An introduction to projective planes

Thematic Program in Commutative Algebra and its Interaction with Algebraic Geometry

In honor of Bernd Ulrich

Undergraduate Colloquium by Juan Migliore

May 31, 2019

1999

Introduction

Two important branches of mathematics are

- algebraic geometry
- combinatorics

These branches interweave and merge in many points. One of these is called projective geometry.

Introduction

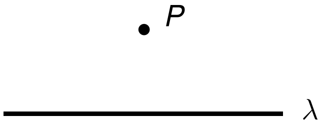
Two important branches of mathematics are

- algebraic geometry
- combinatorics

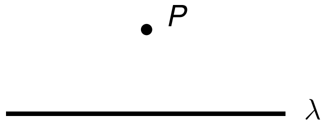
These branches interweave and merge in many points. One of these is called projective geometry.

Today we'll focus on the projective plane, looking at it from different points of view.

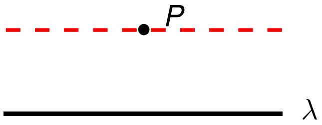
Question. Given a point, P, in the plane and a line, λ, that does not pass through P, Question. Given a point, P, in the plane and a line, λ, that does not pass through P,



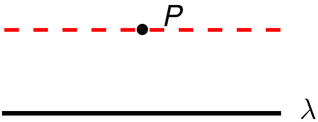
Question. Given a point, P, in the plane and a line, λ, that does not pass through P, how many lines pass through P and do not meet λ?



Question. Given a point, P, in the plane and a line, λ, that does not pass through P, how many lines pass through P and do not meet λ?

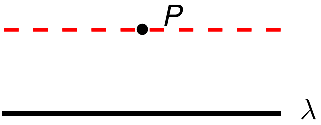


Question. Given a point, P, in the plane and a line, λ, that does not pass through P, how many lines pass through P and do not meet λ?



In high school we learn that the answer is one, the unique line parallel to λ.

Question. Given a point, P, in the plane and a line, λ, that does not pass through P, how many lines pass through P and do not meet λ?



In high school we learn that the answer is one, the unique line parallel to λ.

This is true for the Euclidean plane.

Euclid
c.330 – *c*.260 B.C.

(google images)

But there are non-Euclidean planes, and for these the answer can be “zero”, or it can be “infinitely many.”

But there are non-Euclidean planes, and for these the answer can be “zero”, or it can be “infinitely many.”

One of these is the projective plane.

Here we’ll see that there are no parallel lines, i.e. the answer to the question is “zero.”

But there are non-Euclidean planes, and for these the answer can be “zero”, or it can be “infinitely many.”

One of these is the projective plane.

Here we’ll see that there are no parallel lines, i.e. the answer to the question is “zero.”

So what’s the idea behind the projective plane? We’ll start with the real projective plane, $\mathbb{P}^2_{\mathbb{R}}$.

But there are non-Euclidean planes, and for these the answer can be "zero", or it can be "infinitely many."

One of these is the projective plane.

Here we'll see that there are no parallel lines, i.e. the answer to the question is "zero."

So what's the idea behind the projective plane? We'll start with the real projective plane, $\mathbb{P}^2_{\mathbb{R}}$.

Look at this pair of parallel lines:

You can imagine that if there were points infinitely far from the camera, these two “lines” would meet at one of those points.

You can imagine that if there were points infinitely far from the camera, these two “lines” would meet at one of those points.

And what if there are more than two parallel lines?

All the parallel lines meet in the same “point at infinity!”

All the parallel lines meet in the same “point at infinity!”

This is the point of view of projective geometry (at least over the reals).

All the parallel lines meet in the same “point at infinity!”

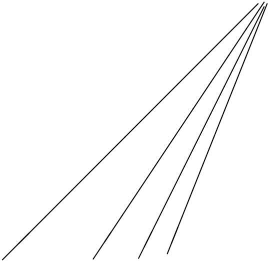
This is the point of view of projective geometry (at least over the reals).

In a sec we’ll look at an axiomatic approach to projective planes, giving rise to different models that satisfy those axioms.

But first let’s follow up this way of getting the real projective plane from the familiar Euclidean plane by adding points at infinity.

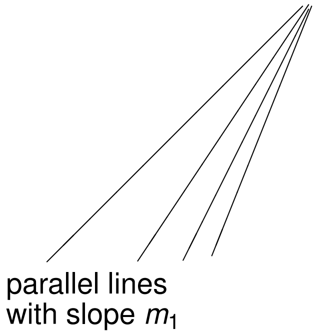
For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



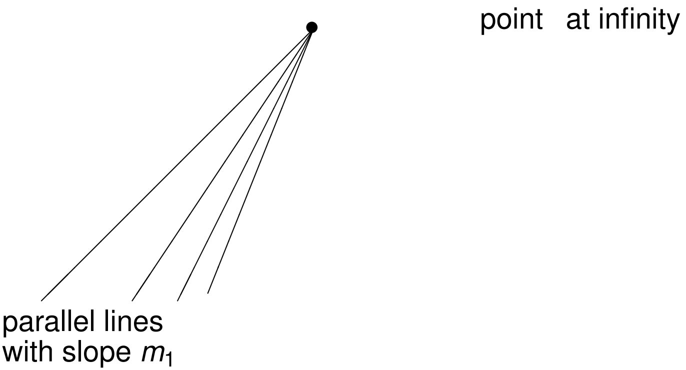
parallel lines
with slope m_1

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



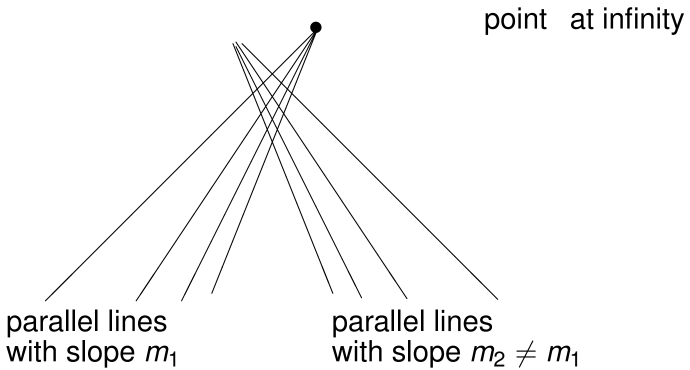
we add a point at infinity where all the lines in the family meet.

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



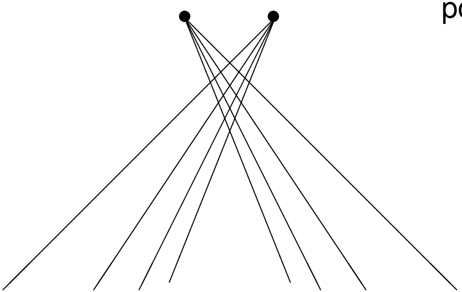
we add a point at infinity where all the lines in the family meet.

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



we add a point at infinity where all the lines in the family meet.

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



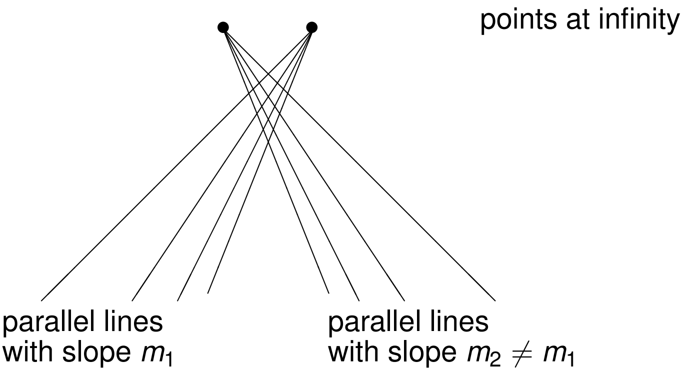
points at infinity

parallel lines
with slope m_1

parallel lines
with slope $m_2 \neq m_1$

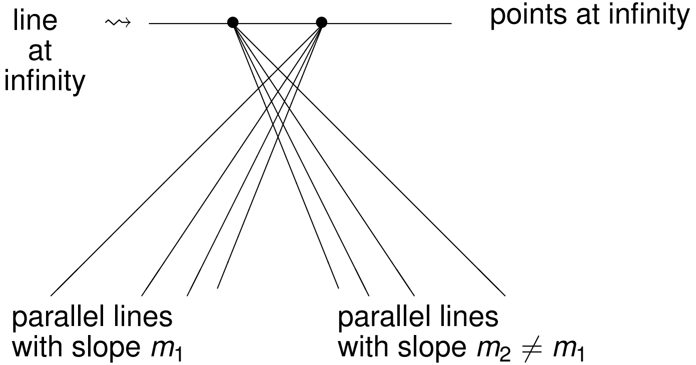
we add a point at infinity where all the lines in the family meet.

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



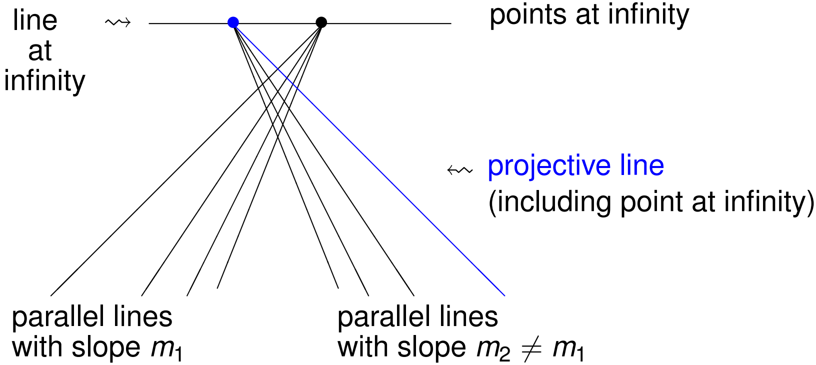
we add a point at infinity where all the lines in the family meet. The set of all points at infinity defines the line at infinity, one for each slope (including ∞).

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



we add a point at infinity where all the lines in the family meet. The set of all points at infinity defines the line at infinity, one for each slope (including ∞).

For each family of parallel lines in the Euclidean plane (i.e. lines that don't meet)



we add a point at infinity where all the lines in the family meet. The set of all points at infinity defines the line at infinity, one for each slope (including ∞).

$\mathbb{P}^2_{\mathbb{R}}$ is the union of the usual Euclidean plane and the (projective) line at infinity:

$\mathbb{P}^2_{\mathbb{R}}$ is the union of the usual Euclidean plane and the (projective) line at infinity:

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}.$$

$\mathbb{P}^2_{\mathbb{R}}$ is the union of the usual Euclidean plane and the (projective) line at infinity:

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}.$$

We've constructed the real projective plane, denoted $\mathbb{P}^2_{\mathbb{R}}$

$\mathbb{P}^2_{\mathbb{R}}$ is the union of the usual Euclidean plane and the (projective) line at infinity:

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}.$$

We've constructed the real projective plane, denoted $\mathbb{P}^2_{\mathbb{R}}$ (the "2" is because the plane has dimension 2, and the $\mathbb{R}$ because Euclidean geometry is based on the real numbers).

$\mathbb{P}^2_{\mathbb{R}}$ is the union of the usual Euclidean plane and the (projective) line at infinity:

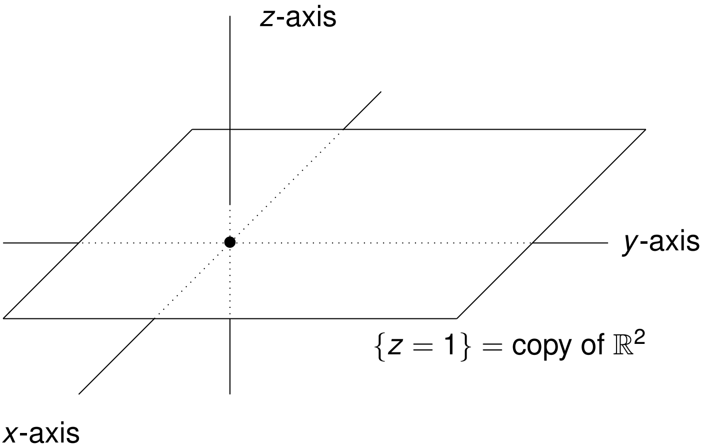
$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}.$$

We've constructed the real projective plane, denoted $\mathbb{P}^2_{\mathbb{R}}$ (the "2" is because the plane has dimension 2, and the $\mathbb{R}$ because Euclidean geometry is based on the real numbers).

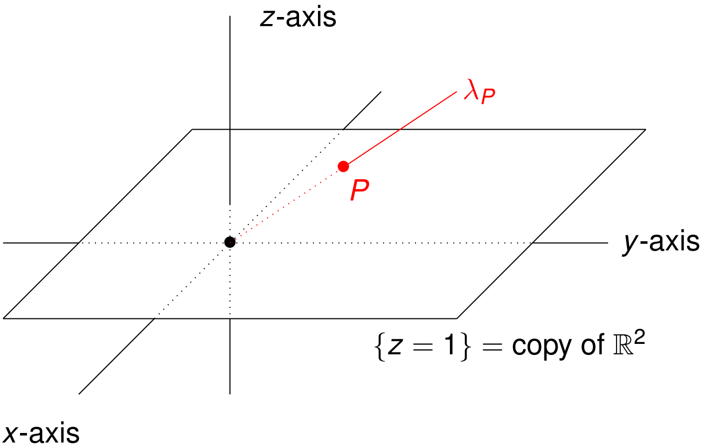
Let's look at a different approach to get $\mathbb{P}^2_{\mathbb{R}}$.

Alternatively, we can define $\mathbb{P}^2_{\mathbb{R}}$ as the set of lines through the origin in $\mathbb{R}^3$.

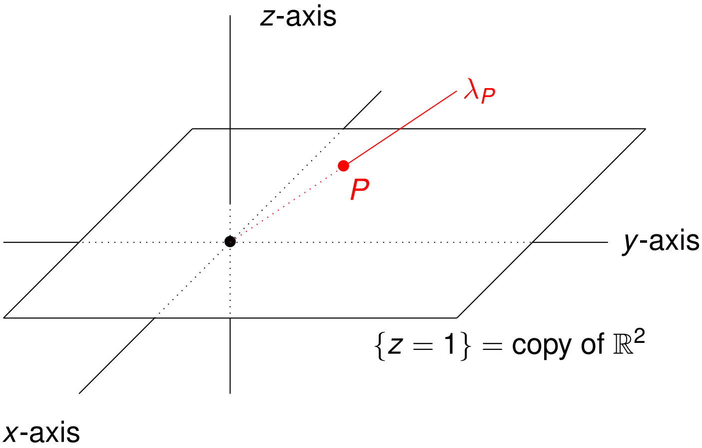
Alternatively, we can define $\mathbb{P}^2_{\mathbb{R}}$ as the set of lines through the origin in $\mathbb{R}^3$.

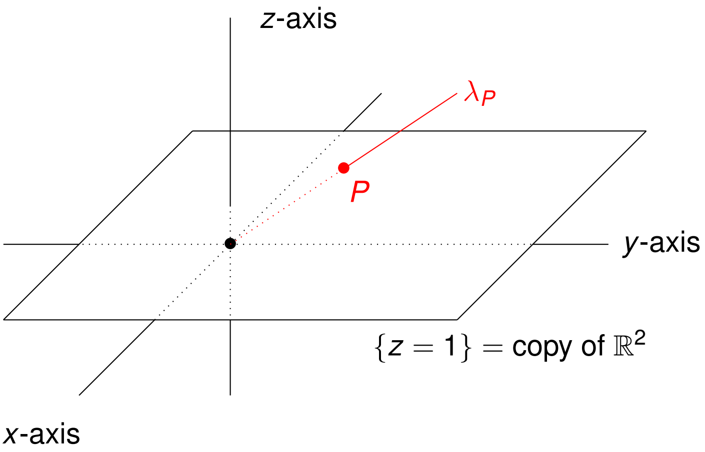


Alternatively, we can define $\mathbb{P}^2_{\mathbb{R}}$ as the set of lines through the origin in $\mathbb{R}^3$.

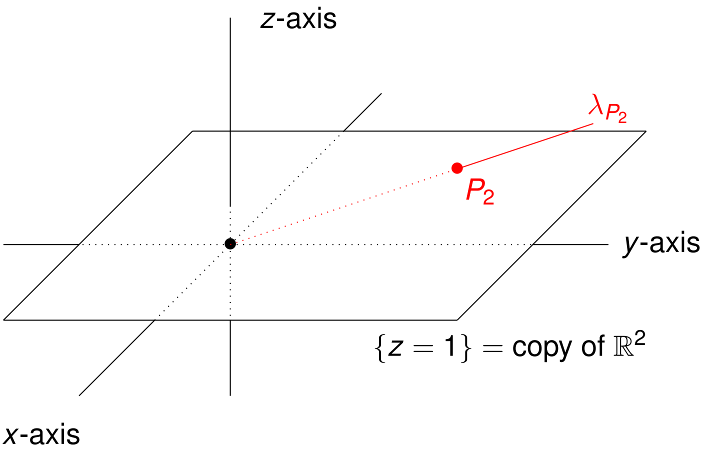


Line λ_P not on (x, y)-plane $\leftrightarrow P \in \mathbb{R}^2$.

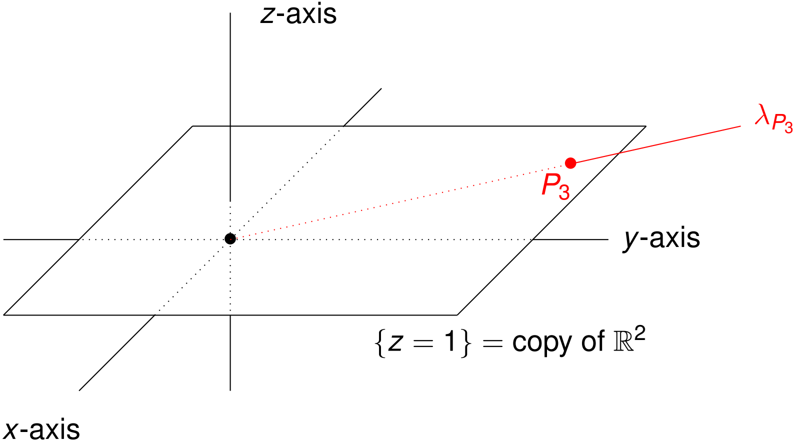

z-axis
λ_P
P
y-axis
$\{z = 1\}$ = copy of $\mathbb{R}^2$
x-axis



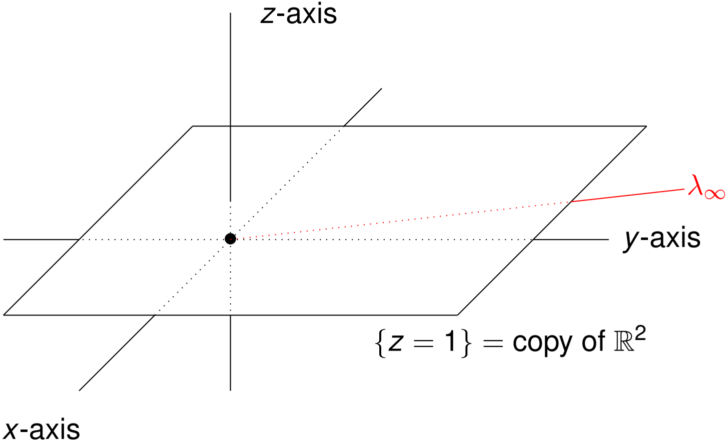
As $P \rightsquigarrow \infty$ in $\mathbb{R}^2$ in any direction,



As $P \rightsquigarrow \infty$ in $\mathbb{R}^2$ in any direction,



As $P \rightsquigarrow \infty$ in $\mathbb{R}^2$ in any direction,



As $P \rightsquigarrow \infty$ in $\mathbb{R}^2$ in any direction, $\lambda_P \rightsquigarrow$ corresponding line through the origin in the (x, y)-plane.

We can identify

$$\{ \text{ lines through the origin in the } (x, y)\text{-plane } \}$$

with the (projective) line at infinity (one point for each “slope”).

We can identify

$$\{ \text{ lines through the origin in the } (x, y)\text{-plane } \}$$

with the (projective) line at infinity (one point for each "slope").

So again, $\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}$

(thinking of $\mathbb{R}^2$ as the plane $\{z = 1\}$ in $\mathbb{R}^3$, and $\mathbb{P}^1_{\mathbb{R}}$ as the set of lines through the origin in $\mathbb{R}^2$).

We can identify

$$\{ \text{ lines through the origin in the } (x, y)\text{-plane } \}$$

with the (projective) line at infinity (one point for each "slope").

So again, $\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}$

(thinking of $\mathbb{R}^2$ as the plane $\{z = 1\}$ in $\mathbb{R}^3$, and $\mathbb{P}^1_{\mathbb{R}}$ as the set of lines through the origin in $\mathbb{R}^2$).

Aside: We can also think of $\mathbb{P}^2_{\mathbb{R}}$ topologically as a sphere with antipodal points identified, but we'll skip this point of view.

Next: such a projective plane comes equipped with a set of coordinates over $\mathbb{R}$.

Next: such a projective plane comes equipped with a set of coordinates over $\mathbb{R}$.

A line through the origin passing through a point $(a, b, c) \in \mathbb{R}^3$ $((a, b, c) \neq (0, 0, 0))$ can be described as

$$\{(ta, tb, tc) \mid t \in \mathbb{R}\}.$$

Next: such a projective plane comes equipped with a set of coordinates over $\mathbb{R}$.

A line through the origin passing through a point $(a, b, c) \in \mathbb{R}^3$ $((a, b, c) \neq (0, 0, 0))$ can be described as

$$\{(ta, tb, tc) \mid t \in \mathbb{R}\}.$$

So

$$\mathbb{P}^2_{\mathbb{R}} = \left\{ [a, b, c] \;\middle|\; \begin{array}{c} (a, b, c) \neq (0, 0, 0) \text{ and} \\ \\ [a, b, c] = [ta, tb, tc] \;\forall t \in \mathbb{R} \end{array} \right\}$$

(E.g. $[1, 2, 3] = [2, 4, 6]$.)

Remark.

1. Given any field k (e.g. $k = \mathbb{R}$), you can construct a projective plane $\mathbb{P}_k^2$ in an analogous way, as the lines through the origin in k^3.

Remark.

1. Given any field k (e.g. $k = \mathbb{R}$), you can construct a projective plane $\mathbb{P}^2_k$ in an analogous way, as the lines through the origin in k^3.

   Such a projective plane comes equipped with a set of coordinates over k in the same way.

Remark.

1. Given any field k (e.g. $k = \mathbb{R}$), you can construct a projective plane $\mathbb{P}^2_k$ in an analogous way, as the lines through the origin in k^3.

   Such a projective plane comes equipped with a set of coordinates over k in the same way.

2. We can also define $\mathbb{P}^n_k$ (where k is any field) as the set of lines through the origin in k^{n+1} and get a coordinate system in the same way.

Remark.

1. Given any field k (e.g. $k = \mathbb{R}$), you can construct a projective plane $\mathbb{P}^2_k$ in an analogous way, as the lines through the origin in k^3.

   Such a projective plane comes equipped with a set of coordinates over k in the same way.

2. We can also define $\mathbb{P}^n_k$ (where k is any field) as the set of lines through the origin in k^{n+1} and get a coordinate system in the same way.

   This allows us to study algebraic varieties by considering the vanishing loci of homogeneous polynomials.

We can then study projective algebraic geometry, including homogeneous ideals and commutative algebra.

We can then study projective algebraic geometry, including homogeneous ideals and commutative algebra.

This is a very beautiful and well-traveled road. But we won't be following this road today.

We can then study projective algebraic geometry, including homogeneous ideals and commutative algebra.

This is a very beautiful and well-traveled road. But we won't be following this road today.

Play

I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I –
I took the one less traveled by,
And that has made all the difference.

from: Robert Frost, "The Road Not Taken"

We can then study projective algebraic geometry, including homogeneous ideals and commutative algebra.

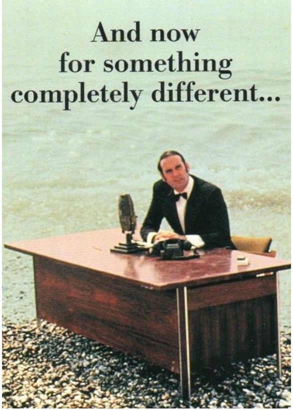
This is a very beautiful and well-traveled road. But we won't be following this road today.

Play

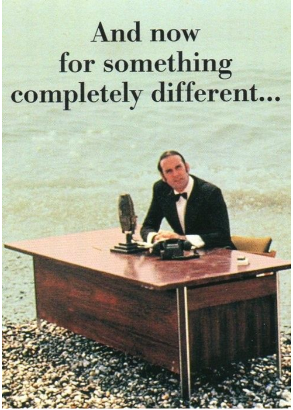
I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I –
I took the one less traveled by,
And that has made all the difference.

from: Robert Frost, "The Road Not Taken"

Or to put it less poetically...



(with apologies to Monty Python).



(with apologies to Monty Python).

Let's give the axiomatic definition of a projective plane, which includes the real projective plane as a special case.

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;
- a collection of subsets, called lines;

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;
- a collection of subsets, called lines;

such that the following axioms hold:

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;
- a collection of subsets, called lines;

such that the following axioms hold:

P1. Given any two distinct points there is exactly one line containing them.

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;
- a collection of subsets, called lines;

such that the following axioms hold:

P1. Given any two distinct points there is exactly one line containing them.

P2. Any two distinct lines meet in exactly one point.

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;
- a collection of subsets, called lines;

such that the following axioms hold:

P1. Given any two distinct points there is exactly one line containing them.

P2. Any two distinct lines meet in exactly one point.

P3. There exist at least three points not on the same line.

The axiomatic definition of a projective plane

A projective plane, denoted $\mathbb{P}^2$, is defined as follows.

- a set S, whose elements are called points;
- a collection of subsets, called lines;

such that the following axioms hold:

P1. Given any two distinct points there is exactly one line containing them.

P2. Any two distinct lines meet in exactly one point.

P3. There exist at least three points not on the same line.

P4. Each line contains at least three points.

Notice that these axioms don't assume the real numbers $\mathbb{R}$, or even any underlying field.

Notice that these axioms don't assume the real numbers $\mathbb{R}$, or even any underlying field.

Exercise. Verify these axioms for our real projective plane $\mathbb{P}^2_{\mathbb{R}}$:

So our real projective plane $\mathbb{P}^2_{\mathbb{R}}$ is, in fact, a projective plane.

A good reference for projective planes and related topics: G. Eric Moorhouse, “Incidence Geometry” (available online).

Also, there are many good textbooks, including a couple by Robin Hartshorne.

Even Wikipedia has interesting information.

A good reference for projective planes and related topics: G. Eric Moorhouse, “Incidence Geometry” (available online).

Also, there are many good textbooks, including a couple by Robin Hartshorne.

Even Wikipedia has interesting information.

Now let’s look at

finite projective planes!

A good reference for projective planes and related topics: G. Eric Moorhouse, “Incidence Geometry” (available online).

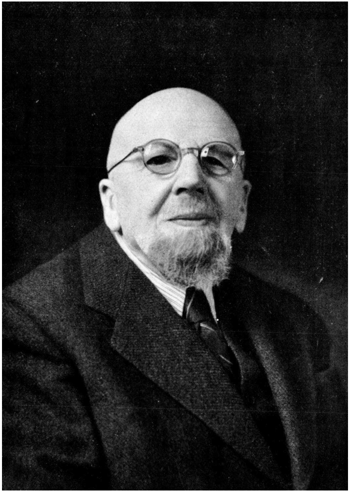
Also, there are many good textbooks, including a couple by Robin Hartshorne.

Even Wikipedia has interesting information.

Now let’s look at

finite projective planes!

One of these was discovered by Gino Fano.



Gino Fano
1871 – 1952
(Biblioteca Digitale Italiana di Matematica)

The Fano projective plane

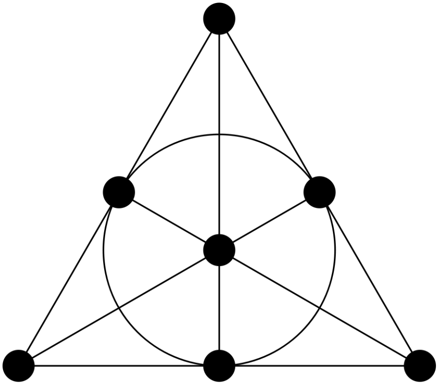
The Fano projective plane

Oops, sorry! That's the symbol for the Deathly Hallows from Harry Potter.

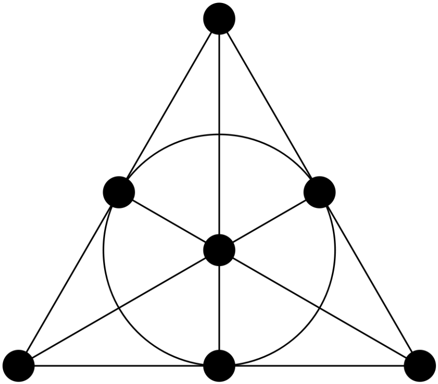
The Fano projective plane

Oops, sorry! That's the symbol for the Deathly Hallows from Harry Potter. Here's the Fano plane:

Example. The Fano projective plane (picture from Wikipedia):

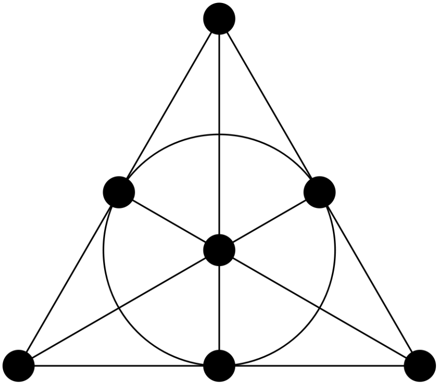


Example. The Fano projective plane (picture from Wikipedia):



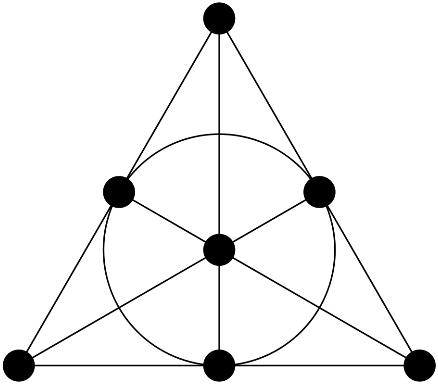
The plane consists of exactly 7 points.

Example. The Fano projective plane (picture from Wikipedia):



The plane consists of exactly 7 points. The lines are subsets of three points, as indicated (there are 7 of them).

Example. The Fano projective plane (picture from Wikipedia):



The plane consists of exactly 7 points. The lines are subsets of three points, as indicated (there are 7 of them).

I leave it to you to verify axioms P1 – P4.

Facts about finite projective planes

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!)

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!) Let's denote this number by $d + 1$. The number d is the order of the projective plane.

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!) Let's denote this number by $d + 1$. The number d is the order of the projective plane.

2. There are $d + 1$ lines through every point.

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!) Let's denote this number by $d+1$. The number d is the order of the projective plane.
2. There are $d+1$ lines through every point.
3. The plane contains d^2+d+1 points.

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!) Let's denote this number by $d + 1$. The number d is the order of the projective plane.
2. There are $d + 1$ lines through every point.
3. The plane contains $d^2 + d + 1$ points.
4. The plane also has $d^2 + d + 1$ lines.

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!) Let's denote this number by $d+1$. The number d is the order of the projective plane.
2. There are $d+1$ lines through every point.
3. The plane contains d^2+d+1 points.
4. The plane also has d^2+d+1 lines. (Notice the duality.)

Facts about finite projective planes

Every finite projective plane satisfies the following properties (derived from the axioms):

1. In a given finite projective plane, any two lines have the same number of points. (!!!!) Let's denote this number by $d + 1$. The number d is the order of the projective plane.

2. There are $d + 1$ lines through every point.

3. The plane contains $d^2 + d + 1$ points.

4. The plane also has $d^2 + d + 1$ lines. (Notice the duality.)

5. It's an open question to know exactly for which integers d there exists a projective plane of order d. Let's see some things that are known, especially for small d.

(a) If p is a prime then there exists a projective plane of order p^n for any $n \geq 1$. (Build it from k^3 as before.)

(a) If p is a prime then there exists a projective plane of order p^n for any $n \geq 1$. (Build it from k^3 as before.)

(b) So orders 2, 3, 4, 5, 7, 8, 9, 11, 13 exist.

(a) If p is a prime then there exists a projective plane of order p^n for any $n \geq 1$. (Build it from k^3 as before.)

(b) So orders 2, 3, 4, 5, 7, 8, 9, 11, 13 exist.

(c) Famous conjecture: Every finite projective plane has order p^n for some prime p and some integer $n \geq 1$.

(a) If p is a prime then there exists a projective plane of order p^n for any $n \geq 1$. (Build it from k^3 as before.)

(b) So orders 2, 3, 4, 5, 7, 8, 9, 11, 13 exist.

(c) Famous conjecture: Every finite projective plane has order p^n for some prime p and some integer $n \geq 1$.

(d) (Bruck-Ryser theorem (1949).) If a finite projective plane of order d exists, and d is congruent to 1 or 2 (mod 4), then d must be the sum of two squares.

(a) If p is a prime then there exists a projective plane of order p^n for any $n \geq 1$. (Build it from k^3 as before.)

(b) So orders 2, 3, 4, 5, 7, 8, 9, 11, 13 exist.

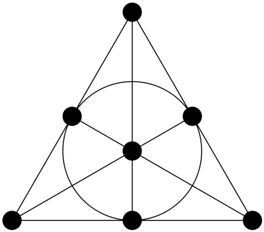
(c) Famous conjecture: Every finite projective plane has order p^n for some prime p and some integer $n \geq 1$.

(d) (Bruck-Ryser theorem (1949).) If a finite projective plane of order d exists, and d is congruent to 1 or 2 (mod 4), then d must be the sum of two squares.

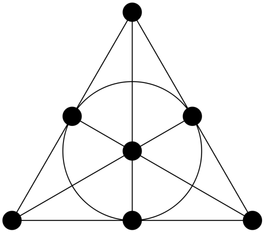
(e) That rules out order 6. Order 10 was ruled out by a complicated computer check. First open case: order 12.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.

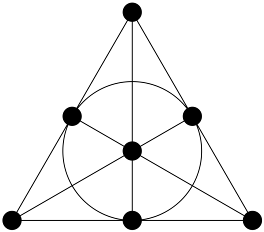


Example. Let's verify Facts # 1 – 4 for the Fano projective plane.



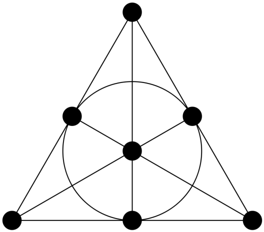
1. Each line has $d + 1 = 3$ points.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.



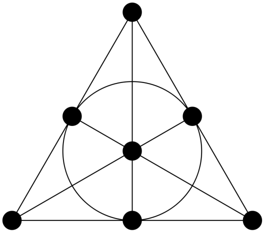
1. Each line has $d + 1 = 3$ points. So the Fano plane has order $d = 2$.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.



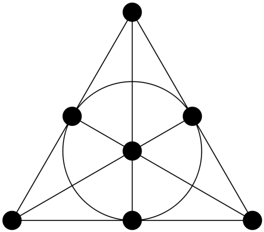
1. Each line has $d + 1 = 3$ points. So the Fano plane has order $d = 2$.
2. Through each point pass $d + 1 = 3$ lines.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.



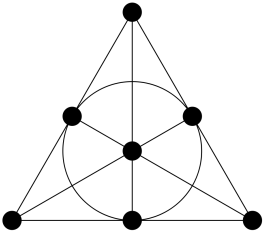
1. Each line has $d + 1 = 3$ points. So the Fano plane has order $d = 2$.
2. Through each point pass $d + 1 = 3$ lines.
3. The plane has $q = 2^2 + 2 + 1 = 7$ points.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.



1. Each line has $d+1=3$ points. So the Fano plane has order $d=2$.
2. Through each point pass $d+1=3$ lines.
3. The plane has $q=2^2+2+1=7$ points.
4. The plane has $q=7$ lines.

Example. Let's verify Facts # 1 – 4 for the Fano projective plane.



1. Each line has $d+1=3$ points. So the Fano plane has order $d=2$.
2. Through each point pass $d+1=3$ lines.
3. The plane has $q=2^2+2+1=7$ points.
4. The plane has $q=7$ lines.

In fact, the Fano plane is $\mathbb{P}^2_{\mathbb{Z}_2}$.

Pure O-sequences

The study of finite projective planes can be approached through algebraic combinatorics.

Pure O-sequences

The study of finite projective planes can be approached through algebraic combinatorics.

One way that this is realized is through the notion of the pure O-sequence corresponding to our projective plane of order d.

Pure O-sequences

The study of finite projective planes can be approached through algebraic combinatorics.

One way that this is realized is through the notion of the pure O-sequence corresponding to our projective plane of order d.

First let's define pure O-sequences in general, then say which ones correspond to finite projective planes.

Let $\mathcal{M}_e = \{m_1, \dots, m_r\}$ be a set of distinct monomials of the same degree e (not necessarily squarefree in general) in some polynomial ring $k[x_1, \dots, x_n]$.

Let $\mathcal{M}_e = \{m_1, \dots, m_r\}$ be a set of distinct monomials of the same degree e (not necessarily squarefree in general) in some polynomial ring $k[x_1, \dots, x_n]$.

For each $1 \le i \le e$, let $\mathcal{M}_i$ be the monomials of degree i that divide at least one of $m_1, \dots, m_r$.

Let $\mathcal{M}_e = \{m_1, \ldots, m_r\}$ be a set of distinct monomials of the same degree e (not necessarily squarefree in general) in some polynomial ring $k[x_1, \ldots, x_n]$.

For each $1 \le i \le e$, let $\mathcal{M}_i$ be the monomials of degree i that divide at least one of $m_1, \ldots, m_r$.

The pure O-sequence associated to $\mathcal{M}_e$ is the sequence

$$(1, |\mathcal{M}_1|, \ldots, |\mathcal{M}_{e-1}|, |\mathcal{M}_e|).$$

Example. Let $R = k[x, y, z]$ and $e = 3$. Let

$$\mathcal{M}_3 = \{x^3, xyz, x^2y, y^3\}.$$

Example. Let $R = k[x, y, z]$ and $e = 3$. Let

$$\mathcal{M}_3 = \{x^3, xyz, x^2y, y^3\}.$$

Then

$$\mathcal{M}_2 = \{x^2, xy, xz, yz, y^2\}$$

Example. Let $R = k[x, y, z]$ and $e = 3$. Let

$$\mathcal{M}_3 = \{x^3, xyz, x^2y, y^3\}.$$

Then

$$\begin{aligned} \mathcal{M}_2 &= \{x^2, xy, xz, yz, y^2\} \\ \mathcal{M}_1 &= \{x, y, z\} \end{aligned}$$

Example. Let $R = k[x, y, z]$ and $e = 3$. Let

$$\mathcal{M}_3 = \{x^3, xyz, x^2y, y^3\}.$$

Then

$$\begin{aligned} \mathcal{M}_2 &= \{x^2, xy, xz, yz, y^2\} \\ \mathcal{M}_1 &= \{x, y, z\} \end{aligned}$$

leading to the pure O-sequence

$$(1, 3, 5, 4).$$

Remark.

Algebraic point of view:

For each degree, collect the monomials not in the corresponding list $\mathcal{M}_i$.

Together these generate a monomial ideal, whose quotient has Hilbert function equal to the pure O-sequence.

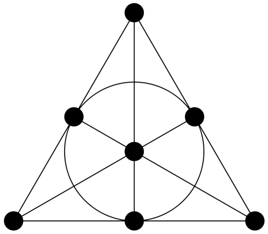
Remark.

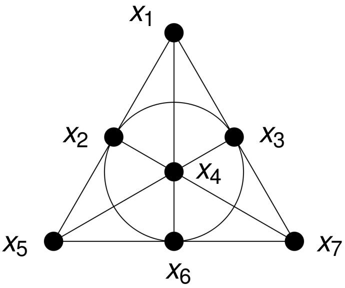
Algebraic point of view:

For each degree, collect the monomials not in the corresponding list $\mathcal{M}_i$.

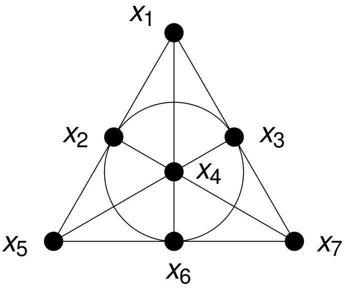
Together these generate a monomial ideal, whose quotient has Hilbert function equal to the pure O-sequence.

Let's see how we can associate a pure O-sequence to a finite projective plane, using the Fano plane as an example.



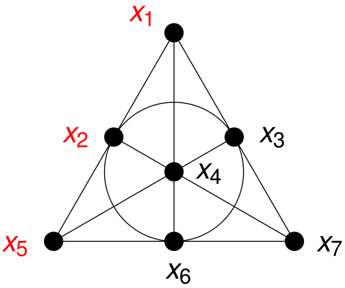


1. Label each point with a different variable. Recall that the plane has $q = 2^2 + 2 + 1 = 7$ points and 7 lines, and order $d = 2$.



1. Label each point with a different variable. Recall that the plane has $q = 2^2 + 2 + 1 = 7$ points and 7 lines, and order $d = 2$.

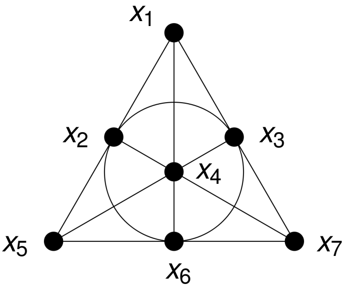
2. Collect and count the (squarefree) monomials of degree $d + 1 = 3$ corresponding to points on a line:



1. Label each point with a different variable. Recall that the plane has $q = 2^2 + 2 + 1 = 7$ points and 7 lines, and order $d = 2$.

2. Collect and count the (squarefree) monomials of degree $d + 1 = 3$ corresponding to points on a line:

$x_1 x_2 x_5$



1. Label each point with a different variable. Recall that the plane has $q = 2^2 + 2 + 1 = 7$ points and 7 lines, and order $d = 2$.

2. Collect and count the (squarefree) monomials of degree $d + 1 = 3$ corresponding to points on a line:

$$x_1x_2x_5,\ x_1x_4x_6,\ x_1x_3x_7,\ x_2x_4x_7,\ x_2x_3x_6,\ x_3x_4x_5,\ x_5x_6x_7$$

3. These monomials will be our set $\mathcal{M}_3$ generating our pure O-sequence. Note $|\mathcal{M}_3| = 7$ (there are 7 lines).

3. These monomials will be our set $\mathcal{M}_3$ generating our pure O-sequence. Note $|\mathcal{M}_3| = 7$ (there are 7 lines).

4. Since two points lie on exactly one line, we get

$$|\mathcal{M}_2| = 7 \cdot \binom{3}{2} = 7 \cdot 3 = 21$$

3. These monomials will be our set $\mathcal{M}_3$ generating our pure O-sequence. Note $|\mathcal{M}_3| = 7$ (there are 7 lines).

4. Since two points lie on exactly one line, we get

$$|\mathcal{M}_2| = 7 \cdot \binom{3}{2} = 7 \cdot 3 = 21 = \binom{7}{2}.$$

3. These monomials will be our set $\mathcal{M}_3$ generating our pure O-sequence. Note $|\mathcal{M}_3| = 7$ (there are 7 lines).

4. Since two points lie on exactly one line, we get

$$|\mathcal{M}_2| = 7 \cdot \binom{3}{2} = 7 \cdot 3 = 21 = \binom{7}{2}.$$

5. Since there are 7 points, we get

$$|\mathcal{M}_1| = 7.$$

3. These monomials will be our set $\mathcal{M}_3$ generating our pure O-sequence. Note $|\mathcal{M}_3| = 7$ (there are 7 lines).

4. Since two points lie on exactly one line, we get

$$|\mathcal{M}_2| = 7 \cdot \binom{3}{2} = 7 \cdot 3 = 21 = \binom{7}{2}.$$

5. Since there are 7 points, we get

$$|\mathcal{M}_1| = 7.$$

   This leads to the pure O-sequence

$$(1, 7, 21, 7).$$

   This is the pure O-sequence associated to the Fano plane.

Fact: A projective plane of order d exists if and only if

$$\left(1, q, q\binom{d+1}{2}, q\binom{d+1}{3}, \ldots, q\binom{d+1}{d}, q\right).$$

is a pure O-sequence, where $q = d^2 + d + 1$, $d \geq 2$.

Fact: A projective plane of order d exists if and only if

$$\left(1, q, q\binom{d+1}{2}, q\binom{d+1}{3}, \ldots, q\binom{d+1}{d}, q\right).$$

is a pure O-sequence, where $q = d^2 + d + 1$, $d \geq 2$.

Note the monomials generating such a sequence must be squarefree.

Fact: A projective plane of order d exists if and only if

$$\left(1, q, q\binom{d+1}{2}, q\binom{d+1}{3}, \ldots, q\binom{d+1}{d}, q\right).$$

is a pure O-sequence, where $q = d^2 + d + 1$, $d \geq 2$.

Note the monomials generating such a sequence must be squarefree.

Easy exercise:

$$q\binom{d+1}{2} = \binom{q}{2}.$$

Fact: A projective plane of order d exists if and only if

$$\left(1, q, q\binom{d+1}{2}, q\binom{d+1}{3}, \ldots, q\binom{d+1}{d}, q\right).$$

is a pure O-sequence, where $q = d^2 + d + 1$, $d \geq 2$.

Note the monomials generating such a sequence must be squarefree.

Easy exercise:

$$q\binom{d+1}{2} = \binom{q}{2}.$$

The above fact is not a trivial argument, although some of our facts are immediate (q points, q lines, . . .).

This provides an algebraic approach to finite projective planes.

See for instance

D. Cook II, J.M., U. Nagel and F. Zanello, *An algebraic approach to finite projective planes*, Journal of Algebraic Combinatorics **43** (2016), 495–519.

We described algebraic properties of algebras associated to finite projective planes, obtained as above.

Some of these properties are related to the characteristic of the field defining the polynomial ring in which we place our monomials.

Email me at

migliore.1@nd.edu

if you want either the CMNZ paper or my slides.

Thank you!

Happy Birthday, Bernd!