Descriptive Set Theory and Model Theory
Third Lecture

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Thematic Program on Model Theory,
Notre Dame, June 2016

Overview of the four lectures:

1. Polish groups and ample generics (w/ A. S. Kechris)
2. Topological rigidity of automorphism groups (w/ A. S. Kechris)
3. Coarse geometry of Polish groups
4. Geometry of automorphism groups

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- provide a geometric picture of topological groups as we have of say f.g. groups, Lie groups and Banach spaces,
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- relate the model theoretical properties of countable structures with the geometry of their automorphism groups.

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From this, we define a *left-invariant* metric on Γ, called the word metric, by

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$$\frac{1}{K}\rho_S - C \;\leqslant\; \rho_{S'} \;\leqslant\; K\rho_S + C$$

for some constants K, C.

To see this, note that since S' is finite, there is some k so that every $s' \in S'$ can be written as a product

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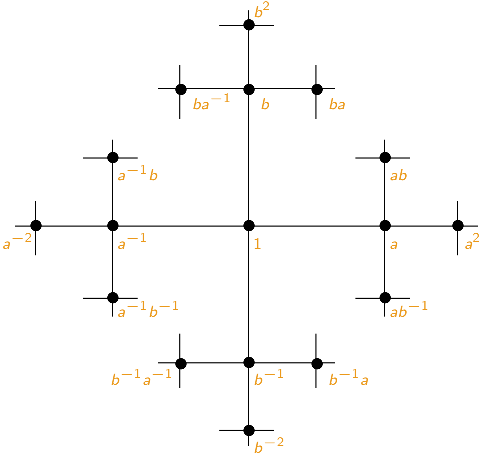
An identical argument shows the other inequality.

For example, let $\mathbb{F}_2$ be the free non-abelian group on generators a, b and set $S = \{1, a, b, a^{-1}, b^{-1}\}$.



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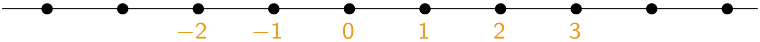

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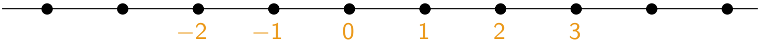

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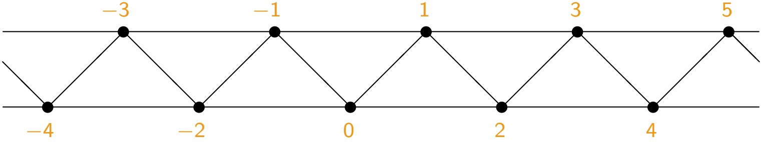

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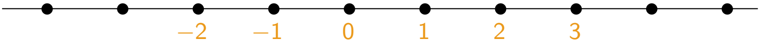

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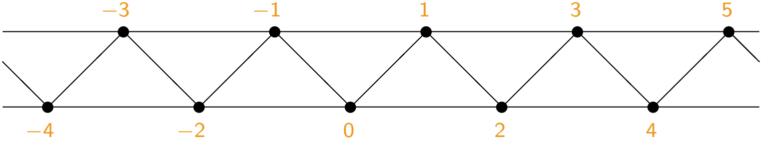

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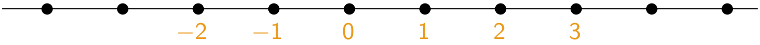

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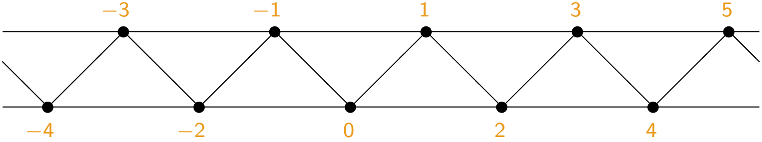

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So there is a clear large scale or quasi-metric geometry inherent to the group, independent of the choice of generating set.

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By the Baire category theorem, some power K^p has non-empty interior, so if K_1, K_2 are two such sets, then

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So, up to quasi-isometry, ρ_K is independent of K.

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Any two such metrics d and d' will be coarsely equivalent, that is,

$$\kappa\big(d(x,y)\big) \leqslant d'(x,y) \leqslant \omega\big(d(x,y)\big)$$

for functions $\kappa, \omega \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \kappa(t) = \infty$.

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Observe that this is weaker than being quasi-isometric, but still a non-trivial notion of equivalence between metrics.

Uniform vs coarse equivalence

To gain a better understanding of the notion of coarse equivalence, observe that two metrics d and d' are coarsely equivalent if and only if, for all sequences $(x_n), (y_n)$,

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For this reason, coarse equivalence is sometimes called uniform equivalence at infinity.

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A uniform space is intended to capture the idea of being uniformly close in a topological space and hence gives rise to concepts of Cauchy sequences and completeness.

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The main point here is that, for a uniform structure, we are interested in E_α for α small, but positive, while, for a coarse structure, α is often large, but finite.

Left-uniform structure on a topological group

If G is a topological group, its left-uniformity $\mathcal{U}_L$ is that generated by entourages of the form

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A basic theorem, due essentially to G. Birkhoff (fils) and S. Kakutani, is that

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d,$$

where the union is taken over all continuous left-invariant écarts d on G, i.e., so that

$$d(zx, zy) = d(x, y).$$

Left-coarse structure on a topological group

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Definition

If G is a topological group, its left-coarse structure $\mathcal{E}_L$ is given by

$$\mathcal{E}_L = \bigcap_{d} \mathcal{E}_d,$$

where the intersection is taken over all continuous left-invariant écarts d on G.

Coarsely bounded sets

The definition of the coarse structure $\mathcal{E}_L$ is not immediately transparent and it is thus useful to have alternate descriptions of it.

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A subset $A \subseteq G$ of a topological group is said to be coarsely bounded if

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One may easily show that the class of coarsely bounded sets is an ideal of subsets of G stable under the operations

$$A \mapsto A^{-1}, \quad (A, B) \mapsto AB \quad \text{and} \quad A \mapsto \overline{A}.$$

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

$$E_A = \{(x, y) \mid x^{-1}y \in A\},$$

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Though this applies to all topological groups, going forward we only consider Polish, that is, separable and completely metrisable topological groups.

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A subset A of a Polish group G is coarsely bounded if and only if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$A \subseteq (FV)^k.$$

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- More generally, in a locally compact σ-compact group, they are the relatively compact subsets.

- Similarly, in the underlying additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$, they are the norm bounded subsets.

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3. *G is locally bounded, i.e., there is a coarsely bounded identity neighbourhood $V \subseteq G$.*

In case d is a compatible left-invariant metric inducing the coarse structure on G, that is,

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For example, on a locally compact group the coarsely proper metrics are simply the proper metrics.

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In the last lecture, we shall present a number of locally bounded automorphism groups.

Quasimetric spaces

Recall that, in a finitely or compactly generated group, the word metrics of finite, respectively, compact generating sets are all quasi-isometric.

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On the other hand, in countable, locally compact or locally bounded Polish group, the coarsely proper metrics are only coarsely equivalent.

How about the word metrics given by coarsely bounded generating sets?

Proposition

Suppose that S and S' are two symmetric closed and coarsely bounded generating sets for a Polish group G.
Then the word metrics ρ_S and $\rho_{S'}$ are quasi-isometric.

Up to quasi-isometry of metrics, we may therefore unequivocally talk about the geometry induced by the word metrics of closed, coarsely bounded generating sets.

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Definition

A map $\phi\colon (M, d_M) \to (N, d_N)$ between metric spaces is said to be a quasi-isometric embedding if there are constants K and C so that

$$\frac{1}{K} \cdot d_M(x, y) - C \leqslant d_N(\phi x, \phi y) \leqslant K \cdot d_M(x, y) + C.$$

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Moreover, ϕ is a quasi-isometry if in addition $\phi[M]$ is cobounded in N, that is, $\sup_{y \in N} d_N(y, \phi[M]) < \infty$.

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It is a small exercise to see that being quasi-isometric is an equivalence relation of metric spaces.

Example

Let Fin denote the normal subgroup of S_∞ consisting of all finitely supported permutations.

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Also, as S_∞ is coarsely bounded, a simple calculation shows that the semidirect product

$$S_\infty \ltimes \texttt{Fin}$$

is quasi-isometric to $\texttt{Fin}$ equipped with the word metric ρ_S given by the generating set

$$S = \{\text{transpositions}\}.$$

Example

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As in the previous example, one may now see that the semidirect product

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Now consider the automorphism group $\mathrm{Aut}(T_\infty)$ and fix some vertex $\mathbf{r} \in T_\infty$.

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Thus,

$$\mathrm{Aut}(T_\infty) \ \approx_{QI} \ T_\infty \ \approx_{QI} \ S_\infty \ltimes \mathbb{F}_\infty.$$

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So, for example, $\mathrm{Isom}(\mathbb{QU})$ and $\mathrm{Aut}(T_\infty)$ are not quasi-isometric and therefore must be non-isomorphic groups.