AMPLE GENERICS IN POLISH GROUPS

CHRISTIAN ROSENDAL

1. Baire category and Comeagre orbits

Suppose G is a group of homeomorphisms of a topological space X. We say that G is *topologically transitive* if for all non-empty open subsets $U, V \subseteq X$ there is $g \in G$ such that $g \cdot U \cap V \neq \emptyset$.

Proposition 1. *The following are equivalent for a group G of homeomorphisms of a Polish space X.*

(i) *G is topologically transitive,*
(ii) *there is a dense orbit $G \cdot x$,*
(iii) *there is a comeagre set of points with dense orbits.*

Proof. That (iii)$\Rightarrow$(ii)$\Rightarrow$(i) is trivial. Also, if $\{U_n\}_{n\in\mathbb{N}}$ is a basis for the topology on X consisting of non-empty open sets and G is topologically transitive, then $G \cdot U_n$ is dense for every n, whereby

$$\bigcap_{n\in\mathbb{N}} G \cdot U_n = \bigcap_{n\in\mathbb{N}} \{x \in X \mid G \cdot x \cap U_n \neq \emptyset\} = \{x \in X \mid G \cdot x \text{ is dense }\}$$

is comeagre, showing (i)$\Rightarrow$(iii). $\square$

If A is a subset of a Polish space X, we let $U(A)$ be the union of all open sets $V \subseteq X$ so that A is comeagre in V, i.e., so that $V \setminus A$ is a meagre subset of X. By Lindelöf's Theorem, we can write $U(A) = \bigcup_n V_n$, where A is comeagre in each of the V_n. Thus $U(A) \setminus A = \bigcup_n V_n \setminus A$ is meagre and thus A is comeagre in $U(A)$. It follows that $U(A)$ is the *largest* open set in which A is comeagre.

Lemma 2 (S. Banach and B. J. Pettis)**.** *Suppose G is a Polish group and $A, B \subseteq G$ are subsets. Then*

$$U(A) \cdot U(B) \subseteq AB.$$

Proof. We note that if $x \in U(A)U(B)$, then the open set

$$V = xU(B)^{-1} \cap U(A) = U(xB^{-1}) \cap U(A)$$

is non-empty and so xB^{-1} and A are comeagre in V. It follows that $xB^{-1} \cap A \neq \emptyset$, whereby $x \in AB$. $\square$

Lemma 3 (E. Effros)**.** *Suppose G is a Polish group acting continuously on a Polish space X and let $x \in X$. Then the following are equivalent:*

(1) *For every identity neighbourhood V, the set $V \cdot x$ is comeagre in a neighbourhood of x.*
(2) *For every identity neighbourhood V, the set $V \cdot x$ is somewhere dense.*
(3) *The orbit $G \cdot x$ is non-meagre.*

Proof. (1)⇒(3) is trivial. Also, for (3)⇒(2), suppose $G \cdot x$ is non-meagre and V is an identity neighbourhood. Then we can find $g_n \in G$ such that $G = \bigcup_n g_n V$, whence $G \cdot x = \bigcup_n g_n V \cdot x$. So some $g_n V \cdot x$, and therefore also $V \cdot x$, is non-meagre and hence somewhere dense.

Finally, for (2)⇒(1), suppose that $V \cdot x$ is somewhere dense for every identity neighbourhood V. Suppose towards a contradiction that for some identity neighbourhood U, the set $U \cdot x$ is meagre, whence there are closed nowhere dense sets $F_n \subseteq X$ covering $U \cdot x$. But then the sets $K_n = \{g \in G \mid g \cdot x \in F_n\}$ are closed and cover U and thus, by the Baire category theorem, some K_n contains a non-empty open set gV, where V is an identity neighbourhood and $g \in G$. So $gV \cdot x \subseteq F_n$ and $V \cdot x$ must be nowhere dense, which is a contradiction.

Now, if V is any identity neighbourhood, let $U \subseteq V$ be a smaller identity neighbourhood such that $U^{-1}U \subseteq V$. Then $U \cdot x$ is comeagre in some neighbourhood of a point $g \cdot x$, where $g \in U$, and thus $g^{-1}U \cdot x \subseteq V \cdot x$ is comeagre in a neighbourhood of x. □

Lemma 4. *Suppose G is a Polish group acting continuously on a Polish space X. Then the following are equivalent:*

1. *There is a non-meagre orbit $\mathcal{O} \subseteq X$.*
2. *There is a non-empty open set $O \subseteq X$ with the following property: For all open $\emptyset \neq V \subseteq O$ and identity neighbourhood $U \subseteq G$, there is a smaller open $\emptyset \neq W \subseteq V$ such that the action of U on W is topologically transitive, i.e., for any non-empty open $W_0, W_1 \subseteq W$ there is $g \in U$ such that $gW_0 \cap W_1 \neq \emptyset$.*

Moreover, if $\mathcal{O}$ is an orbit comeagre in an open set $O \subseteq X$, then (2) holds for O.

Proof. (1)⇒(2): If $\mathcal{O} \subseteq X$ is a non-meagre orbit, let $O \subseteq X$ be a non-empty open set in which $\mathcal{O}$ is comeagre. Now, if $V \subseteq O$ is non-empty open and $U \subseteq G$ is a neighbourhood of 1, pick $x \in V \cap \mathcal{O}$ and choose an open neighbourhood $U_0 \subseteq U$ of 1 such that $U_0 U_0^{-1} \subseteq U$. Then, by Lemma 3, $U_0 \cdot x$ is dense in some open neighbourhood $W \subseteq V$ of x and it follows that the action of U on W is topologically transitive.

(2)⇒(1): Suppose $O \subseteq X$ is an open set satisfying the assumption in (2). Fix a neighbourhood basis $\{U_n\}_{n\in\mathbb{N}}$ at $1 \in G$ and a basis $\{V_n\}_{n\in\mathbb{N}}$ for the induced topology on O consisting of non-empty open sets. Now, for every n and m, let $W_{n,m} \subseteq V_n$ be a non-empty open subset such that the action of U_m^{-1} on $W_{n,m}$ is topologically transitive. Then $W_m = \bigcup_n W_{n,m}$ is open dense in O since it intersects every V_n. Also, for any $V_k \subseteq W_{n,m}$, $W_{n,m} \cap \left(U_m^{-1} \cdot V_k\right)$ is open dense in $W_{n,m}$, and so

$$D_{n,m} = W_{n,m} \cap \bigcap_{V_k \subseteq W_{n,m}} \left(U_m^{-1} \cdot V_k\right)$$

is comeagre in $W_{n,m}$. Note also that if $x \in D_{n,m}$, then for any $V_k \subseteq W_{n,m}$, $U_m \cdot x \cap V_k \neq \emptyset$, showing that $U_m \cdot x$ is dense in $W_{n,m}$. We notice that $D_m = \bigcup_n D_{n,m}$ is comeagre in O and that for any $x \in D_m$, $U_m \cdot x$ is somewhere dense. It follows that for any x belonging to the comeagre subset $\bigcap_m D_m \subseteq O$, and for any k, $U_k \cdot x$ is somewhere dense, which by the previous lemma implies that $G \cdot x$ is non-meagre. □

Combining Lemmas 1 and 4, we have the following characterisation of the existence of comeagre orbits.

Proposition 5. *Suppose G is a Polish group acting continuously on a Polish space X. Then there is a comeagre orbit on X if and only if*

(1) *the action of G is topologically transitive, and*
(2) *for any non-empty open $V \subseteq X$ and identity neighbourhood $U \subseteq G$, there is a smaller non-empty open set $W \subseteq V$ on which the action of U is topologically transitive.*

Proof. That these conditions are implied by the existence of a comeagre orbit is immediate from Lemma 4. Conversely, (2) implies that some orbit $G \cdot x$ is comeagre in an open set O, while (1) implies that there is a comeagre set of points with dense orbits. In particular, the orbit of x is both dense in X and comeagre in O. But then the orbit must be nowhere meagre and therefore comeagre. $\square$

2. Ample generics

Suppose G is a Polish group acting continuously on a Polish space X. Then for any positive integer n, we can define the diagonal action $G \curvearrowright X^n$ by

$$g \cdot (x_1, \ldots, x_n) = (g \cdot x_1, \ldots, g \cdot x_n).$$

Definition 6. *Suppose G is a Polish group acting continuously on a Polish space X. We say that the action has* ample generics *if for every $n \geqslant 1$ there is a comeagre orbit in X^n under the diagonal action of G.*

We shall refer to elements $(x_1, \ldots, x_n)$ of the comeagre orbit of dimension n as generics.

Easy examples of such actions are, for example, the natural action of S_∞ on Cantor space $\mathcal{P}(\mathbb{N})$. The generic in $\mathcal{P}(\mathbb{N})^n$ is then simply a tuple $(A_1, \ldots, A_n)$ of subsets of $\mathbb{N}$ so that every intersection

$$A_1^{\epsilon_1} \cap \ldots \cap A_n^{\epsilon_n}$$

is infinite and coinfinite, where $\epsilon_i \in \{-1, 1\}$, $A^{-1} = \mathbb{N} \setminus A$ and $A^1 = A$.

For more interesting examples, consider a Polish group G acting on itself by conjugation. Then the diagonal action is given by

$$g \cdot (h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1}).$$

If this action has ample generics, we simply say that G has ample generics itself.

We shall now present some of the main consequences of ample generics.

Lemma 7. *Let $G \curvearrowright X$ be a Polish group acting continuously on a Polish space with ample generics and suppose $A, B \subseteq X$ are arbitrary subsets such that*

- *A is non-meagre,*
- *B is nowhere meagre.*

Then, if $\overline{x} = (x_1, \ldots, x_n) \in X^n$ is generic and $V \ni 1$ is open, there are $g \in V$, $y \in A$ and $z \in B$ such that $(x_1, \ldots, x_n, y)$ and $(x_1, \ldots, x_n, z)$ are generic, while

$$g \cdot (x_1, \ldots, x_n, y) = (x_1, \ldots, x_n, z).$$

Notice that the last condition implies that $g \in G_{x_i}$ for $i \leqslant n$, where G_{x_i} is the pointwise stabiliser of x_i.

Proof. Let $\mathcal{O} \subseteq X^{n+1}$ be the comeagre orbit of dimension $n+1$. Then, by Kuratowski-Ulam,

$$\forall^* \overline{u} \in X^n \ \forall^* y \in X \quad (\overline{u}, y) \in \mathcal{O}.$$

Also, for any $\overline{u} \in X^n$ and $g \in G$,

$$\begin{aligned} \forall^* y \in X \ \ (\overline{u}, y) \in \mathcal{O} &\Rightarrow \forall^* y \in X \ \ (g \cdot \overline{u}, g \cdot y) \in \mathcal{O} \\ &\Rightarrow \forall^* z \in X \ \ (g \cdot \overline{u}, z) \in \mathcal{O}. \end{aligned}$$

So

$$D = \{\overline{u} \in X^n \mid \forall^* y \in X \ (\overline{u}, y) \in \mathcal{O}\}$$

is comeagre and G-invariant. Thus, D contains the comeagre orbit in X^n and hence $\overline{x} \in D$. It follows that

$$(*) \qquad \forall^* y \in X \ (x_1, \ldots, x_n, y) \in \mathcal{O}.$$

So, as A is non-meagre, we can find $y \in A$ such that $(x_1, \ldots, x_n, y) \in \mathcal{O}$. Notice now that

$$G_{\overline{x}} \cdot y = \{z \in X \mid (x_1, \ldots, x_n, z) \in \mathcal{O}\},$$

which is comeagre in X by $(*)$.

Now, since $(G_{\overline{x}} \cap V) \cdot y$ covers $G_{\overline{x}} \cdot y$ by countably many translates, the set $(G_{\overline{x}} \cap V) \cdot y$ is somewhere comeagre in X and hence intersects B. So letting $z \in B \cap (G_{\overline{x}} \cap V) \cdot y$, we can find $g \in G_{\overline{x}} \cap V$, such that

$$g \cdot y = z,$$

whereby

$$g \cdot (\overline{x}, y) = (\overline{x}, z),$$

proving the lemma. $\square$

Lemma 8. *Suppose $G \curvearrowright X$ is a Polish group acting continuously on a Polish space with ample generics and that $A_n, B_n \subseteq X$ are respectively non-meagre and nowhere meagre. Then there is a continuous map*

$$\alpha \in 2^{\mathbb{N}} \mapsto h_\alpha \in G$$

such that if $\alpha|_n = \beta|_n$ but $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$h_\alpha \cdot A_n \cap h_\beta \cdot B_n \neq \emptyset.$$

Proof. Using Lemma 7, we define by induction on the length of $s \in 2^{<\mathbb{N}} \setminus \{\emptyset\}$, points $x_s \in X$ and group elements $f_s \in G$ such that for all s,

(1) $(x_{s|1}, x_{s|2}, \ldots, x_s)$ is generic,
(2) $x_{s0} \in A_{|s|}$ and $x_{s1} \in B_{|s|}$,
(3) for all $\alpha \in 2^{\mathbb{N}}$, the infinite product $f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \ldots$ converges,
(4) $f_{s0} = 1$,
(5) $f_{s1} \cdot (x_{s|1}, x_{s|2}, \ldots, x_s, x_{s1}) = (x_{s|1}, x_{s|2}, \ldots, x_s, x_{s0})$.

Set $h_\alpha = f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \ldots$. And notice also that by (5), if $t \sqsubset s$, then $f_s \cdot x_t = x_t$. It follows that for all $\alpha, \beta \in 2^{\mathbb{N}}$, if $\alpha|_n = \beta|_n = s$, $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$\begin{aligned} h_\alpha \cdot x_{\alpha|n+1} &= f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \ldots \cdot x_{\alpha|n+1} \\ &= f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \cdots f_{\alpha|n+1} \cdot x_{\alpha|n+1} \\ &= f_{s|1} f_{s|2} \cdots f_s f_{s0} \cdot x_{s0} \\ &= f_{s|1} f_{s|2} \cdots f_s \cdot x_{s0} \\ &= f_{s|1} f_{s|2} \cdots f_s f_{s1} \cdot x_{s1} \\ &= f_{\beta|1} f_{\beta|2} \cdots f_{\beta|n} f_{\beta|n+1} \cdot x_{\beta|n+1} \\ &= f_{\beta|1} f_{\beta|2} \ldots \cdot x_{\beta|n+1} \\ &= h_\beta \cdot x_{\beta|n+1}. \end{aligned}$$

Since $x_{\alpha|n+1} \in A_n$ and $x_{\beta|n+1} \in B_n$, we have

$$h_\alpha \cdot A_n \cap h_\beta \cdot B_n \neq \emptyset$$

as required. □

Though we do not have any interesting applications of this lemma in the context of general actions with ample generics, when applied to Polish groups with ample generics, the consequences are quite intriguing.

Theorem 9 (Fundamental Theorem for Ample Generics). *Let G be a Polish group with ample generics and $\{k_i A_i f_i\}_{i \in \mathbb{N}}$ a covering of G, where $k_i, f_i \in G$ and $A_i \subseteq G$ are arbitrary subsets of G. Then there is an i such that*

$$A_i^{-1} A_i A_i^{-1} A_i^{-1} A_i A_i^{-1} A_i A_i A_i^{-1} A_i$$

is an identity neighbourhood.

Proof. By leaving out all terms $k_i A_i f_i$, such that A_i is meagre, and reenumerating, we can suppose that

(1) for every i there are infinitely many n such that

$$k_i A_i f_i = k_n A_n f_n,$$

(2) $\bigcup_i k_i A_i f_i$ is comeagre,
(3) each $f_i^{-1} A_i f_i$ is non-meagre.

Notice also that if there is some n such that

$$A_n^{-1} A_n A_n A_n^{-1} A_n$$

is somewhere comeagre, then by Lemma 2 we would be done. So assume towards a contradiction that this fails. Then

$$B_n = G \setminus (f_n A_n^{-1} A_n A_n A_n^{-1} A_n f_n^{-1})$$

is nowhere meagre. So by Lemma 8 there is a continuous mapping

$$\alpha \in 2^{\mathbb{N}} \mapsto h_\alpha \in G$$

so that if $\alpha|_n = \beta|_n$ but $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$h_\alpha f_n^{-1} A_n f_n h_\alpha^{-1} \cap h_\beta B_n h_\beta^{-1} \neq \emptyset,$$

i.e.,

$$h_\alpha f_n^{-1} A_n f_n h_\alpha^{-1} \not\subseteq h_\beta f_n A_n^{-1} A_n A_n A_n^{-1} A_n f_n^{-1} h_\beta^{-1}.$$

Now, the mapping

$$(g,\alpha) \in G \times 2^{\mathbb{N}} \mapsto g^{-1} h_\alpha \in G$$

is continuous and open, and therefore inverse images of comeagre sets are comeagre. So, as $\bigcup_{i\in\mathbb{N}} k_i A_i f_i$ is comeagre in G, we have by the Kuratowski-Ulam Theorem that

$$\forall^* g \in G\ \forall^* \alpha \in 2^{\mathbb{N}} \quad g^{-1} h_\alpha \in \bigcup_{i\in\mathbb{N}} k_i A_i f_i.$$

So pick some $g \in G$ with

$$\forall^* \alpha \in 2^{\mathbb{N}} \quad g^{-1} h_\alpha \in \bigcup_{i\in\mathbb{N}} k_i A_i f_i,$$

and find some i such that

$$\{\alpha \in 2^{\mathbb{N}} \mid g^{-1} h_\alpha \in k_i A_i f_i\}$$

is dense in some basic open set

$$N_t = \{\alpha \in 2^{\mathbb{N}} \mid t \subseteq \alpha\}.$$

Let now $n > |t|$ be such that $k_i A_i f_i = k_n A_n f_n$ and find $\alpha, \beta \in N_t$ such that

$$h_\alpha, h_\beta \in g k_n A_n f_n$$

and $\alpha|_n = \beta|_n$ while $\alpha(n) = 0$ and $\beta(n) = 1$. Then, if $h_\alpha = g k_n a f_n$ and $h_\beta = g k_n b f_n$, where $a, b \in A_n$, we have

$$\begin{aligned} h_\beta^{-1} h_\alpha f_n^{-1} A_n f_n h_\alpha^{-1} h_\beta &= f_n^{-1} b^{-1} a A_n a^{-1} b f_n^{-1} \\ &\subseteq f_n A_n^{-1} A_n A_n A_n^{-1} A_n f_n^{-1}. \end{aligned}$$

But this clearly contradicts

$$h_\alpha f_n^{-1} A_n f_n h_\alpha^{-1} \not\subseteq h_\beta f_n A_n^{-1} A_n A_n A_n^{-1} A_n f_n^{-1} h_\beta^{-1}$$

and hence proves the theorem. □

Corollary 10. *If G is a Polish group with ample generics, then any homomorphism $\pi\colon G \to H$ from G into a Polish group H is continuous.*

Proof. Suppose $\pi\colon G \to H$ is a homomorphism and that V is an identity neighbourhood in H. Pick a symmetric open identity neighbourhood W in H so that $W^{10} \subseteq V$. Then, as W covers H by countably many left-translates, also $U = \pi^{-1}(W)$ covers G by countably many left-translates. It therefore follows from Theorem 9 that U^{10} is an identity neighbourhood so that $\pi[U^{10}] \subseteq W^{10} \subseteq V$. This shows continuity of π at 1, whence π is a continuous map. □

Corollary 11. *Suppose G has ample generics. If $\{k_i H_i f_i\}_{i\in\mathbb{N}}$ is a covering of G by two-sided translates of subgroups, then some H_i is open.*

In particular, every countable index subgroup is open.

3. Further results on ample generics

Theorem 12 (J. Mycielski)**.** *Let X be a non-empty perfect Polish space and $R \subseteq X \times X$ a comeagre set. Then there is a homeomorphic copy $C \subseteq X$ of Cantor space such that $(x, y) \in R$ for all $x, y \in C$, $x \neq y$.*

Proof. Fix a compatible complete metric d on X. Note first that if $V_0, V_1 \subseteq X$ are non-empty open subsets and $D \subseteq X \times X$ is dense open, then there are disjoint, non-empty open subsets $U_0 \subseteq V_0$, $U_1 \subseteq V_1$ such that $U_0 \times U_1 \subseteq D$. Moreover, by shrinking the U_i further, one can ensure that the U_i have arbitrarily small diameter.

Now, suppose that $R \subseteq X \times X$ is comeagre and find a decreasing sequence $(D_n)_{n\in\mathbb{N}}$ of dense open subsets of $X \times X$ such that $R \supseteq \bigcap_{n\in\mathbb{N}} D_n$. We then define a Cantor scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ on X satisfying

(1) each U_s is a non-empty open set of diameter $< \frac{1}{|s|+1}$,
(2) $\overline{U_{s0}} \cap \overline{U_{s1}} = \emptyset$ for all $s \in 2^{<\mathbb{N}}$,
(3) $U_s \times U_t \subseteq D_n$ for all $s, t \in 2^n$, $s \neq t$.

Letting $f\colon 2^{\mathbb{N}} \to X$ be defined by $\{f(x)\} = \bigcap_{n\in\mathbb{N}} U_{x|n}$, we see that f is continuous, injective and that $(f(x), f(y)) \in \bigcap_{n\in\mathbb{N}} D_n \subseteq R$ for all $x \neq y$. Let now $C = f[2^{\mathbb{N}}]$. □

Theorem 13. *Suppose G is a Polish group with ample generics and $A \subseteq G$ is a symmetric subset containing* 1*. Then either A admits a continuum of disjoint left translates in G or A^{12} is an identity neighbourhood.*

Proof. Suppose A does not admit a continuum of disjoint left translates. Note first that if A^2 is meagre, then the binary relation R on G given by

$$xRy \Leftrightarrow x^{-1}y \in A^2$$

is meagre too, since the mapping $(x, y) \in G^2 \mapsto x^{-1}y \in G$ is surjective, continuous and open. But, by Theorem 12, if R is meagre, then there is a Cantor set $C \subseteq G$ such that for distinct $x, y \in C$, $(x, y) \notin R$, i.e., $xA \cap yA = \emptyset$, contradicting the assumption on A. So A^2 must be non-meagre.

We claim that A^6 must be somewhere comeagre. For if not, let $A_n = A^2$ and set $B_n = G \setminus A^6$, which then is nowhere meagre. Applying Lemma 8 to this sequence of pairs, we find an injective mapping $\alpha \in 2^{\mathbb{N}} \mapsto h_\alpha \in G$ such that if $\alpha|_n = \beta|_n$ but $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$h_\alpha A_n h_\alpha^{-1} \cap h_\beta B_n h_\beta^{-1} \neq \emptyset.$$

It follows that for distinct $\alpha, \beta \in 2^{\mathbb{N}}$, we have

$$h_\beta^{-1} h_\alpha A^2 h_\alpha^{-1} h_\beta \cap G \setminus A^6 \neq \emptyset,$$

and so, as A^2 is symmetric, $h_\beta^{-1} h_\alpha \notin A^2$, whereby $h_\alpha A \cap h_\beta A = \emptyset$, again contradicting the assumption on A.

Thus, A^6 is somewhere comeagre and therefore, by Lemma 2, A^{12} is a neighbourhood of 1. □

We notice that the above result gives rise to a notion of smallness, namely admitting a continuum of disjoint translates, in Polish groups, which is not closed under unions and hence does not correspond to an ideal.

Of course, ample generics is not something you are likely to find in many groups, and, in fact, most bigger Polish transformation groups even have meagre conjugacy classes. To see this, we can state a fairly general condition that implies that all conjugacy classes in a non-discrete Polish group are meagre. Namely,

Proposition 14. *Suppose $G \neq \{1\}$ is Polish such that for all infinite $S \subseteq \mathbb{N}$ and all open $V \ni 1$, the set*

$$A(S,V) = \{g \in G \mid \exists n \in S \; g^n \in V\}$$

is dense in G. Then all conjugacy classes in G are meagre.

Proof. Let $V_0 \supseteq V_1 \supseteq \ldots \ni 1$ be a neighbourhood basis at the identity and note that for every infinite $S \subseteq \mathbb{N}$

$$\begin{aligned} C(S) &= \{g \in G \mid \exists (s_n) \subseteq S \; g^{s_n} \to 1\} \\ &= \{g \in G \mid \forall k \; \exists s \in S \setminus [1,k] \; g^s \in V_k\} \\ &= \bigcap_k A(S, V_k). \end{aligned}$$

Then $C(S)$ is comeagre and conjugacy invariant. So if $\mathcal{O} \subseteq G$ were some comeagre conjugacy class, we would have

$$\mathcal{O} \subseteq \bigcap_{\substack{S \subseteq \mathbb{N} \\ \text{infinite}}} C(S).$$

But then if $g \in \mathcal{O}$, any sequence g^{n_i}, $n_i < n_{i+1}$, would have a subsequence converging to 1, and so $g = 1$ and $\mathcal{O} = \{1\}$. This contradicts that $\mathcal{O}$ is comeagre in $G \neq \{1\}$. $\square$

Among the groups that satisfy this condition are, for example, $\mathrm{Aut}([0,1],\lambda)$, $\mathrm{Isom}(\mathbb{U})$ and $\mathcal{U}(\ell_2)$.

The principal examples of Polish groups known with ample generics are non-Archimedean Polish groups, i.e., automorphism groups of countable structures. However, recent examples due independently to M. Malicki and F. LeMaître–A. Kaïchouh are not of this sort.

Theorem 15 (LeMaître–Kaïchouh)**.** *There are connected Polish groups with ample generics.*

On the other hand, P. Wesolek was recently able to show that no locally compact group has a comeagre conjugacy class, thus establishing that this is only a phenomenon for large Polish groups.

Theorem 16 (Wesolek)**.** *No non-trivial locally compact group has a comeagre conjugacy class.*

Department of Mathematics, Statistics, and Computer Science (M/C 249), University of Illinois at Chicago, 851 S. Morgan St., Chicago, IL 60607-7045, USA

E-mail address: rosendal.math@gmail.com

URL: http://homepages.math.uic.edu/~rosendal