Quasi-isometries, harmonic functions and boundaries of DL-graphs, SOL, and treebolic spaces

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Question by W. at Ljublana-Leoben graph theory seminar (mid-early 1980ies):

Is there a locally finite, vertex-transitive graph that does not look vaguely like a Cayley graph of a group ?

Made more precise after Gromov (1988) had introduced notion of quasi-isometry.

Is there a locally finite, vertex-transitive graph that is not quasi-isometric a Cayley graph of some finitely generated group ?

Question stated in two papers Soardi and Woess [Math. Zeitschrift, 1990], Woess [Discrete Math., 1991].

Recent answer by Eskin, Fisher and Whyte [Annals of Math., 2012].

Outline

- Review involved concepts.
- Describe construction of counterexample by DIESTEL AND LEADER.
- Review results by ESKIN, FISHER AND WHYTE
- Explain related structures related sturctures:
  treebolic space and SOL.
- Outline results on random walks and harmonic functions.
- Describe extended constructions and related results.

- All graphs in this talk are connected, locally finite and infinite, carry integer-valued graph metric.
- If G is a finitely generated group and S a finite, symmetric set of generators, then the Cayley graph $X(G,S)$ has vertex set G, and
$$x \sim y \iff y = xs\,,\ s \in S\,.$$
- A quasi-isometry (rough isometry) between metric spaces (X_1, d_1) and (X_2, d_2) is $\varphi : X_1 \to X_2$ with
$$A^{-1} d_1(x_1, y_1) - B \le d_2(\varphi x_1, \varphi y_1) \le A\, d_1(x_1, y_1) + B \quad \forall\, x_1, y_1 \in X_1$$
$$d(y_2, \varphi X_1) \le B \quad \forall\, y_2 \in X_2\,.$$
- Bi-Lipschitz, if $B = 0$.

Examples

- Any two Cayley graphs of the same f.g. group are bi-Lipschitz.
- If G_1 and G_2 have a common subgroup with finite index in each of the two then they are quasi-isometric.
- The integer lattices $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ are not quasi-isometric when $d_1 \neq d_2$.
- Let T be a tree with $2 \leq deg(\cdot) \leq M$ and a finite upper bound on the lengths of all unbranched paths, then T is quasi-isometric with the regular tree with degree 3.
- A group is quasi-isometric with a tree if and only if it is virtually free Gromov (1988), Woess (1986/89).

Transitive graphs

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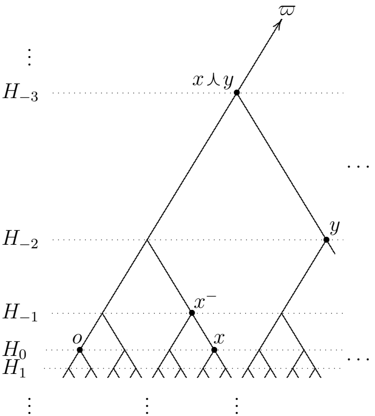
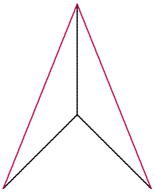
- The Automorphism group of a graph is the group of neighbourhood perserving bijections of the vertex set.
- A graph is called (vertex) transitive if its automorphism group acts transitively.
- Cayley graphs of groups are transitive.
- Example of an (intrinsically infinte) transitive non-Cayley graph: grandmother graph:

  Start with upper half plane drawing of homogeneous tree $\mathbb{T}_p$ with degree $p + 1$.

Grandmother graph



$\mathfrak{h}(x) = k :\Leftrightarrow x \in H_k$ level function

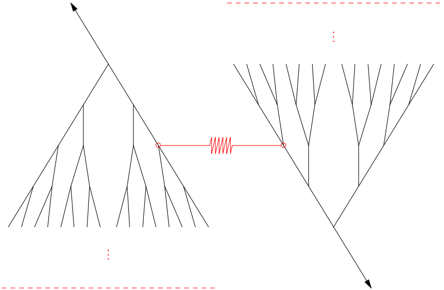


$\mathrm{Aff}(\mathbb{T}) = \{g \in \mathrm{Aut}(\mathbb{T}) : g(x^-) = (gx)^- \text{ for all } x\}$ affine group of $\mathbb{T}$.

$= \mathrm{Aut}(\text{grandma graph})$ $\Rightarrow$ grandma graph is not a Cayley graph.

But it is quasi-isometric to a Cayley graph!

- In the mid-early 1990ies, DIESTEL AND LEADER proposed a construction of transitive graphs which they conjectured to be non-q.i. to any Cayley graph. Conjecture published in 2001.



- $$\mathrm{DL}(p,q) = \left\{ x_1 x_2 \in \mathbb{T}_p \times \mathbb{T}_q : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0 \right\}$$

  Neighbourhood: $x_1 x_2 \sim y_1 y_2 :\Leftrightarrow x_i \sim y_i \quad (i = 1, 2)$

- DIESTEL AND LEADER considered $\mathrm{DL}(2,3)$
- MÖLLER AND P. NEUMANN (2001, private communication by M.) observed (for $q = 2$) that $\mathrm{DL}(q,q)$ is a Cayley graph of the lamplighter group $\mathbb{Z}(q) \wr \mathbb{Z}$.
- Solution of quasi-isometry question:

Theorem. [ESKIN, FISHER AND WHYTE, 2012]

If $q \neq p$ then $\mathrm{DL}(p,q)$ is not quasi-isometric with any finitely generated group.

- A quasi-isometry DL $\to$ DL is called height respecting if it permutes the “horizontal” level sets $H_k \times H_{-k}$ up to bounded distance.

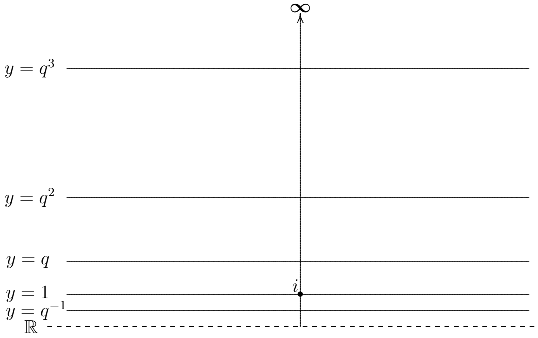
Theorem. [Eskin, Fisher and Whyte, 2012]

If $q \neq p$ then every (A, B)-quasi isometry is at $C(A, B)$-bounded distance from a height respecting one.

Implies that horizontal levels (and their distances) are distorted only up to uniform bounds.

Treebolic space

- $\mathbb{T}_p$ metric graph, edges ≡ intervals of length 1
- Extend level function $\mathfrak{h}$ linearly to interior points of edges (now real-valued)
- $\mathbb{H}_q$ ($q > 1$ real) sliced hyperbolic plane:

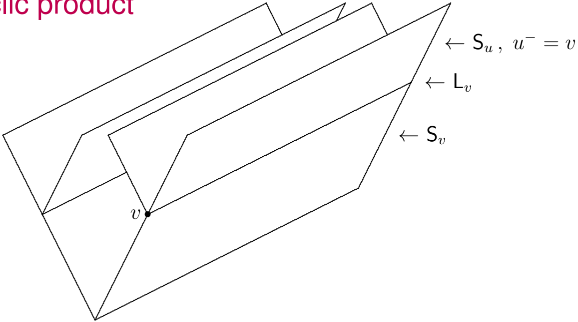


height function of $z = x + \mathfrak{i}\,y$:

$\mathfrak{h}(z) = \log_q y$

$$\mathrm{HT}(p,q) = \left\{ \mathfrak{z} = (w,z) \in \mathbb{T}_p \times \mathbb{H}_q : \mathfrak{h}(w) = \mathfrak{h}(z) \right\}$$

"Horocyclic product"



- If $q = p$, the Baumslag-Solitar group

$$\mathrm{BS}(p) = \left\{ \begin{pmatrix} p^m & k/p^l \\ 0 & 1 \end{pmatrix} : k,l,m \in \mathbb{Z} \right\} = \langle a, b \mid a\,b = b^p\,a \rangle$$

acts on $\mathrm{HT}(p,p)$ by isometries & with compact quotient.

- Quasi-isometry classification of BS(p) $(2 \le p \in \mathbb{Z})$ by [Farb and Mosher, 1998 + 1999] uses action on HT(p,p).

- If $p \neq q$ then there is no finitely generated group of isometries that acts with compact quotient on HT(p,q).

  Conjecture.

  In that case, HT(p,q) is not quasi-isometric to any finitely generated group. [Almost sure.]

- Hyperbolic upper half plane $\mathbb{H} = \{x + i\,w : x, w \in \mathbb{R},\ w > 0\}$
  $\longrightarrow$ logarithmic model $z = \log w$, coordinates $(x, z) \in \mathbb{R}^2$.

- Change curvature to $-p^2$ $\longrightarrow \mathbb{H}(p)$ is $\mathbb{R}^2$,

  length element $ds^2 = d_p s^2 = e^{-2pz}\,dx^2 + dz^2$.

- Height function of $\mathbf{x} = (x, z)$ is $\mathfrak{h}(\mathbf{x}) = z$

$$\begin{aligned}\mathrm{Sol}(p,q) &= \big\{\mathbf{x}_1\mathbf{x}_2 \in \mathbb{H}(p) \times \mathbb{H}(q) : \mathfrak{h}(\mathbf{x}_1) + \mathfrak{h}(\mathbf{x}_2) = 0\big\} \\ &= \big\{(x,y,z) \in \mathbb{R}^3 : (x,z) \in \mathbb{H}(p),\ (y,-z) \in \mathbb{H}(q)\big\}\end{aligned}$$

with length element

$$ds^2 = d_{p,q}s^2 = e^{-2pz}\,dx^2 + e^{2qz}\,dy^2 + dz^2 .$$

$$\mathcal{S} = \mathcal{S}(p,q) = \left\{ \mathfrak{g} = \begin{pmatrix} e^{pc} & a & 0 \\ 0 & 1 & 0 \\ 0 & b & e^{-qc} \end{pmatrix}, \quad a,b,c \in \mathbb{R} \right\}$$

- Lie group $\equiv \mathrm{Sol}(p,q)$, $\qquad \mathfrak{g} \longleftrightarrow (a,b,c)$.
- Isometric, fixed-point-free action on Sol(p,q) ($\equiv$ group product):

$$(a,b,c) \cdot (x,y,z) = \big(e^{pc}x + a, e^{-qc}y + b, c + z\big).$$

- The group $\mathcal{S}(p,p) = \mathcal{S}(1,1)$ contains many co-compact lattices (discrete subgroups acting with compact quotient).

Quasi-isometry questions: solution by analogous methods as for $\mathrm{DL}(p,q)$.

Theorem. [Eskin, Fisher and Whyte, 2012]

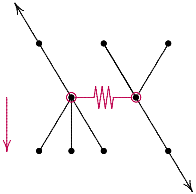
If $q \neq p$ then $\mathrm{Sol}(p,q)$ is not quasi-isometric with any (Cayley graph of a) finitely generated group.

Random processes, harmonic functions

Class of random processes adapted to the geometry, with vertical drift parameters.

- On DL(p,q) : random walk with transition matrix P_α $(0 < \alpha < 1)$

probability $\frac{\alpha}{p}$ on each edge



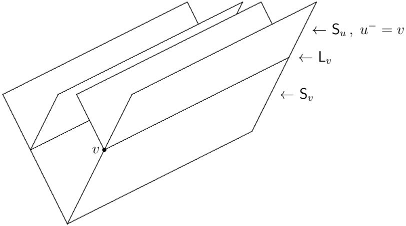
probability $\frac{1-\alpha}{q}$ on each edge

- On Sol(p,q) : Brownian motion induced by Laplacian

$$\mathfrak{L}_a = \frac{1}{2}\Big(e^{2pz}\,\partial_x^2 + e^{-2qz}\,\partial_y^2 + \partial_z^2\Big) + a\,\partial_z\,.$$

Random processes, harmonic functions

On $\mathrm{HT}(p,q)$: Brownian motion ≡ process induced by Laplacian $\mathfrak{L}_{\alpha,\beta}$ ($\alpha \in \mathbb{R}, \beta > 0$) that takes care of singularities at bifurcation lines.



For $\mathfrak{z} = (x + i\,y, w) \in \mathrm{HT}^o$

$$\mathfrak{L}_{\alpha,\beta} f(\mathfrak{z}) = y^2(\partial_x^2 + \partial_y^2) f(\mathfrak{z}) + \alpha\, y\, \partial_y f(\mathfrak{z})$$

acting on suitable function space.

"Nice" functions in its domain must be

- continuous on HT
- twice continuously differentiable on each S_v

(up to the boundary lines, not nec. continuous from both sides at L_v)

- satisfy on each L_v the Kirchhoff condition

$$\partial_y f_v(\mathfrak{z}-) = \beta \cdot \sum_{u:u^-=v} \partial_y f_u(\mathfrak{z}+)\,, \qquad f_v = f\big|_{\mathsf{S}_v}$$

- In all three cases, the respective operator has natural projections onto the first as well as on the second of the two spaces that make up the horocylic product.
- Also, there is the "vertical" projection onto the line.

Theorem. [Woess, 2005]; [Brofferio, Salvatori and Woess, 2011]; [Bendikov, Saloff-Coste, Salvatori and Woess, 2014]

Every positive harmonic function for the respective operator has the form

$$h(x_1 x_2) = h_1(x_1) + h_2(x_2),$$

where for $j = 1, 2$, h_j is a non-negative harmonic function for the projected operator on the 1st, resp. 2nd one of the two spaces that make up the horocyclic product.

Methods

- Green kernel $G(x,z) = \sum_{n=0}^{\infty} p^{(n)}(x,z)$, resp. $= \int_{n=0}^{\infty} p_t(x,z)\,dt$
  - is invariant under (transitive) group of “vertical” isometries,
  - satisfies uniform local Harnack inequality

$$G(x,z)\mathrm{m}(z) \le C_d\, G(x,z')\,\mathrm{m}(z')$$

  whenever $d(z,z') \le d$ and $d(z,x)$, $d(z',x) \ge 10(d+1)$.

- h positive harmonic function is called minimal, if

  (1) $h(o) = 1$ (o reference point on level 0)

  (2) Whenever $h \ge \bar{h} \ge 0$ with $\bar{h}$ harmonic, then $\bar{h}/h = \mathrm{const}$.

- Every minimal harmonic function is a limit of Martin kernels :

$$h = \lim_{n\to\infty} K(\cdot, z_n), \quad \text{where} \quad d(o,z_n) \to \infty \quad \text{and} \quad K(x,z) = \frac{G(x,z)}{G(o,z)}$$

Methods

- If $z_n = z_{n,1} z_{n,2}$, first suppose $\inf_n \mathfrak{h}(z_{n,1}) = c > -\infty$.
  Let τ_1 be a level-isometry of the first factor ($\mathbb{T}$ or $\mathbb{H}$). Then $\tau(z_1 z_2) = \tau_1(z_1) z_2$ is an isometry of the horocyclic product.

- Geometry $\Rightarrow$ $d(\tau z_n, z_n) = d_1(\tau_1 z_{n,1}\, , z_{n,1}) \le d = d_c$, whence

$$K(\tau x, z_n) = \frac{G(\tau x, z_n)}{G(\tau x, \tau z_n)} \frac{G(\tau x, \tau z_n)}{G(o, z_n)} \le C_d\, K(x, z_n)$$

- $\Rightarrow$ $h(\tau x) \le C_d\, h(x)$ $\Rightarrow$ $h(\tau x)/h(x) = \text{const.}$

- Additional use of Harnack inequality
  $\Rightarrow$ $h(\tau x) = h(x)\,,$ $h(x_1 x_2)$ depends only on x_2 .

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- Second, if $\sup_n \mathfrak{h}(z_{n,1}) < \infty$, then analogously $h(x_1 x_2)$ depends only on x_1 .
- Every harmonic function $h \geq 0$ is an integral over minimal ones :

$$h = \int_{\mathcal{M}_{\min}} K(\cdot, \xi)\, d\nu^h(\xi)$$

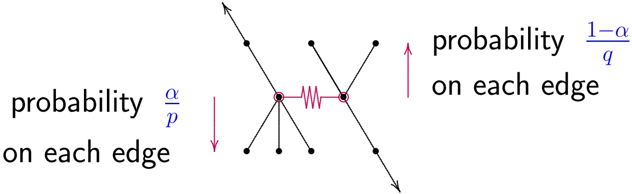
- Martin boundary $\mathcal{M}$: boundary in compactification where
  - each $K(x, \cdot)$ extends continuously;
  - extended functions separate boundary points.

$$\mathcal{M}_{\min} = \{\xi \in \mathcal{M} : K(\cdot, \xi) \quad \text{minimal}\}.$$

- $\mathcal{M}_{\min} \setminus \mathcal{M}_{\min,1} \subset \mathcal{M}_{\min,2} \quad \Rightarrow$

$$h = \int_{\mathcal{M}_{\min,1}} K(\cdot, \xi)\, d\nu^h(\xi) + \int_{\mathcal{M}_{\min} \setminus \mathcal{M}_{\min,1}} K(\cdot, \xi)\, d\nu^h(\xi)$$

- For P_α on $\mathsf{DL}(p,q)$ $(0 < \alpha < 1)$:



- All minimal harmonic functions (on the two trees) are known explicitly [Woess, 2005].
- The Martin compactification is fully described [Brofferio and Woess, 2005].

  It depends on the vertical drift $a = 2\alpha - 1$.

- For $\mathfrak{L}_a = \frac{1}{2}\Big(e^{2pz}\,\partial_x^2 + e^{-2qz}\,\partial_y^2 + \partial_z^2\Big) + a\,\partial_z$ on $\mathrm{Sol}(p,q)$ with vertical drift paramter a:

- All minimal harmonic functions (on the two hyperbolic planes) are known explicitly [Brofferio, Salvatori and Woess, 2011].

  They are modified Poisson kernels.

  Compare also with [Raugi, 1996] (for random walks) and [Lyons and Sullivan, 1984] (for BM) on $\mathrm{Sol}(1,1)$.

- We do not (yet) have the full Martin compactification ≡ directions of convergence of Martin kernels.

More on minimal harmonic fctns & Martin bdry

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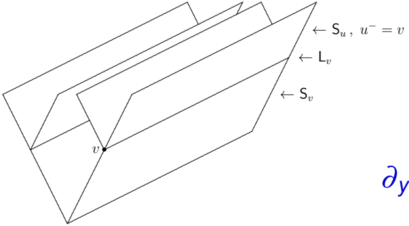
- For $\mathfrak{L}_{\alpha,\beta}$ on $\mathsf{HT}(p,q)$ ($\alpha \in \mathbb{R}, \beta > 0$):



For $\mathfrak{z} = (x + i\,y, w) \in \mathsf{HT}^o$

$$\mathfrak{L}_{\alpha,\beta} f(\mathfrak{z}) = y^2(\partial_x^2 + \partial_y^2) f(\mathfrak{z}) + \alpha\, y\, \partial_y f(\mathfrak{z})$$

$$\partial_y f_v(\mathfrak{z}-) = \beta \cdot \textstyle\sum_{u:u^-=v} \partial_y f_u(\mathfrak{z}+)\,, \quad f_v = f\big|_{\mathsf{S}_v}$$

- All minimal harmonic functions coming from $\mathbb{T}$ are known explicitly,
  but those coming from “sliced” $\mathbb{H}$ are known explicitly (modified Poisson kernels) only when $\beta\, p = 1$. [Bendikov, Saloff-Coste, Salvatori and Woess, 2014/15].
- We do not (yet) have the full Martin compactification.

Bounded harmonic functions

Theorem. [Woess, 2005]; [Brofferio, Salvatori and Woess, 2011]; [Bendikov, Saloff-Coste, Salvatori and Woess, 2014]

In all three cases, the weak Liouville property holds (all bounded harmonic functions are constant; the Poisson boundary is trivial) if and only if $a = 0$ (vertical drift).

Martin compactification

- Geometric compactifictation
  - $\widehat{\mathbb{T}}$ end compactification
  - $\widehat{\mathbb{H}}$ hyperbolic compactification (≡ closed unit disk)
- Horocyclic compactifictation
  - $\mathbb{T}$ has special end ϖ. $\mathbb{H}$ has special bdry point $\infty =: \varpi$.

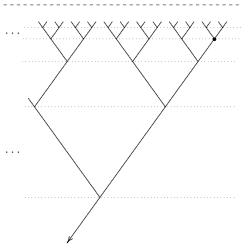
  Both cases: replace ϖ by

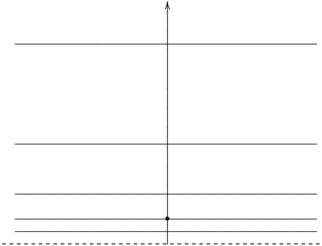
  ⋆ ϖ_k , $k \in \mathbb{Z} \cup \{\pm\infty\}$ (tree, discrete case), resp.

  ⋆ ϖ_t , $t \in [-\infty, \infty]$ (metric tree, resp. hyp. plane)

  - topology refines geometric compactification; $z_n \to \varpi_t$ if

  $z_n \to \varpi$ in geometric compactification, and $\mathfrak{h}(z_n) \to t$.





Geometric / horocyclic compactification of horocyclic product: closure in the direct product of the geometric / horocyclic compactifications of the two factor spaces.

Martin compactification

Conjecture. In all three cases: dependence on vertical drift

If $a = 0$ then Martin compactification = geometric compactification.

If $a \neq 0$ then Martin compactification = horocyclic compactification.

For P_α on $\mathrm{DL}(p, q)$, conjecture is a
Theorem of [Brofferio and Woess, 2005]

More work & future work

- Horocyclic product of more than 2 trees :

  $$\mathsf{DL}(p_1,\ldots,p_r) = \left\{x_1 \cdots x_r \in \mathbb{T}_{p_1} \times \cdots \times \mathbb{T}_{p_r} : \mathfrak{h}(x_1) + \cdots + \mathfrak{h}(x_r) = 0\right\}$$

  with suitable neighbourhood relation.

- Comprehensive study by [Bartholdi, Neuhauser and Woess, 2008].

  Comprises prominent group whose Cayley graph is $\mathsf{DL}(p,p,p)$, see recent work of [Amchislavska and Riley, 2014/15].

- Open question : is $\mathsf{DL}(2,2,2,2)$ a Cayley graph ?

More work & future work

- Levelled trees: $\mathbb{T}_{p,r}$ – each vertex has p incoming and r outgoing edges.
- Levelled product with “sliced” $\mathbb{H}_q$ – if $p < r$ then non-amenable Baumslag-Solitar group $\mathrm{BS}(p,r) = \langle a, b \mid a\, b^p = b^r\, a \rangle$ acts on resulting treebolic space with $q = r/p$.
- [Cuno and Sava, 2015] : Poisson boundary of random walks on $\mathrm{BS}(p,r)$.