Lecture Notes on Complexity and Periodic Orbits

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1 Introduction

These are supplemental lecture notes for the Undergraduate Summer School on Boundaries and Dynamics at Notre Dame University. They are incomplete. As of now, there is insufficient background for some of the topics, a complete absence of citations, some undefined objects, and several of the proofs could use greater details. Furthermore, these notes have not been vetted, so is a hope that students will help with improvements, both with finding mistakes and typos, but also through stylistic criticism.

That said, the goal of these notes is to expose some of the intriguing properties of a class of complicated dynamical systems. This is done by examining one famous example: the "2-1-1-1"-hyperbolic toral automorphism. This example is a prototype of hyperbolic behavior and has insights into dynamical systems that are both smooth and complicated. The two properties exposed here are stable sets and the growth rate of periodic orbits. Although, it is incomplete in these notes themselves, part of the course will be to hint at the relationship of these two.

2 Definitions and First Examples

2.1 Definition of a Dynamical System

One should consider this a basic definition that provides good intuition for understanding dynamical systems from both a historic and philosophical point of view. There exists greater notions of dynamical systems but we restrict to actions of $\mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$ with $+$ denotes the standard addition.

Definition 2.1. *A* dynamical system*, $(X, \mathcal{T}, \phi)$, is a set, X, an additive group or semi-group, $\mathcal{T} = \mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$, and a function, $\phi : X \times \mathbf{T} \to X$, which satisfies the properties*

$$\phi(x, t+s) = \phi(\phi(x, s), t) \tag{1}$$

for any $x \in X$ and $s, t \in \mathcal{T}$. In general, we assume that $\phi(\cdot, 0)$ is the identity map, that is that $\phi(x, 0) = x$ for all $x \in X$.

For a dynamical system, $(X, \mathcal{T}, \phi)$, we call the set X the ***phase space***, $\mathcal{T}$ the ***time*** or the ***notion of time***, and ϕ the ***rule***. Notice that when $t \in \mathcal{T}$ is fixed, we obtain a function $\phi(\cdot, t) : X \to X$.

Exercise 2.2. *For a dynamical system, $(X, \mathcal{T}, \phi)$, prove that if $\mathcal{T} = \mathbb{R}$ or $\mathbb{Z}$, then the assumption that $\phi(\cdot, 0)$ is the identity is unnecessary and that $\phi(\cdot, -t)$ is the inverse function of $\phi(\cdot, t)$ for any $t \in \mathcal{T}$.*

Based on $\mathcal{T}$, there are some some natural divisions within dynamical systems. One is discrete vs. continuous. A ***discrete dynamical system*** is one in which $\mathcal{T} = \mathbb{N}$ or $\mathbb{Z}$ (if you are familiar with topology, this is due to the topology of $\mathbb{Z}$ being discrete). The system can be given via a single function $f : X \to X$. Here, we have $f(x) = \phi(x, 1)$ and we notice then that $f^n(x) = \phi(x, n)$. In these cases, we occasionally refer to the dynamical system (X, f), and the reader is to assume that $\mathcal{T} = \mathbb{Z}$ if f is invertible and $\mathcal{T} = \mathbb{N}$ if f is not. A ***continuous dynamical system*** is one where $\mathcal{T} = \mathbb{R}$. We typically refer to such a system as a ***flow***.

A second natural division is invertible vs. non-invertible. Here the split is not on the topology of time, but whether time is a group or just a semi-group. When $\mathcal{T} = \mathbb{Z}$ or $\mathbb{R}$, the dynamical system is ***invertible*** and when $\mathcal{T} = \mathbb{N}$ it is ***non-invertible***. Because of their importance to this chapter we stress the next definitions.

Definition 2.3. *For a dynamical system $(X, \mathcal{T}, \phi)$, and any $x \in X$, the* orbit *of x is*

$$\mathcal{O}(x) := \{\phi(x,t) : t \in \mathcal{T}\} . \tag{2}$$

Furthermore, an $x \in X$ is called a ***a periodic point*** *if there is a $t \in \mathcal{T}$ with $t > 0$ and $\phi(x,t) = x$. In this case, the t is called the* ***period*** *of x and the $\mathcal{O}(x)$ is called a* ***periodic orbit***. *If $\mathcal{O}(x) = \{x\}$, we say that x is a* ***fixed point***.

Notice that a fixed point is periodic of any period. We recall a couple definitions. For a function $f : X \to X$, the ***graph*** of f is $\Gamma(f) := \{(x, f(x)) : x \in X\} \subset X \times X$. For a set X, the ***diagonal*** in $X \times X$ is the subset $\Delta(X) := \{(x,x) : x \in X\}$.

Exercise 2.4. *For a dynamical system, $(X, \mathcal{T}, \phi)$, prove there is a natural one-to-one correspondence between periodic points of period $t > 0$ and points of intersection of $\Gamma(\phi^t)$ and $\Delta(X)$ (see Figure 1).*

Exercise 2.5. *Consider the dynamical system $([0,1], f)$ where $f(x) = x^2$ for $x \in [0,1]$. What is the natural notion of time? What is $\mathcal{O}(x)$? Show that 0 and 1 are fixed points, but there are no other periodic points. Finally, if $x \neq 1$, show that $\lim_{n\to\infty} f^n(x) = 0$.*

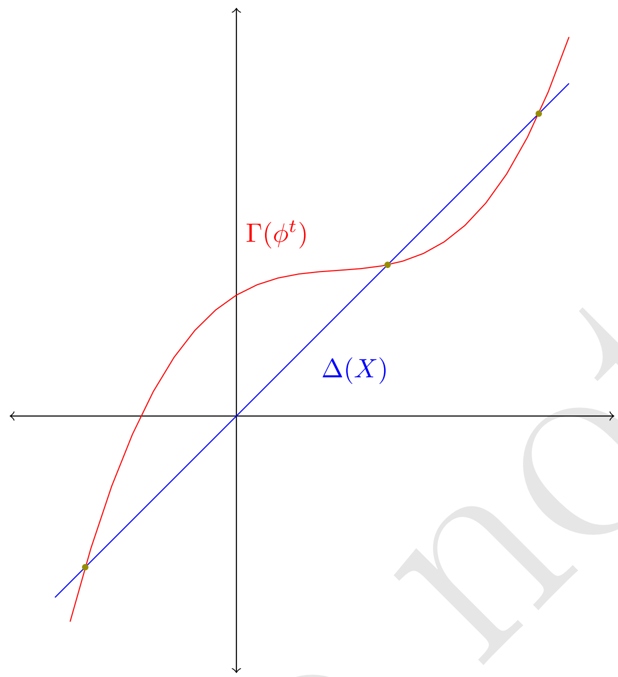

Figure 1: The intersections of $\Gamma(\phi^t)$ and $\Delta(X)$ correspond to periodic points.

**** Exercise 2.6.** *Consider the dynamical system* $([0,1], f)$ *where* f *is a continuous function with* $f(0) = f(1) = 0$ *and* $f(0.5) = 1$*. Prove there are an infinite number of periodic points.*

Mathematically, the phase space, X, is often assumed to have some mathematical structure. In these settings, the rule is usually assumed to preserve some of this structure. Standard examples, with the terminology used to describe them, are:

- X is a metric or topological space this, called *topological dynamics*,
- X is a differentiable manifold, called *smooth dynamics*, and
- X is measure space, called *ergodic theory*.

One should not in anyway interpret these as being exclusive. For example, a differentiable manifold automatically has topological structure, which in turn has measurable structure.

Side Discussion 2.7. *When trying to use dynamical tools to make predictions in real world settings, establishing the proper phase space is the first step in being able to make interpretations. Then one would choose a notion of time that is appropriate. From these two, usually physical laws or general theory in the field are used to establish an appropriate rule. This is an interesting topic, but, unfortunately, largely outside the scope of this chapter. In this chapter, we simply give examples that are mathematically defined.*

2.2 Circle Rotations

The phase space for a circle rotation is called a "topological circle" or a 1-torus, which we define as $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. This can also be written, $\mathbb{T}^1 = \mathbb{R}/\sim$, where $x \sim y$ if $x - y \in \mathbb{Z}$. Notice this is (at least topologically) the same as $[0,1]/\sim$, where $x \sim x$ and $0 \sim 1$. Some would call this the interval $[0,1]$ with 0 and 1 "glued" together. For $x \in \mathbb{R}$, we use the notation $[x] \in \mathbb{T}^1$ to denote the equivalence class of $[x]$. We state a few useful properties of the one-torus that rely on more machinery than we have available to prove. This relies on more than simply the theory of equivalence classes.

Unproven Fact 2.8. *The map $d_{\mathbb{T}^1} : \mathbb{T}^1 \times \mathbb{T}^1$ defined by*

$$d_{\mathbb{T}^1}([x],[y]) = \inf \left\{ d_{\mathbb{R}}(\tilde{x},\tilde{y}) : \tilde{x} \in [x] \text{ and } \tilde{y} \in [y] \right\}, \tag{3}$$

where $d_{\mathbb{R}}$ is the standard metric in $\mathbb{R}$, gives a metric on $\mathbb{T}^1$.

This exercise shows why one calls it a topological circle.

** **Exercise 2.9.** *Consider the map $\psi : \mathbb{R} \to \mathbb{C}$, given by $\psi(\theta) = e^{i2\pi\theta}$ for $\theta \in \mathbb{R}$. Show the following:*

(a) $\psi(\theta) = \psi(\tilde{\theta})$ *if and only if* $\theta - \tilde{\theta} \in \mathbb{Z}$*,*

(b) there exists a well defined 1-1 function $\tilde{\psi} : \mathbb{T}^1 \to \mathbb{C}$ *with the property that* $\tilde{\psi}([x]) = \tilde{(}x)$ *for any* $x \in \mathbb{T}^1$*, and*

(c) $\tilde{\psi}$ *is a homeomorphism from* $\mathbb{T}^1 \to \{z \in \mathbb{C} : |z| = 1\}$*.*

Notice that for any $x, y \in \mathbb{R}$ and $\tilde{x} \in [x]$ and $\tilde{y} \in [y]$, then $\tilde{x} + \tilde{y} \in [x + y]$. This means that the addition respects the equivalence classes. For $\alpha \in \mathbb{R}$ define the function $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$ by $R_\alpha([x]) = [\alpha + x]$ for $[x] \in \mathbb{T}^1$.

Definition 2.10. *For* $\alpha \in \mathbb{R}$*, the* ***circle rotation****, of angle* α *is the dynamical system* $(\mathbb{T}^1, R_\alpha)$*.*

Notice that this is an invertible, discrete system, hence $\mathcal{T} = \mathbb{Z}$. The dynamics of these systems splits into a dichotomy which is illustrated in the next two exercises. These exercises illustrate a relationship between dynamics and number theory.

Exercise 2.11. *Prove that if* $\alpha \in \mathbb{Q}$*, then the dynamical system* $(\mathbb{T}^1, R_\alpha)$ *has the property that every point is periodic of the same period. What is the period?*

**** Exercise 2.12.** *Prove that if* $\alpha \notin \mathbb{Q}$*, then the dynamical system* $(\mathbb{T}^1, R_\alpha)$ *has the property that every point has a dense orbit.*

2.3 The Shift Map

The shift map on two symbols (symbols are defined below) is one of the simplest examples of a symbolic system. There are two major reasons to study symbolic systems. One reason is that they are relatively easy to define and study. Consequently, much is known about them. The second, more important reason, is that many other systems can be "re-imagined" as symbolic systems. Results then often have applications outside the original setting. We give two highly related systems, one invertible, the other non-invertible, that have simple to define dynamics. We start with the invertible system, as it is the more important one for our discussions.

The invertible system is usually referred to as "two-sided", for reasons that should be clear after both systems are defined. The phase space is $\Sigma_2 := \{0,1\}^{\mathbb{Z}}$, understood as a countable product of the two element set $\{0,1\}$ (called the ***symbols***) indexed by $\mathbb{Z}$. For $i \in \mathbb{Z}$ and $x \in \Sigma_2$, we use the notation $x_i \in \{0,1\}$ to denote the i-th coordinate of x. We also use the shorthand $x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$. The ***left shift map*** or just ***shift map*** is $\sigma : \Sigma_2 \to \Sigma_2$, where, for $x \in \Sigma_2$, $\sigma(x)$ has coordinates of $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$.

For the non-invertible system, we use $\Sigma_2^+ := \{0,1\}^{\mathbb{N}}$ with a similar notation and shorthand conventions. The shift map is effectively the same and hence we do not change notation or terminology. We have $\sigma : \Sigma_2^+ \to \Sigma_2^+$, where, for $x \in \Sigma_2^+$, $\sigma(x)$ has coordinates of $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$.

Definition 2.13. *The **full two-sided shift on two symbols** is the dynamical system (Σ_2, σ). The **full one-sided shift on two symbols** is the dynamical system (Σ_2^+, σ).*

The map is cleaner to express in the shorthand for the one-sided shift. In this case, the shorthand gives $\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. For example, we have $\sigma(0,1,0,0,1,\ldots) = (1,0,0,1,\ldots)$.

These spaces can be given a metric that makes the shifts continuous. For $x, y \in \Sigma_2$ (or $x, y \in \Sigma_2^+$), the metric is $d_\Sigma(x,y) = 2^{-k}$ where $k = \min\{|i| : x_i \neq y_i\}$. For example, for $x = (0,1,1,1,0,\ldots) \in \Sigma_2^+$ and $y = (0,1,1,1,1,\ldots) \in \Sigma_2^+$, the distance is $d_\Sigma(x,y) = 2^{-4}$.

Exercise 2.14. *Show that d_Σ is a metric on Σ_2 and that the shift map σ is continuous with respect to this metric.*

** **Exercise 2.15.** *Show that Σ_2^+ is homeomorphic to the standard cantor set.*

We now consider a system that illustrates how a symbolic system can give information about a non-symbolic system, $(\mathbb{T}^1, E_2)$, where $E_2 : \mathbb{T}^1 \to \mathbb{T}^1$ is given by $E_2([x]) = [2x]$. This is called the ***doubling map***.

Exercise 2.16. *Show that E_2 is a two-to-one map. Furthermore, show that there is exists a map $\pi : \Sigma_2^+ \to \mathbb{T}^1$ one-to-one off of a countable set and with $\pi(\sigma(x)) = E_2(\pi(x))$ for any $x \in \Sigma_2^+$.*

2.4 The "2-1-1-1"-hyperbolic toral automorphism

The "2-1-1-1"-hyperbolic toral automorphism is one of the most famous examples in elementary dynamical systems. Although it is very particularly defined, some of the structure observed in the "2-1-1-1"-hyperbolic toral automorphism is in fact very typical of classes of complicated systems. We cannot fully discuss the depth of the previous statement here, as it requires a solid understanding of differentiable manifolds. We do address some of this structure in Sections 3 and 4, but we do not explain how this generalizes.

The phase space is the two-torus (see Figure 2), $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where the equivalence for $x, y \in \mathbb{R}^2$ is given by $x \sim y$ if and only if $x - y \in \mathbb{Z}^2$. One can see that this is, both as a set and topologically, the same as $\mathbb{T}^1 \times \mathbb{T}^1$. In other words, the two-torus is the twice product of the one-torus. Similar to the one-torus setting, this can be viewed as $\mathbb{T}^2 = ([0,1] \times [0,1])/ \sim$ where $\begin{pmatrix} 0 \\ x_2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$ for any $x_1, x_1 \in [0,1]$. Note this implies that $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by rules of equivalence relations. Similar to T^1, for $x \in \mathbb{R}$, we use the notation $[x] \in \mathbb{T}^2$ to denote the equivalence class of $[x]$.

Unproven Fact 2.17. *The map* $d_{\mathbb{T}^2} : \mathbb{T}^2 \times \mathbb{T}^2$ *defined by*

$$d_{\mathbb{T}^2}([x],[y]) = \inf \{ d_{\mathbb{R}^2}(\tilde{x},\tilde{y}) : \tilde{x} \in [x] \text{ and } \tilde{y} \in [y] \}, \tag{4}$$

where $d_{\mathbb{R}^2}$ *is the standard metric in* $\mathbb{R}^2$*, gives a metric on* $\mathbb{T}^2$*.*

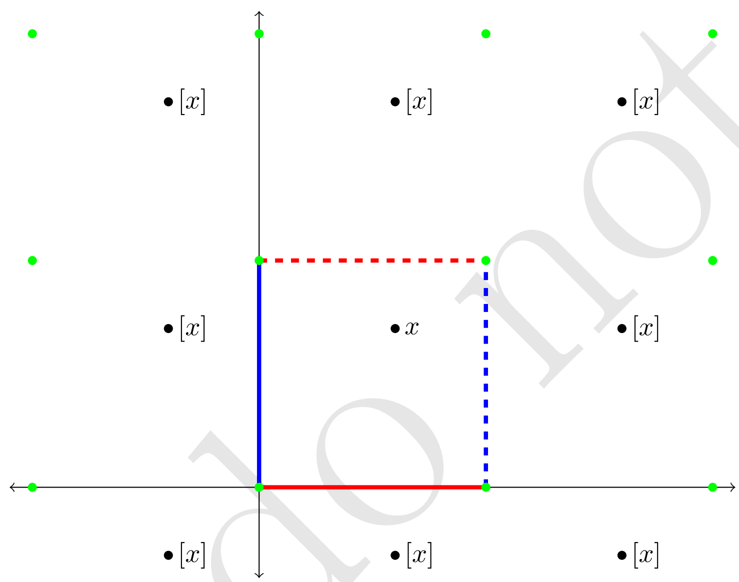

Figure 2: The two torus, $\mathbb{T}^2$. Illustrating elements of an equivalence class and the preferred fundamental domain.

Consider the linear map $A_{\text{hyp}} : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$A_{\text{hyp}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5}$$

In Exercise 2.22 below, one shows that linear maps on $\mathbb{R}^2$ with integer coefficients induce well defined maps on $\mathbb{T}^2$. The map A_{hyp} is one such map (and so is its inverse). Thus, the induced $A_{\text{hyp}} : \mathbb{T}^2 \to \mathbb{T}^2$ with

$$A_{\text{hyp}}\left([x]\right) = \left[A_{\text{hyp}}(x)\right],$$

is well defined and invertible. It can also be seen to be continuous.

Definition 2.18. *The* ***"2-1-1-1"-hyperbolic toral automorphism*** *is the dynamical system* $(\mathbb{T}^2, A_{hyp}, \mathbb{Z})$.

The linear map A_{hyp} has two eigenvalues. Listed with our corresponding selection of eigenvecters are:

$$\begin{aligned}
\lambda_{\text{hyp}} &= \frac{3+\sqrt{5}}{2} \text{ with eigenvecter,} \\
v_{\lambda_{\text{hyp}}} &= \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} \\ \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \end{pmatrix}, \text{ and} \qquad (6) \\
\lambda_{\text{hyp}}^{-1} &= \frac{3-\sqrt{5}}{2} \text{ with eigenvecter,} \\
v_{\lambda_{\text{hyp}}^{-1}} &= \begin{pmatrix} \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} \end{pmatrix}. \qquad (7)
\end{aligned}$$

Notice that $\|v_{\lambda_{\text{hyp}}}\| = \|v_{\lambda_{\text{hyp}}^{-1}}\| = 1$ and $0 < \lambda_{\text{hyp}}^{-1} < 1 < \lambda_{\text{hyp}}$. In Section 4, there is deeper quantitative discussion on periodic points and periodic orbits. Here we give a more qualitative discussion that shows that periodic points are countably infinite and dense. We start with a lemma explaining how to characterize periodic points. The proof follows immediately from the definition of the equivalence classes defining $\mathbb{T}^2$.

Lemma 2.19. *For the "2-1-1-1"-hyperbolic toral automorphism,* A_{hyp}*, an* $[x] \in \mathbb{T}^2$ *is periodic of period* n *if and only if* $(A_{hyp}^n - I_2)(x) \in \mathbb{Z}^2$*, where* $I_2 : \mathbb{R}^2 \to \mathbb{R}^2$ *is the identity linear map and* $x \in \mathbb{R}^2$ *is any representative of* $[x]$.

Part of the proof of the next proposition is a nice illustration of the "pigeon hole principle". Vaguely, the pigeon hole principle refers to techniques that rely on the finiteness of choice to force eventual reuse.

Proposition 2.20. *For the "2-1-1-1"-hyperbolic toral automorphism, A_{hyp}, an $[x] \in \mathbb{T}^2$ is periodic if and only if any representative $x \in [x]$ has rational coefficients.*

Proof. ($\Rightarrow$) Assume that $[x] \in \mathbb{T}^2$ is a periodic point with period n for the "2-1-1-1"-hyperbolic toral automorphism. Then by Lemma 2.19, for a representative $x \in [x]$, we have $(A^n_{\text{hyp}} - I_2)(x) = m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$. From standard linear algebra and Equations 6 and 7, we have that $(A^n_{\text{hyp}} - I_2)$ has eigenvalues $\lambda^n_{\text{hyp}} - 1 > 0$ and $\lambda^{-n}_{\text{hyp}} - 1 < 0$. Hence it is invertible making $x = (A^n_{\text{hyp}} - I_2)^{-1} m$. The matrix $(A^n_{\text{hyp}} - I_2)$ has integer coefficients. This implies that $(A^n_{\text{hyp}} - I_2)^{-1}$ has rational coefficients and so does $x = (A^n_{\text{hyp}} - I_2)^{-1} m$.

($\Leftarrow$) For this part of the proof, for any $[x] \in \mathbb{T}^2$ we systematically choose the representative $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in ([0,1) \times [0,1)) \cap [x]$ with $0 \leq x_1, x_2 < 1$. If any representative has rational coefficients, then so does our chosen one. By choosing a common denominator, we can represent $x = \begin{pmatrix} \frac{l_1}{k} \\ \frac{l_2}{k} \end{pmatrix}$, with $k, l_1, l_2 \in \mathbb{N}$ and $0 \leq l_1, l_2 \leq k - 1$. Since A_{hyp} has integer coefficients, it respects the class with rational coefficients that has common denominator k. As there are only a finite number in $([0,1) \times [0,1))$, k^2. After a finite number of operations it must eventually repeat. $\square$

Corollary 2.21. *For the "2-1-1-1"-hyperbolic toral automorphism, $(\mathbb{T}^2, A_{hyp}, \mathbb{Z})$, periodic points are countably infinite and dense.*

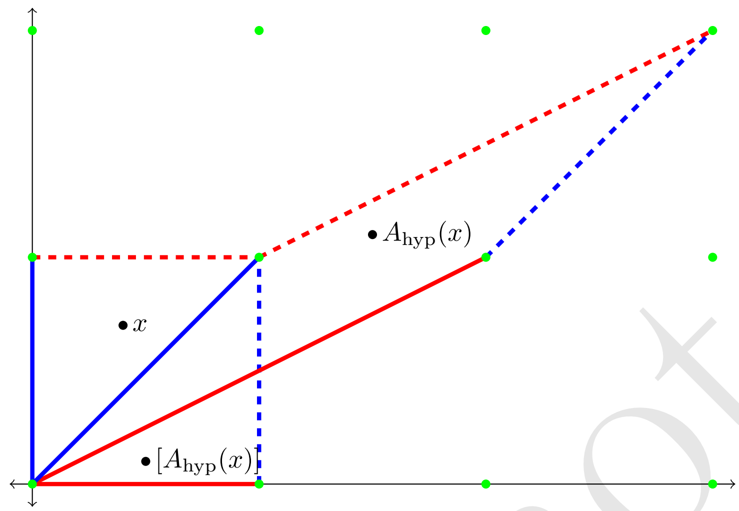

Figure 3: Action of the "2-1-1-1"-hyperbolic toral automorphism.

Exercise 2.22. *Let* $B : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear map with integer coordinates for its matrix. Prove that* $B : \mathbb{T}^2 \to \mathbb{T}^2$ *given by* $B([x]) = [B(x)]$ *is a well defined operation. Furthermore, show that there is an inverse* $B^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$ *if and only if* $\det(B) = \pm 1$.

This following exercise, when combined with Proposition 3.7, is profoundly related to some important ergodic theory results. Those results are outside the scope of this chapter, however the exercise is approachable.

**** Exercise 2.23.** *Let* $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ *denote the standard projection. Prove that for any* $x \in \mathbb{R}^2$, $\pi(\{x + tv_{\lambda_{hyp}} : t \in \mathbb{R}\})$ *is dense in* $\mathbb{T}^2$. *(Hint: This relates to Equation 6 and Exercise 2.12 (see Figure 4.)*

Mostly for fun, we explore another interesting number theoretic question. Recall the Fibonacci numbers defined by $F_0 = 1$, $F_1 = 1$ and the rest inductively defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (note: $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, ...).

Exercise 2.24. *Prove that*

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{pmatrix}.$$

Use this to show that $F_{2n+2}F_{2n} - F_{2n+1}^2 = 1$.

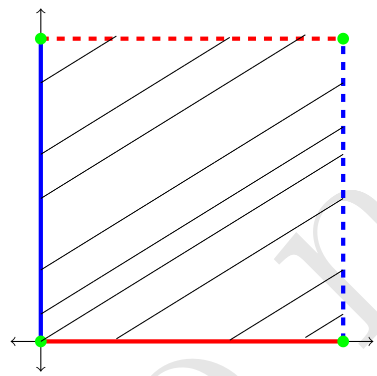

Figure 4: The projection, $\pi\left(\left\{\vec{0} + tv_{\lambda_{\text{hyp}}} : t \in [0, 5.877]\right\}\right)$.

3 Complex systems and structures

3.1 Topological Entropy

Lets say we have a dynamical system (X, f), with f continuous. For reasons used in the definition, we assume that X is a compact metric space with metric d. For any $x, y \in X$, let

$$d_n^f(x, y) := \max_{j=0,\ldots,n-1} \left\{ d(f^j(x), f^j(y)) \right\}. \tag{8}$$

Exercise 3.1. *Prove that* d_n^f *defines a metric and that the topology induced by* d_n *is the same as the topology induced by* d.

Notice from the previous exercise that X, d_n^f is also a compact metric space.

Let $\epsilon > 0$ and fixed. We use the notation, $B^f_{n,\epsilon}(x) = \left\{y : d_n^f(x,y) < \epsilon\right\}$, to denote the ϵ-balls with respect to the d_n^f metric. For an open cover, $\mathtt{O}_{\text{cov}}$, of X, we say that $\mathtt{O}_{\text{cov}}$ is an (f, n, ϵ)***-cover*** if every $O \in \mathtt{O}_{\text{cov}}$ there is an $x \in X$ such that $O \subset B^f_{n,\epsilon}(x)$. We define

$$S_d(f, \epsilon, n) := \min\left\{\#\mathtt{O}_{\text{cov}} : \mathtt{O}_{\text{cov}} \text{is an } (f, n, \epsilon)\text{-cover}\right\}, \tag{9}$$

and

$$h_d(f, \epsilon) := \limsup_{n\to\infty} \frac{\ln S_d(f, \epsilon, n)}{n}. \tag{10}$$

Definition 3.2. *The* ***topological entropy****, $h_{top}(f)$ for a dynamical system, (X, f), is defined as*

$$h_{top}(f) := \lim_{\epsilon\to 0} h_d(f, \epsilon).$$

Given the definition, the following fact seems surprising, at first anyway. It comes from compactness.

Unproven Fact 3.3. *The entropy does not depend on the metric d, only the topology.*

Unproven Fact 3.4. *The entropy for the "2-1-1-1"-hyperbolic toral automorphism is $\ln \lambda_{hyp} > 0$ and the entropy for the two sided shift is $\ln 2 > 0$.*

3.2 Stable and Unstable Sets

From the definition of entropy, one notices immediately that positive entropy means that points are being forced "apart". However, it is generally the case that positive entropy results in points that contract together. As one might expect, this is not the "typical" behavior of two points. In order to develop this understanding, we are going to define the stable (and unstable) sets for for a point. The uses of these objects in dynamical systems, especially smooth dynamically systems, are numerous. There are many relationships between stable sets and entropy (both topological and metric), measure theory, periodic orbits, or differential geometry, for example.

Definition 3.5. *Let X be a metric space with metric d. For a dynamical system $(X, \mathcal{T}, \phi)$ and a point $x \in X$, the **stable set** of x is*

$$\mathcal{W}^s(x) := \left\{ y \in X : \lim_{t \to \infty} d\left(\phi(x,t), \phi(y,t)\right) = 0 \right\}. \tag{11}$$

*Furthermore, if the dynamical system is invertible, the **unstable set** of x is*

$$\mathcal{W}^u(x) := \left\{ y \in X : \lim_{t \to -\infty} d\left(\phi(x,t), \phi(y,t)\right) = 0 \right\}. \tag{12}$$

Notice that the definition of the unstable sets requires the system to be invertible. We define the local stable for discrete systems. This is not essential. Recall the d_n^f from the definition of entropy in Subsection 3.1.

Definition 3.6. *Let X be a metric space with metric d. For a discrete dynamical system (X, f) and a point $x \in X$, and $\epsilon > 0$, the ϵ-**local stable set** of x is*

$$\mathcal{W}^s_\epsilon(x) := \mathcal{W}^s(x) \bigcap \left\{ y \in X : \lim_{n \to \infty} d_n^f(x,y) \le \epsilon \right\}. \tag{13}$$

*Furthermore, if the dynamical system is invertible, the ϵ-**local unstable set** of x is*

$$\mathcal{W}^u_\epsilon(x) := \mathcal{W}^u(x) \bigcap \left\{ y \in X : \lim_{n \to -\infty} d_n^f(x,y) \le \epsilon \right\}. \tag{14}$$

This can be rewritten with,

$$\mathcal{W}_\epsilon^s(x) = \mathcal{W}^s(x) \bigcap \bigcap_{k=0}^{\infty} f^{-k}\left(\bar{B}_\epsilon\left(f^k(x)\right)\right). \tag{15}$$

Similar intersection can be used for $\mathcal{W}_\epsilon^u(x)$ if f is invertible. We now move into exploring these objects for the “2-1-1-1”-hyperbolic toral automorphism. The proofs of the next two propositions are intertwined and the proof of Proposition 3.8 is not rigorous.

Proposition 3.7. *For the “2-1-1-1”-hyperbolic toral automorphism,* $(\mathbb{T}^2, A_{hyp}, \mathbb{Z})$ *any* $[x] \in \mathbb{T}^2$ *has* $\mathcal{W}^s([x]) = \pi(\{x + tv_{\lambda_{hyp}^{-1}} : t \in \mathbb{R}\})$ *and* $\mathcal{W}^u([x]) = \pi(\{x + tv_{\lambda_{hyp}} : t \in \mathbb{R}\})$.

Proof. We show this for $\mathcal{W}^s([x])$. The proof for $\mathcal{W}^u([x])$ works the same using A_{hyp}^{-1}. Notice that for any $t \in \mathbb{R}$, we have $A_{\text{hyp}}^n(x + tv_{\lambda_{\text{hyp}}^{-1}}) = A_{\text{hyp}}^n(x) + t\lambda_{\text{hyp}}^{-n} v_{\lambda_{\text{hyp}}^{-1}}$. Hence we have, $d_{\mathbb{T}^2}(A_{\text{hyp}}^n([x]), A_{\text{hyp}}^n([x + tv_{\lambda_{\text{hyp}}^{-1}}])) \leq d_{\mathbb{R}^2}(A_{\text{hyp}}^n(x), A_{\text{hyp}}^n(x + tv_{\lambda_{\text{hyp}}^{-1}})) = \lambda_{\text{hyp}}^{-n}|t||| \xrightarrow{n\to\infty} 0$. This implies that $\pi(\{x + tv_{\lambda_{\text{hyp}}^{-1}} : t \in \mathbb{R}\}) \subset \mathcal{W}^s([x])$.

Now for the reverse inequality, consider $[y] \in \mathcal{W}^s([x])$. For any $\epsilon > 0$, more or less by definition, there is an $n_{[y]} \in \mathbb{N}$ with $A_{\text{hyp}}^{n_{[y]}}([y]) \in \mathcal{W}_\epsilon^s\left(A_{\text{hyp}}^{n_{[y]}}([x])\right)$. By Proposition 3.8, this implies that $A_{\text{hyp}}^{n_{[y]}}([y]) \in \pi(\{A_{\text{hyp}}^{n_{[y]}}([x]) + tv_{\lambda_{\text{hyp}}^{-1}} : -\epsilon \leq t \leq \epsilon\})$. Consequently,

$$\begin{aligned}
[y] &\in A_{\text{hyp}}^{-n_{[y]}}\left(\pi\left(\left\{A_{\text{hyp}}^{n_{[y]}}([x]) + tv_{\lambda_{\text{hyp}}^{-1}} : -\epsilon \leq t \leq \epsilon\right\}\right)\right) \\
&= \pi\left(\left\{[x] + tv_{\lambda_{\text{hyp}}^{-1}} : -\lambda_{\text{hyp}}^{n_{[y]}}\epsilon \leq t \leq \lambda_{\text{hyp}}^{n_{[y]}}\epsilon\right\}\right) \\
&\subset \pi\left(\left\{[x] + tv_{\lambda_{\text{hyp}}^{-1}} : t \in \mathbb{R}\right\}\right).
\end{aligned}$$

This completes the proof. $\square$

Proposition 3.8. *For the “2-1-1-1”-hyperbolic toral automorphism,* $(\mathbb{T}^2, A_{hyp}, \mathbb{Z})$

and $\epsilon > 0$ sufficiently small, any $[x] \in \mathbb{T}^2$ has $\mathcal{W}^s_\epsilon([x]) = \pi(\{x + tv_{\lambda^{-1}_{hyp}} : -\epsilon \leq t \leq \epsilon\})$ and $\mathcal{W}^u_\epsilon([x]) = \pi(\{x + tv_{\lambda_{hyp}} : -\epsilon \leq t \leq \epsilon\})$.

We choose not to prove this rigorously, but to rely on the picture given in Figure 5. This picture relies on Equation 15 and the local behavior. The axis shown are the eigendirections.

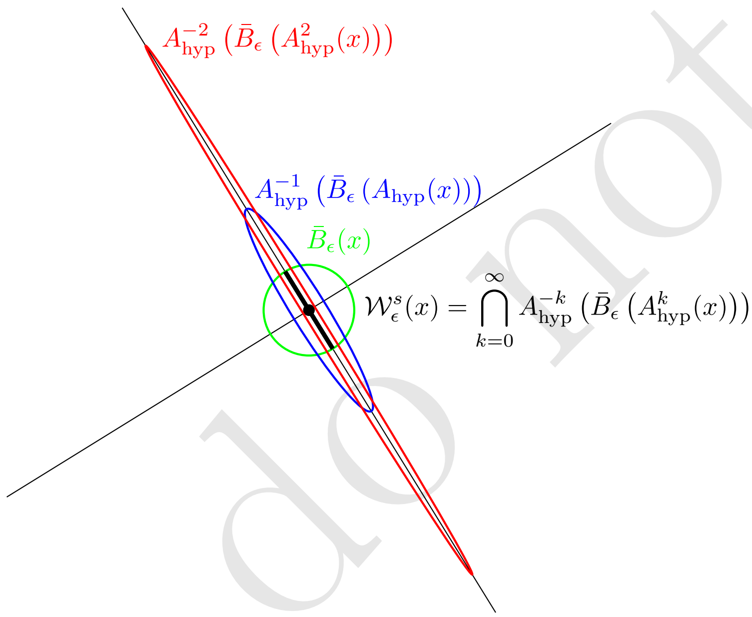

Figure 5: For the “2-1-1-1”-hyperbolic toral automorphism, the intersection gives the local stable set.

This is a little easier to work with in the other main example, the shift map.

Exercise 3.9. *For the two-sided shift map on two symbols $(\Sigma_2, \sigma, \mathbb{Z})$ and $x \in \Sigma_2$, find and demonstrate what $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ are. What happens for the local stables?*

4 Counting Periodic Orbits

In this section, we explore a relationship between entropy and the growth rate of periodic orbits.

4.1 Margulis' Theorem with algebraic counting

Notation 4.1. *For a dynamical system (X, f) and $n \in N$, we use the notations P_n and to denote the number of periodic points of period n and first period n respectively. We use the notation $P_n^{\mathcal{O}}$ to denote the number of periodic* orbits *of period less than or equal to n.*

Theorem 4.2 (Main result)**.** *For the cat map, $(\mathbb{T}^2, A_{hyp}, \mathbb{Z})$, the periodic orbits satisfy the following asymptotic formula*

$$\lim_{n\to\infty} \frac{(\lambda_{hyp} - 1)nP_n^{\mathcal{O}}}{\lambda_{hyp}^{n+1}} = 1.$$

Proof of this theorem is broken up into several parts. The techniques for proving this are accessible, but the proof is non-trivial.

Proposition 4.3 (Picks' Theorem)**.** *Let P be a polygon in $\mathbb{R}$ with vertexes in $\mathbb{Z}^2$. Then $Area(P) = i_P + \frac{b_P}{2} - 1$ where i_P and b_P are the number of integer coordinates in the interior and boundary of P respectively.*

**** Exercise 4.4.** *Prove Pick's Theorem.*

We use a version of the Dominated Convergence theorem, stated in Proposition 4.5.

Proposition 4.5 (Dominated Convergence Theorem)**.** *Let a_{ij} be a sequence of sequences in $\mathbb{R}$ with a coordinate wise convergence, $\lim\limits_{i\to\infty} a_{ij} = a_j$. Let b_j be a sequence in $\mathbb{R}$ satisfying $0 \leq |a_{ij}| \leq b_j$ for all i, and $\sum\limits_{j=1}^{\infty} b_j$ exists and is* *finite. Then, the following sums,* $\sum_{j=1}^{\infty} a_{ij}$, $\sum_{j=1}^{\infty} a_j$ *exist and are finite for all* i *and* $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} a_j$.

Exercise 4.6. *Prove the dominated convergence theorem.*

Theorem 4.7. *For the cat map,* $(\mathbb{T}^2, A_{hyp}, \mathbb{Z})$ *and* $n \in \mathbb{N}$, $P_n = \lambda^n + \lambda^{-n} - 2$.

Proof. For any $[x] \in \mathbb{T}^2$, the $[x]$ is periodic of period n if and only if $A_{\mathrm{hyp}}^n(x) - x \in \mathbb{Z}^2$. To count these, take the representatives $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ with $0 \leq x_1, x_2 < 1$. For such x, $A_{\mathrm{hyp}}^n \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2$ if and only if $(A_{\mathrm{hyp}}^2 - I_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. The number of these corresponds to the number of lattice points of $\mathbb{Z}^2$ contained in the parallel piped $(A^n - I_2)([0,1) \times [0,1))$. Through a proper use of Pick's theorem, this is the area of the parallel piped. From standard linear algebra, this is $|\det(A^n - I_2)| = \lambda_{\mathrm{hyp}}^n + \lambda_{\mathrm{hyp}}^{-n} - 2$. $\square$

The core part of the proof is in fact already done.

Lemma 4.8. *For* $\lambda > 1$, *we have* $\lim_{n\to\infty} \frac{n}{\lambda^n} \sum_{j=1}^{n} \frac{\lambda^j}{j} = \frac{1}{1 - \lambda^{-1}}$.

Proof. This proof follows from an application of the of the Dominated Convergence theorem. Define the sequence $b_k = (k+1)\lambda^{-k}$. Note that following sum exists and is finite $\sum_{k=1}^{\infty} b_k$. We can rewrite the sum

$$\begin{aligned} \frac{n}{\lambda^n} \sum_{j=1}^{n} \frac{\lambda^j}{j} &= \sum_{j=1}^{n} \frac{n}{j} \lambda^{j-n} \\ &= \sum_{k=0}^{n-1} \frac{n}{n-k} \lambda^{-k}. \end{aligned}$$

Define for $k < n$, $a_{nk} = \frac{n}{n-k}\lambda^{-k}$ and if $k \geq n$, $a_{nk} = 0$. Note that $\dfrac{n}{n-k} \leq k+1$ for $n > k$. Therefore $a_{nk} \leq (k+1)\lambda^{-k} = b_k$. Furthermore, $\lim\limits_{n\to\infty} a_{nk} = \lambda^{-k}$. By construction and the Dominated Convergence theorem

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{\lambda^n} \sum_{j=1}^{n} \frac{\lambda^j}{j} &= \sum_{k=0}^{\infty} a_{nk} \\ &= \sum_{k=0}^{\infty} \lambda^{-k} \\ &= \frac{1}{1-\lambda^{-1}}.\end{aligned}$$

This completes the proof. $\square$

Exercise 4.9. *For $\lambda > 1$, let c_j be an increasing positive sequence with $\lim\limits_{j\to\infty} j\lambda^{-j}c_j = 0$. Prove that $\lim\limits_{n\to\infty} \dfrac{n}{\lambda^n} \sum\limits_{j=1}^{n} c_j = 0$.*

We now have all of the tools to prove Theorem 4.2.

Main Result. Notice that $P_n^{\mathcal{O}} = \sum\limits_{j=1}^{n} \dfrac{P_j^{\text{first}}}{j}$. Let $b_j = \frac{P_j^{\text{first}}}{j} - \lambda^j$, $j = 1, 2, 3, \ldots$. The result from Theorem 4.7. includes periodic points whose period divides j. But such periods must be less than or equal to $\frac{j}{2}$ and we can get the bound $|b_j| \leq 2j + \sum_{i\leq\frac{j}{2}} 2\lambda^i \leq j(2 + \lambda^{\frac{j}{2}}) = c_j$. By Exercise 4.9, $\lim\limits_{n\to\infty} \dfrac{n}{\lambda^n} \sum\limits_{j=1}^{n} c_j = 0$. This together with Lemma 4.8 completes the result. $\square$

We close this section with the "exercise egregium". The goal is to take the cat map results of Corollary 2.21 and Theorems 4.7 and 4.2 and find corresponding results for the two sided shift.

** **Exercise 4.10.** *For the full two sided shift (Σ_2, σ) do the following:*

(i) prove that $P_n = 2^n$,

(ii) prove that the periodic points are countable and dense, and

(iii) find the asymptotic growth rate for $P_n^{\mathcal{O}}$.

4.2 A version of G. Margulis' original counting method

The goal of this section is to give an idea of the counting method, not to be fully rigorous. Much of this argument is "proof by pictures." This will eventually be the the summit of these notes, exposing a famous counting argument of G. Margulis. Unfortunately, it is not complete at this time.

We use a fact whose proof is well beyond the scope of this chapter. This fact is that the cat map is measure theoretically mixing. So for any measurable sets, $B_1, B_2 \subset \mathbb{T}^2$, $\lim_{n\to\infty} \mu(B_1 \cap A^n(B_2)) = \mu(B_1)\mu(B_2)$.

For any $\epsilon_1, \epsilon_2 > 0$, consider the rectangle $R_{\epsilon_1,\epsilon_2} \subset \mathbb{R}^2$ whose vertexes are given by $(0,0)$, $\epsilon_1 v_{\lambda_{\text{hyp}}}$, $\epsilon_2 v_{\lambda_{\text{hyp}}^{-1}}$, and $\epsilon_1 v_{\lambda_{\text{hyp}}} + \epsilon_2 v_{\lambda_{\text{hyp}}^{-1}}$, (see Figure 6). Notice that (i) the edges are parallel to stable and unstable sets for the cat map and (ii) $A(R_{\epsilon_1,\epsilon_2}) = R_{\lambda\epsilon_1,\frac{\epsilon_2}{\lambda}}$.

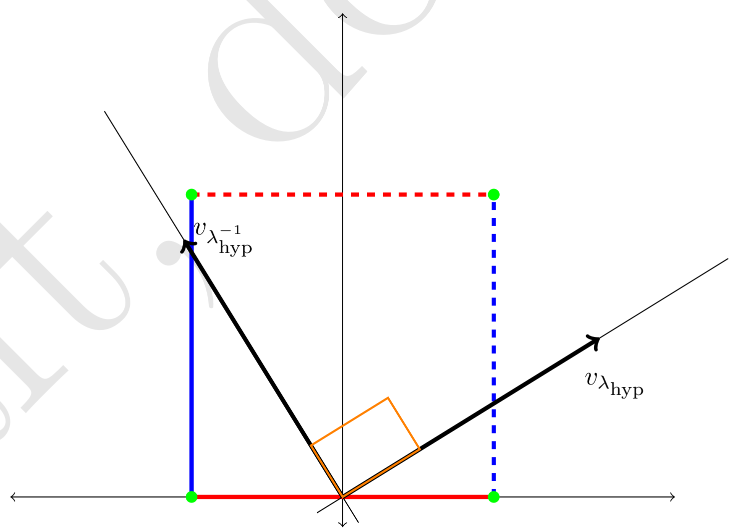

Figure 6: The rectangle $R_{\epsilon_1,\epsilon_2}$ for $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.2$.

We are interested in the way that such rectangles intersect with images of themselves under the map A^n_{hyp} (see Figure 7).

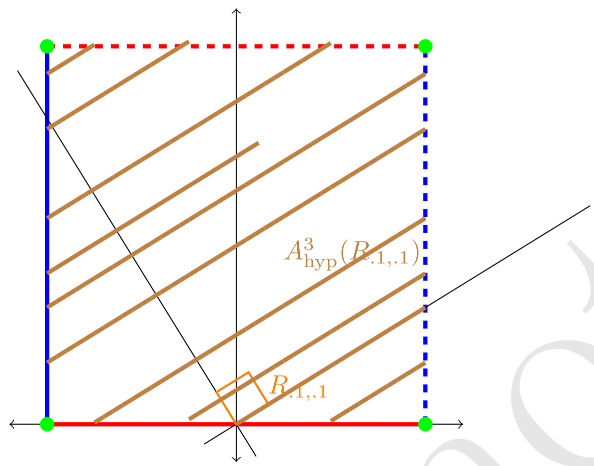

Figure 7: The rectangles $R_{.1,.1}$ and $A^3_{\text{hyp}}(R_{.1,.1})$.

5 Appendix: Background Material

The appendix is not complete.

5.1 A few set theoretic remarks

For a set X, we denote $\mathcal{P}(x)$, to be the power set, that collection of subsets of X. For any set, X, we use $\#X$ to denote the cardinality of X.

5.2 Partitions and equivalence relationships

A ***partition*** of a set X is a collection of subsets $P \subset \mathcal{P}(X)$ with two properties

(i) the sets are pairwise disjoint, that is $Y \cap \tilde{Y} = \emptyset$ for any two distinct $Y, \tilde{Y} \in P$, and

(ii) they cover X, that is $X = \bigcup_{Y \in P} Y$.

5.3 Metric and topological spaces

Definition 5.1. *A* ***metric space****,* (X, d)*, is a set* X*, with a* ***metric*** $d : X \times X \to \mathbb{R}$ *which satisfies the following properties for any* $x, y, z \in X$*:*

(a) $d(x, y) \geq 0$ *,*

(b) $d(x, y) = 0$ *if and only if* $x = y$*,*

(c) $d(x, y) = d(y, x)$

(d) $d(x, z) \leq d(x, y) + d(y, z)$*.*

Exercise 5.2. *Prove that the following are metric spaces:* $(\mathbb{R}, d)$ *where* $d(x, y) := |x - y|$*.*

Two important topics related to metric spaces are limits and open sets. These are the generalization of limits and open intervals found in any first semester calculus class.

We start with the foundation of open sets in the metric setting, open balls.

Definition 5.3. *For a metric space* (X, d)*, for any* $x \in X$ *and* $\epsilon > 0$*, the* ***open ball of radius*** ϵ ***about*** x *is*

$$B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\} .$$

In a metric space (X, d), a subset $O \subset X$ is called ***open*** if for every $x \in X$ there exists a $\epsilon > 0$ such that $B_\epsilon(x) \subset O$. The first thing to note is that any open ball is an open set.

Exercise 5.4. *Prove that in a metric space, any open ball is an open set.*

Open sets in a metric space satisfy some interesting properties that are used to motivate the definition of a topological space.

Proposition 5.5. *For a metric space (X, d), the following hold:*

(a) $\emptyset$ and X are open,

(b) if $\varnothing$ is a collection of open sets then $\bigcup_{O \in \varnothing} O$ is open, and

(c) if $O_1, \ldots, O_n$ are open then $\bigcap_{j=1}^{n} O_j$ is open.

Exercise 5.6. *Prove Proposition 5.5.*

Definition 5.7. *A **topological space** is a set X with a collection of subsets $\mathfrak{T} \subset \mathcal{P}(X)$, which satisfies the properties*

(a) $\emptyset, X \in \mathfrak{T}$,

(b) if $\varnothing \subset \mathfrak{T}$, then $\bigcup_{O \in \varnothing} O \in \mathfrak{T}$, and

(c) if $O_1, \ldots, O_n \in \mathfrak{T}$, then $\bigcap_{j=1}^{n} O_j \in \mathfrak{T}$.

The $\mathfrak{T}$ is called the ***topology*** of X and any $O \in \mathfrak{T}$ is called an ***open set***. Note that Proposition 5.5 shows that a metric space induces a topological space and that the definition of open is consistent. A subset $C \subset X$ is called ***closed*** if $C^c \in \mathfrak{T}$.

A topological space turns out to be the most natural setting for the notion of continuous function. A function $f : X \to Y$ between topological spaces is ***continuous*** if $f^{-1}(O)$ is open in X whenever $O \subset Y$ is open. This coincides with the notion of a function being continuous from first semester calculus.

5.4 Being "the same"

5.5 Differential Manifolds

5.6 Asymptotic behavior and related rates