and recursion theory. *Journal of Symbolic Logic*, 37(2):268–280, 1972.

[10] Carl G. Jockusch and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:33–56, 1972.

[11] Alexander P. Kreuzer. Primitive recursion and the chain antichain principle. *Notre Dame Journal of Formal Logic*, 53(2):245–265, 2012.

[12] Antonio Montalbán. Open questions in reverse mathematics. *Bulletin of Symbolic Logic*, 17(03):431–454, 2011.

[13] Ludovic Patey. Combinatorial weaknesses of Ramseyan principles. In preparation. Available at http://ludovicpatey.com/media/research/combinatorial-weaknesses-draft.pdf, 2015.

[14] Ludovic Patey. Iterative forcing and hyperimmunity in reverse mathematics. In Arnold Beckmann, Victor Mitrana, and Mariya Soskova, editors, *CiE. Evolving Computability*, volume 9136 of *Lecture Notes in Computer Science*, pages 291–301. Springer International Publishing, 2015.

[15] Ludovic Patey. Partial orders and immunity in reverse mathematics. 2016.

[16] Ludovic Patey. The weakness of being cohesive, thin or free in reverse mathematics. *Israel J. Math.*, 216(2):905–955, 2016.

[17] Ludovic Patey. Ramsey-like theorems and moduli of computation. *arXiv.org*, January 2019.

[18] Joseph G. Rosenstein. *Linear orderings*, volume 98 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.

[19] David Seetapun and Theodore A. Slaman. On the strength of Ramsey's theorem. *Notre Dame Journal of Formal Logic*, 36(4):570–582, 1995.

[20] Joseph R. Shoenfield. On degrees of unsolvability. *Annals of Mathematics*, 69(03):644–653, May 1959.

[21] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Cambridge University Press, 2009.

[22] Robert M. Solovay. Hyperarithmetically encodable sets. *Trans. Amer. Math. Soc.*, 239:99–122, 1978.

[23] E. Specker. Ramsey's theorem does not hold in recursive set theory. In *Logic Colloquium '69 (Proc. Summer School and Colloq., Manchester, 1969)*, pages 439–442. North-Holland, Amsterdam, 1971.

[24] Richard P. Stanley. *Catalan numbers*. Cambridge University Press, New York, 2015.

[25] Wei Wang. Some logically weak Ramseyan theorems. *Advances in Mathematics*, 261:1–25, 2014.

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana U.S.A.
Email address: Peter.Cholak.1@nd.edu

Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex
Email address: ludovic.patey@computability.fr