Thin sets

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$RT^{n}_{<\infty,l}$-encodable

- Let $c$ be a coloring of all finite sets of size $n$ (all subsets of $\omega$) by finitely many colors, not necessarily computable.
- A set $T$ is $l$-thin iff $c$ uses at most $l$ colors to color all the sets of size $n$ from $T$ and $T$ is infinite. So $|c([T]^n)| \leq l$.
- A set $S$ is $RT^{n}_{<\infty,l}$-encodable iff there is a coloring $c$ (as above) such that every $l$-thin set $T$ computes $S$, i.e. $S \leq_T T$.

**Question**

What sets are $RT^{n}_{<\infty,l}$-encodable? $RT^{2}_{2,1}$? $RT^{1}_{2,1}$? $RT^{3}_{5,4}$? $RT^{4}_{14,13}$?
$RT^n_{<\infty,l}$-encodable sets are always hyperarithmetic.

- Assume $c$ witness that $S$ is $RT^n_{<\infty,l}$-encodable.
- Given $X$ there is an infinite thin set $H$ for $c$ such that $H \subseteq X$.
- A set $S$ is *computably encodable* if for every infinite set $X$, there is an infinite subset $H$ of $X$ such that $H$ computes $S$.
- By theorems of Jockusch and Soare and Solovay, the computably encodable sets are exactly the hyperarithmetic sets.
The $RT^2_{<\infty,1}$-encodable sets includes all hyperarithmetic sets

- The 1-thin sets are exactly the homogenous sets.
- (Solovay) $S$ is hyperarithmetical iff $S$ has a modulus, i.e. a function $g$ such that, for all functions $h$, if $g \leq h$ then $S \leq_T h$.
- The interval $[x, y]$ is $g$-large iff $g(x) < y$.
- $c(x, y) = 1$ iff $[x, y]$ is $g$-large. (An unbalanced coloring.)
- Let $H$ be a homogenous set for $c$. Fix $x \in H$. Then, for almost all $y \in H$, $[x, y]$ is $g$-large. So, for all $y \in H$, $[x, y]$ is $g$-large.
- Hence $g \leq_T p_H$.

For every hyperarithmetical set $S$ there is a coloring (of the same Turing degree as $S$) such that every homogenous set computes $S$. 
The $RT_{<\infty,l}^n$-encodable sets, for $l < 2^{n-1}$

**Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer)**

For $l < 2^{n-1}$, the $RT_{<\infty,l}^n$-encodable sets are exactly the hyperarithmetic sets.

Again code in a modulus into all thin sets of a coloring.

For $2^{n-1} \leq l$, the coding does not work. In particular, for $n = 3$ and $l = 4$, the coding does not work.

**Definition**

A problem $P$ comes in instance-solution $(E_P, S_P)$ pairs.
\( \text{RT}_{2,1}^1 \)-encodable

**Theorem (Dzhafarov and Jockusch)**

*Only the computable sets are RT\(_{2,1}^1\)-encodable.*

Let \( c : \omega \to 2 \). Let \( A = c^{-1}(1) \).

**Theorem (Dzhafarov and Jockusch)**

*Given \( A \) and a noncomputable \( X \). There an infinite \( G \) such that \( X \not\preceq_T G \) and either \( G \subseteq A \) or \( G \subseteq \overline{A} \).*

We will work forward the proof of this theorem over next few slides.

**Definition (Strong Cone Avoidance of \( P \))**

*Given an instance \( E \) of problem \( P \) and a noncomputable set \( X \), there is a \( P \)-solution \( S \) to \( E \) that \( X \not\preceq_T S \).*

**Corollary**

\( \text{RT}_{2,1}^1 \) satisfies strong cone avoidance. \( \text{RT}_{2,1}^2 \) does not.
**Theorem**

Let $T \subseteq 2^{<\omega}$ be infinite tree. Then $T$ has an infinite path.

**Lemma**

A tree is finite iff there is an $l$ such that for all $\sigma \in 2^l$, $\sigma \notin T$. This is $\Sigma^T_1$ or c.e. in $T$. Moreover this is uniform.

Given $X$ the characteristic function of $X$ is a tree with a single path $X$. So WKL does not satisfy strong cone avoidance.
Cone Avoidance

Definition (Cone Avoidance of $P$)
Cone avoidance for a principle $P$ says every set $Z$, every non-$Z$-computable set $X$ and every $Z$-computable instance $E_P$, there is a solution $S_P$ such that $X \not\leq_T Z \oplus S_P$-computable.

Theorem (Cone Avoidance of WKL)
WKL satisfy cone avoidance. I.e. for all infinite trees $T \leq_T I$ and all $X \not\leq_T I$, there is a path $Z$ such that $X \not\leq_T Z \oplus I$. 
Forcing – Infinite trees and generic paths

The forcing conditions are the infinite trees $\tilde{T}$ such that $\tilde{T} \subseteq T$ and $\tilde{T} \leq_T I$. The forcing extension is inclusion. A set of conditions is dense if every condition can be extended into the dense set. A object $G$ (here a tree) is sufficiently generic if it is the limit of conditions meeting enough dense sets.

**Lemma**

*For all $k$, the set of subtrees such that almost all nodes in the subtree extend some finite segment of size $n \geq k$ is dense. So a generic tree is a path though $T$.**
Cone Avoidance

Let $\Phi$ be a Turing functional. Enough to extend a condition $\tilde{T}$ to force $\Phi^{G \oplus I} \neq X$.

- **Non commitment**: For some $n$, the subtree 
  \[ \{ \sigma \in \tilde{T} \mid \Phi^{\sigma \oplus I}(n) \uparrow \} \] is infinite. Then, by finite use principle, $\Phi^{G \oplus I}(n) \uparrow$. This tree is computable in $I$.

- **Commitment**: There is a $n$ and $\sigma \in \tilde{T}$ such that $\Phi^{\sigma \oplus I}(n) \neq X(n)$ and the subtree 
  \[ \{ \tau \in \tilde{T} \mid \tau \preceq \sigma \text{ or } \sigma \preceq \tau \} \] is infinite.

- **Otherwise**. Then, for all $n$, there is an $l$ such that, for all $\sigma \in \tilde{T} \cap 2^l$, $\Phi^{\sigma \oplus I}(n) \downarrow = X(n)$. Therefore $X \leq_T I$. Contradiction.
Theorem
Given A and noncomputable X. There a G such that either
\( X \not\leq_T G \cap A \) and \( G \cap A \) is infinite or \( X \not\leq_T G \cap \overline{A} \) and \( G \cap \overline{A} \) is infinite.

Use conditions \((F, I)\) where \( F \) is finite, \( I \) is infinite, \( \max F < \min I \), and \( X \not\leq_T I \). \((\tilde{F}, \tilde{I})\) extends \((F, I)\) if \( F \subseteq \tilde{F} \subseteq F \cup I \) and \( \tilde{I} \subseteq I \). WLOG \( I \cap A \) and \( I \cap \overline{A} \) are both infinite. With enough genericity, both \( G \cap A \) and \( G \cap \overline{A} \) are infinite.
Strong Cone Avoidance of $RT_{2,1}^1$

Must extend $(F, I)$ to show either $\Phi^{G \cap A} \neq X$ or $\Psi^{G \cap \overline{A}} \neq X$.

**Definition**

Let $P_{n,k}$ be the tree of $Z \subseteq I$ such that there is no $E \subseteq Z$ with $\Phi^{(F \cap A) \cup E}(n) \downarrow = k$ and no $E \subseteq (I - Z)$ with $\Psi^{(F \cap \overline{A}) \cup E}(n) \downarrow = k$.

**Lemma**

These trees are uniformly computable in $(F, I)$. So $P_{n,k} \leq_T I$.

**Commitment:** For some $n$ and $k \neq X(n)$, $P_{n,k}$ is finite. So $I \cap A \notin P_{n,k}$. So a finite extension is enough.
Non commitment

Let $S = \{(n,k) \mid P_{n,k} \text{ is finite}\}$. $S$ is c.e. in $(F,I)$. For all $n$, $(n, 1 - X(n))$ not in $S$. If, for all $n$, $(n, X(n))$ in $S$, then $X$ is computable from $(F,I)$ or just $I$. Since $X \not\leq_T I$, there must be an $(n, X(n))$ not in $S$.

Fix such an $n$. $P_{n,X(n)}$ is infinite. Use cone avoidance of WKL to find a $Z \in P_{n,X(n)}$ such that $X \not\leq_T Z \oplus I$. If $Z$ is infinite extend to $(F,Z)$. Otherwise use $(F,I - Z)$.

This is called thinning the reservoir.
The $RT^n_{<\infty,l}$-encodable sets, for big $l$

**Theorem (Wang)**

For big $l$ (in terms of $n$), the $RT^n_{<\infty,l}$-encodable sets are exactly the computable sets. For $n = 2$, $l = 2$ and, for $n = 3$, $l = 5$.

Use the strong cone avoiding of $RT^n_{<\infty,l}$, for big $l$.

This is an inductive forcing proof and relies on (strong) cone avoiding of earlier and other principles, like $COH$, $WKL$, $RT^1_{<\infty,1}$, $RT^2_{3,1}$, etc. Use Mathias like conditions like we used above.
A recap for $n = 1, 2, 3$

Only the computable sets are $RT^1_{<\infty,1}$-encodable. Same for $RT^2_{<\infty,2}$-encodable and $RT^3_{<\infty,5}$-encodable.

The hyperarithmetic sets are $RT^2_{<\infty,1}$-encodable and same for $RT^3_{<\infty,3}$-encodable.

What about $RT^3_{5,4}$?
Definition (Strong Nonarithmetical Cone Avoidance)
Given an instance $E$ of problem $P$ and a set $X$ not arithmetical, there is a $P$-solution $S$ to $E$ such that $X$ is not $S$-arithmetical.

Theorem
$RT_{5,4}^3$ satisfies strong nonarithmetical cone avoidance. So does $RT_{<\infty,2^n-1}^n$.

Very carefully choose the reservoir.
Corollary

At best only the arithmetic sets are $RT^{3}_{5,4}$-encodable.

Another coding is needed to code the arithmetic sets. By necessity this coding will also provide a counterexample to strong cone avoiding.

Theorem

There is a $\Delta^0_2$ coloring $c : [\omega]^3 \rightarrow 5$ such that every 4-thin set for $c$ computes $0'$. 

Corollary

Only the arithmetic sets are $RT^{3}_{5,4}$-encodable.
Our first attempt

Let $g$ be a modulus of $0'$.

- Recall $[a, b]$ is $g$-large iff $g(a) \leq b$. Otherwise it is $g$-small.
- Let $i(x, y) = 1$ if $[x, y]$ is $g$-large and 0 otherwise.
- Let $c(x, y, z) = \langle i(x, y), i(y, z), i(x, z) \rangle$. This is a 5 coloring, some colors are missing.
- Apply $RT^3_{5,4}$ to $c$ to get a thin set $T$.
- If any color but $\langle 0, 0, 1 \rangle$ is missed, $T$ or a reduction of $T$ has all $g$-large intervals and hence computes $0'$. The principal function dominates the modulus for $0'$.
- Need to learn more about missing the color $\langle 0, 0, 1 \rangle$. 
GAP

A set $H$ is $g$-transitive iff, for all $x < y < z$ in $H$, if $[x, y]$ and $[y, z]$ are $g$-small so is $[x, z]$. GAP is the statement that, for all $g$, an infinite $g$-transitive set exists. So the existence of a 4-thin set (when colored as above) without the color $\langle 0, 0, 1 \rangle$.

Theorem

GAP satisfies strong cone avoidance. Hence the above coloring does not show the arithmetic sets are $RT^3_{5,4}$-encodable. Also GAP follows from $RT^2_{2,1}$.

Perhaps should known this coding would fail since all hyperarithmetic sets have a modulus and there are nonarithmetic hyperarithmetic sets. Needed to use some fact about just the arithmetic sets.
Back to $RT^3_{5,4}$ and coding left c.e. increasing functions

Refine the above coloring $c$. We need to make it harder to avoid the color $\langle 0, 0, 1 \rangle$. So we have to color more triples with color $\langle 0, 0, 1 \rangle$ and less with color $\langle 0, 0, 0 \rangle$.

The modulus $g$ of $0'$ is a left c.e. increasing function with approximations $g_0, g_1, \ldots$ (the approximations are increasing).

- Define $j(x, y, z)$ is 1 iff $[x, z]$ is $g$-large or $[x, y]$ is $g_z$-large.
- Let $c(x, y, z) = \langle i(x, y), i(y, z), j(x, y, z) \rangle$. A 5 coloring.
- Apply $RT^3_{5,4}$ to $c$ to get a thin set $T$.
- For all possible missed colors, a reduction of $T$ has all $g$-large intervals and hence computes $0'$.

**Theorem**

$c$ is $\Delta^0_2$ and every 4-thin set for $c$ computes $0'$.
The Catalan and Schröder numbers

The \( n \)th Catalan number is the number of paths from \((0,0)\) to \((n,n)\) that take steps \((0,1)\) and \((1,0)\), and don’t go above main diagonal; the \( n \)th Schröder number is the same, except the paths are also allowed to take \((1,1)\) steps.
Bounds for $n > 3$, recap

**Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer)**

For $l < 2^{n-1}$, the $\text{RT}^n_{<\infty, l}$-encodable sets are exactly the hyperarithmetic sets.

**Theorem**

$\text{RT}^n_{<\infty, 2^{n-1}}$ satisfies strong nonarithmetical cone avoidance.

**Corollary**

At best only the arithmetic sets are $\text{RT}^n_{<\infty, 2^{n-1}}$-encodable.

**Theorem (Wang)**

For $l$ the $n$th Schröder number, $\text{RT}^n_{<\infty, l}$ satisfies strong cone avoidance.
Bounds for $n > 3$

Let $d_0, d_1, \ldots$ be the sequence of Catalan numbers. In particular, $d_0 = 1, d_1 = 1, d_2 = 2, d_3 = 5, d_4 = 14, d_5 = 42, d_6 = 132, d_7 = 429$.

**Theorem**

$RT_n^{<\infty,d_n}$ satisfies strong cone avoidance.

**Corollary**

Only the computable sets are $RT_n^{<\infty,d_n}$-encodable.

**Theorem**

For all $n$, there is a $\Delta^0_2$ coloring of $[\omega]^n$ such that every $d_n - 1$-thin set computes $0'$.

**Corollary**

The arithmetic sets are $RT_n^{<\infty,d_n-1}$-encodable.
Computable Coding via a Modulus

Question
How necessary is it to code via the use of a modulus?

Theorem
Fix a function $g$. Let $f$ be a computable instance of $RT^{n}_{k+1,k}$ such that every infinite $k$-thin set computes a function dominating $g$. Then for every infinite $f$-thin set $H$, $p_H$ is a modulus for $g$.

Our coding examples are not computable but arithmetic. Using the Limit Lemma they can be reflected into computable colorings with the same properties.
Arbitrary Coding via a Modulus

Theorem (Liu and Patey)

Fix a function $g$. Let $f$ be an instance of $RT^n_{k+1,k}$ such that every infinite $k$-thin set computes a function dominating $g$ via some fixed Turing functional. Then for every infinite $f$-thin set $H$, $p_H$ is a uniform modulus for $g$.

For $l = 1$, our coding examples for $RT^n_{<\infty,l}$ are uniform. But for larger $l$ they are uniform in the missed color. Can the hypothesis of the above theorem be weaken to reflect this uniformity?

Lemma (Liu and Patey)

There is an instance for $RT^2_{2,1}$ all of those homogenous sets compute $0'$ but by not computing a function dominating a modulus for $0'$. 
Definition
A function \( f : \mathbb{N} \to \mathbb{N} \) is Z-hyperimmune if it is not dominated by any function computable in Z. An infinite set \( A = \{x_0 < x_1 < \ldots\} \) is Z-hyperimmune if its principal function, \( p_A(n) = x_n \), is Z-hyperimmune.

Definition
A problem \( P \) admits the preservation of \( p \) hyperimmunities if for every set \( Z \) and every collection \( \{f_s : s < p\} \) of Z-hyperimmune functions, every instance \( E_P \leq_T Z \) has a solution \( S_P \) such that, for every \( s \leq m, f_s \) is \( Z \oplus S_P \)-hyperimmune.

So \( S_P \) does not contain the information needed to dominate any of the \( f_s \)'s.
What does preservation of hyperimmunities get us?

Theorem (Patey)
\( RT^2_{<\infty,k} \) admits preservation of \( k \), but not \( k + 1 \), hyperimmunities.
Hence \( RT^2_{<\infty,k+1} \) does not “follow” from \( RT^2_{<\infty,k} \).

Definition
The thin set theorem for \( n \) and \( \omega \), \( TS^n \), says for all \( \omega \)-colorings, 
\( c : [\mathbb{N}]^n \to \omega \), there is an infinite set \( T \) such that \( |c([T])^n| \neq \omega \).

Theorem (Patey)
\( TS^n \) preserves \( k \)-hyperimmunities, for every \( k \in \omega \), but not \( \omega \)-hyperimmunities.
Hyperimmune-free degrees

Definition
A Turing degree $d$ is hyperimmune-free iff, for all $f \leq_T d$, $f$ is not hyperimmune.

Lemma
Every instance of a problem $P$ has a solution of hyperimmune-free degree iff $P$ preserves all continuum many hyperimmune functions.

Corollary (Jockusch and Soare)
$WKL$ preserves all continuum many hyperimmune functions.

Patey also showed that problems where all instances have generic or random solutions admit the preservation of countable many hyperimmunities.
Cone Avoidance and preservation of 1-hyperimmunity

Theorem (Downey, Greenberg, Harrison-Trainor, Patey, and Turestsky)

A problem admits the preservation of 1-hyperimmunity iff the problem satisfies cone avoidance.

Corollary

$RT^n_{<\infty, C_n}$ admits the preservation of 1-hyperimmunity. Moreover this bound is tight.
Preservation of $p$-hyperimmunities

**Theorem**

$RT^n_{<\infty, p^n C_n}$ admits the preservation of $p$-hyperimmunities. Moreover this bound is tight.

So $RT^n_{<\infty, (p+1)^n C_n}$ does not follow from $RT^n_{<\infty, p^n C_n}$. 
Nonhyperarithmetic hyperimmune functions

Definition
A problem $P$ admits the preservation of $p$ nonhyperarithmetic hyperimmunities if for every set $Z$ and every collection \{f_s : s < p\} of $Z$-nonhyperarithmetic $Z$-hyperimmune functions, every instance $E_P \leq_T Z$ has a solution $S_P$ such that, for every $s \leq m, f_s$ is $Z \oplus S_P$-hyperimmune (and also likely $Z \oplus S_P$-nonhyperarithmetic).

Theorem
$RT^n_{<\infty,2^n}$ preserves one nonhyperarithmetic hyperimmunity. Moreover this bound is tight.
Questions

Question
For which $\ell$ does $\text{RT}_n^{<\infty,\ell}$ preserve $p$ hyperimmunities and $q$ nonhyperarithmetic hyperimmunities?

Question
Is there an $n$ and $\ell$ such that $\text{RT}_n^{<\infty,\ell+1}$ follows from $\text{RT}_n^{<\infty,\ell}$.