Exact Discontinuous Solutions of Exner's Bed Evolution Model: Simple Theory for Sediment Bores

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Abstract: Determining the evolution of the bed of a river or channel due to the transport of sediment was first examined in a theoretical context by Exner in 1925. In his work, Exner presents a simplified bed evolution model derived from the conservation of fluid mass and an "erosion" equation that is commonly referred to as the sediment continuity or Exner equation. Given that Exner's model takes the form of a nonlinear hyperbolic equation, one expects, depending on the given initial condition of the bed, the formation of discontinuities in the solution in finite time. The analytical solution provided by Exner for his model is the so-called classical or genuine solution of the initial-value problem, which is valid while the solution is continuous. In this paper, using the general theory of nonlinear hyperbolic equations, we consider generalized solutions of Exner's classic bed evolution model thereby developing a simple theory for the formation and propagation of discontinuities in the bed or so-called sediment bores.

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Introduction

Fluid flowing over the bed of a river, channel, or estuary acts to deform the shape of the bed by transporting sediment. Determining the evolution of a given bed configuration due to the motion of the fluid and the resulting sediment transport was first examined in a theoretical context by Exner (1925). Exner's work in this area can properly be considered a classical treatment of the problem. It appears in many texts (see, for example, Graf 1971; Leliavsky 1955; Raudkivi 1967; Sleath 1984; Yang 1996), and a generalization of the "erosion equation" presented in his work, which is a statement of the conservation of sediment mass, is often referred to in the literature as the Exner equation of sediment continuity or simply the Exner equation. It is the foundation of estuarine and river morphodynamics.

From equations for the conservation of fluid and sediment mass, and through a number of simplifying assumptions, Exner derives a simplified bed evolution model that takes the form of a nonlinear hyperbolic scalar equation. Despite the relative simplicity of this model, the results obtained are, to a limited extent, in good agreement with what is observed in nature. Examples provided by Exner to this effect are the prediction of dune formation from an initially symmetric mound or hump and the prediction of the patterns of scour and deposition that occur from changes in river or channel cross sections.

The analytical solution provided by Exner for his model is the so-called classical or genuine solution of the initial-value problem, which is valid while the solution is continuous. However, it is well known that solutions of nonlinear hyperbolic equations, depending on the given initial conditions, may develop discontinuities in finite time and a classical solution ceases to exist. Since Exner's equation is a conservation law (the conservation of sediment mass), which is fundamentally an integral relation, it may be satisfied by solutions that are discontinuous, so-called weak or generalized solutions (for the general theory, see, for example, Lax 1973; Whithman 1974). This is in direct analogy to the hyperbolic conservation equations of gas dynamics or shallow water theory, whose solutions are well known to form discontinuities that satisfy a given entropy condition, referred to as shock waves in gas dynamics and hydraulic jumps (stationary discontinuities) or bores (advancing discontinuities) in shallow water theory.

Needham and Hey (1991) and Zanre and Needham (1996) consider the formation of discontinuities in alluvial sediment beds by analyzing a system of equations comprised of the shallow water equations and the sediment continuity equation for bed load. Specifically, they show that this system of equations is hyperbolic and by considering discontinuous solutions of this system they develop a theory for what they call "sediment bores," in analogy to hydraulic bores of classical shallow water theory. Mathematically, sediment bores are represented as discontinuities in the bed. Physically, as with hydraulic bores in the free surface, sediment bores in the bed are characterized by a rapid change in the vertical over a relatively short horizontal distance. They form as a result of sustained higher upstream than downstream sediment transport rates and will form, for example, as flow enters into a dredged section or as the result of a dam break or landslide that suddenly injects a large volume of sediment into a river or channel. In the geophysical literature, this type of bed form is often referred to as a planar delta form. Needham and Hey (1991), and the references therein, describe some particular cases of this

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type of bed form occurring in rivers. Fraccarollo and Capart (2002) consider formations of this type (referred to as erosional bores) in both a theoretical and experimental framework in the context of an erosional dam break problem. Here, as in Needham and Hey (1991) and in analogy to hydraulic bores, the term sediment bore will be retained.

In this paper, we develop a similar, yet simpler, theory for the formation and propagation of sediment bores, not by considering the full system of three equations, but by considering discontinuous solutions to Exner's simplified bed evolution model, which captures the key features of the bed evolution process. This is much in the same way that the inviscid Burgers equation is the simplest model equation that captures some of the key features of gas dynamics. This analogy, of course, between Burgers' and Exner's equations, is especially useful given the fact that both are nonlinear hyperbolic scalar equations derived from conservation laws, and can therefore be analyzed using the same mathematical theory. Much of this theory was originally developed in the context of studying the inviscid Burgers equation and then later generalized to nonlinear hyperbolic scalar equations with arbitrary convex flux functions (see, for example, Lax 1973, and the references therein). We apply this general theory to the analysis of Exner's equation, thereby developing a simple theory for sediment bores. The results of this analysis are qualitatively similar to those obtained by Needham and Hey (1991) and Zanre and Needham (1996) and provide a degree of physical insight into the problem that can often be lost in the tedious mathematical details of an analysis of the full system of governing equations. The analysis considered here provides a simplified setting to examine the complex problem of bed evolution and will contribute to the overall theoretical understanding of this problem.

We begin our analysis by deriving Exner's model in the fundamental form of an integral conservation law, which is equivalent to the differential form of the equation as transcribed by Exner upon the assumption that the sediment bed is smooth (continuously differentiable). We then consider the classical solution of the initial-value problem and demonstrate, using Exner's aforementioned dune example, how the intersection of characteristics leads to the breakdown of the classical solution and the formation of a discontinuity. The corresponding parabolic equation of Exner's model is then considered by adding a slope-dependent sediment transport term, which leads to the concept of the so-called vanishing viscosity solution. We then consider generalized solutions of Exner's model. From the integral form of Exner's equation we derive the so-called Rankine–Hugonoit jump condition and the entropy condition that govern discontinuous solutions. With these conditions, we consider generalized solutions of three specific problems for Exner's model including the solution of the so-called Riemann problem. Finally, we conclude this paper by summarizing the results and commenting on their importance in the application and verification of numerical models of bed evolution.

Derivation of Exner's Model

Consider a fixed control volume that extends from $x=x_1$ to $x=x_2$ (refer to Fig. 1) in a river or channel of constant width with a bed that evolves over time due to the transport of sediment as bed load, which is that portion of transported sediment load that remains in contact with the bed. Although not considered here, Exner also extended his model to channels of variable width. The equations of the conservation of fluid and sediment mass for depth-averaged flow can be written in integral form as

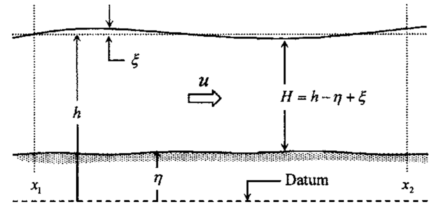

Fig. 1. Definition sketch for Exner's model

$$\frac{d}{dt}\int_{x_1}^{x_2} H dx + [Hu]_{x_1}^{x_2} = 0 \tag{1}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \eta dx + [q_s]_{x_1}^{x_2} = 0 \tag{2}$$

where u=depth-averaged velocity of the fluid; H=total depth of the fluid; η=bed elevation measured vertically upwards from a horizontal datum located below the bed; q_s=volumetric flux of transported bed load sediment per unit width of channel; and $[f]_{x_1}^{x_2}=f(x_2)-f(x_1)$.

Eq. (1) is recognized as the depth-averaged continuity equation, the equation of the conservation of mass for an incompressible fluid assuming depth-averaged flow. Eq. (2) represents the conservation of sediment mass for an incompressible sediment mixture that is homogeneous in structure, i.e., the sediment bed is idealized as a continuous medium with no porosity. This is the so-called sediment continuity or Exner equation. Implicit in these two equations is the assumption that the fluid and sediment remain distinct and separate from one another. That is, there is no mixing of the two media outside of a negligible boundary layer across the interface. This assumption is consistent with the notion of bed load transport.

Exner's model can be derived from these two equations by invoking the three main assumptions of his theory. First, it is assumed that the fluid flow is nearly steady, and therefore from Eq. (1) the fluid flow is approximated as a constant

$$q_f \equiv Hu \approx \text{constant} \tag{3}$$

where we have used the notation q_f to denote the fluid flux. The second assumption is that the deviation of the fluid surface, ζ, from the height of the fluid surface at rest, h, is small enough that it may be neglected in computing the total depth of the fluid, i.e., $H \approx h-\eta$ (see Fig. 1), or what amounts to a rigid lid assumption. From these first two assumptions the fluid velocity, u, can be expressed as

$$u = \frac{q_f}{h-\eta} \tag{4}$$

Exner's third, and perhaps crudest, assumption is that the sediment flux is given by the simple relation

$$q_s = \alpha u \tag{5}$$

where α=constant of proportionality. That is, the sediment flux is assumed to be a linear power law of the velocity. This equation is based on Exner's simple observation that an increase (decrease) in the velocity of the fluid increases (decreases) the capacity of the fluid to transport sediment.

While this observation is certainly true, most empirical sediment transport formulas that are used in practice take a slightly more complicated form

$$q_s = \alpha(u, H, \ldots)u^n \tag{6}$$

where n is typically greater than 1; and α=empirically derived function that is dependent on the flow parameters, sediment properties, and a number of constants. Many of these formulas can be reasonably approximated by taking α as a constant, as Exner has done. The more limiting aspect of Exner's assumption is the use of $n=1$. In this paper, we will retain this assumption for simplicity of presentation and to maintain consistency with Exner's original work. The theoretical results obtained, however, can easily be extended to arbitrary values of n and will not, qualitatively at least, change the findings.

Substituting Eq. (4) into Eq. (5) and then in turn substituting this result into Eq. (2) results in the governing equation of Exner's model

$$\frac{d}{dt}\int_{x_1}^{x_2} \eta dx + \left[\frac{\alpha q_f}{h-\eta}\right]_{x_1}^{x_2} = 0 \tag{7}$$

where q_f and α are both constants so the problem has been reduced to a single equation in terms of the single variable, η. Assuming that η and q_s are smooth, and with some manipulation, this can be written in the form

$$\int_{x_1}^{x_2} \left[\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha q_f}{h-\eta}\right)\right] dx = 0 \tag{8}$$

which for arbitrary $\Delta x = x_2 - x_1$ is equivalent to requiring

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha q_f}{h-\eta}\right) = 0 \tag{9}$$

This is the differential form of Exner's model as presented in his original work, which is a hyperbolic conservation law with a so-called flux function given by

$$f(\eta) \equiv \frac{\alpha q_f}{h-\eta} \tag{10}$$

It is assumed that both α and $q_f > 0$. Additionally, it is assumed that h is sufficiently large so that $(h-\eta) > 0$ for all time so that the singularity of f will not be of concern. Physically, this means that we are assuming that the sediment bed never breaches the free surface of the fluid. With these restrictions, f=convex function of η (i.e., $f_{\eta\eta} > 0$) and the existence and uniqueness of (entropy-satisfying, generalized) solutions to Exner's model follow from a proof of the existence and uniqueness for a scalar conservation law in one dimension with an arbitrary convex flux function (see Lax 1973; Oleinik 1959). We consider these solutions in the next section.

Solutions to Exner's Model

Continuous Solutions

Given that Exner's model takes the form of a hyperbolic equation, it is most clearly analyzed using the notion of characteristics. Consider the initial-value problem of Exner's model

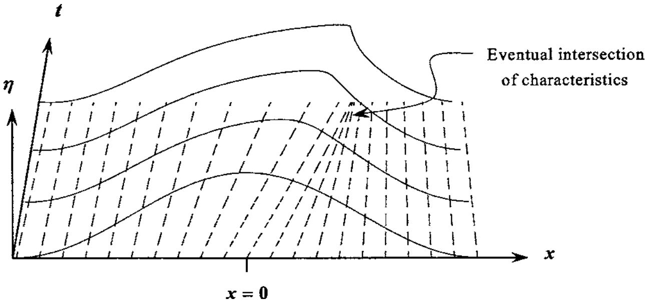

Fig. 2. Evolution of an initially symmetric mound according to Eq. (13) and a sketch of the characteristics in the space–time plane

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha q_f}{h-\eta}\right) = 0, \quad \eta(x,0) = \eta_0(x), \quad -\infty < x < \infty \tag{11}$$

Assuming that the initial condition, η_0, is smooth and that the solution itself remains smooth for a given time interval, then constant values of the initial condition simply propagate along the characteristics in space–time with a speed

$$c(\eta) \equiv \frac{\partial f}{\partial \eta} = \frac{\alpha q_f}{(h-\eta)^2} \tag{12}$$

and the solution to the initial-value problem is given implicitly by

$$\eta(x,t) = \eta_0[x - c(\eta)t] \tag{13}$$

This is the so-called classical or genuine solution, and its derivation is based on the assumption of smoothness of the solution.

In the present context the speed as given by Eq. (12) may be interpreted as the bed celerity, the propagation speed of the bed. Examining this expression a few basic facts about the evolution of the bed are immediately apparent:

1. Higher fluid flow rates result in more rapid changes in the bed;
2. A given bed profile will evolve more rapidly in shallower water (smaller values of h) than in deeper water when subjected to the same fluid flow rate; and
3. For a given bed profile and fluid flow rate, higher parts of the bed will propagate faster than lower parts.

Points 1 and 2 are fairly obvious and straightforward. Point 3, however, warrants additional consideration. Depending on the given initial conditions, the nonlinear dependence of the bed celerity on the bed elevation could lead to the formation of a discontinuity in the solution, which occurs when characteristics first intersect. By a simple argument (see, for example, Lax 1973), it can be shown that for a convex flux function this will occur if any part of the initial condition has a negative slope.

As an example of the formation of a discontinuity, consider the problem examined by Exner, that of an initially symmetric mound. In the region to the left of the characteristic emanating from the point $x=0$, $t=0$, characteristics spread out in a rarefaction wave, which in the present context corresponds to the process of erosion, i.e., the downstream portion of the mound erodes. In the region to the right of this, the characteristics compress together, eventually intersecting, and the solution ceases to be smooth (see Fig. 2). The time when this occurs is referred to as the breaking time, t_b. After this time the differential form of Exner's equation no longer holds (by definition the solutions must be smooth), and we must return to the integral form of the equation as given by Eq. (7), which is also valid for discontinuous

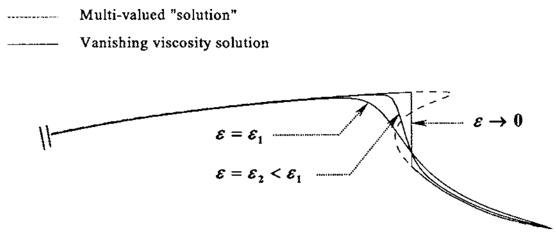

Fig. 3. Sketch of the multivalued "solution" and the concept of the vanishing viscosity solution

solutions. Extension of the classical solution beyond the breaking time results in the unphysical multivalued "solution" as demonstrated in Exner's original paper and as shown here in Fig. 3. Exner recognizes that this solution is unphysical, and, in fact, he states that his model would be subject to the proviso that the bed slopes should not exceed the angle of repose of the bed material (Exner 1925), i.e., a "smoothing effect" would be present in nature that is not accounted for in his model.

This leads us to consider the so-called vanishing viscosity solution of a parabolic equation (see, for example, Leveque 2002). Suppose a small diffusive or viscous term is added to Exner's model

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha q_f}{h-\eta}\right) = \varepsilon\frac{\partial^2\eta}{\partial x^2} \tag{14}$$

where ε=positive constant diffusivity parameter, which would theoretically be a function of the angle of repose of the bed material. Diffusive terms of this type in the sediment continuity equation have appeared in the literature (see, for example, Watanabe 1987; Daly and Porporato 2005). Their inclusion in the equation can be justified by the argument that the sediment transport is dependent on the bed slope, that is, downward (upward) slopes result in an increase (decrease) in sediment transport rates due to gravity effects. This idea may be expressed by modifying the original sediment transport formula, q_{s0}, by an additional bed slope term

$$q_s = q_{s0} - \varepsilon\frac{\partial\eta}{\partial x} \tag{15}$$

which produces the desired results due to gravity effects. Substitution of this equation into Exner's equation produces the additional diffusive term of Eq. (14). The resulting equation is now parabolic instead of hyperbolic, and it can be shown that for all time $t>0$ this equation will have a unique, smooth solution. Furthermore, it can be shown that as $\varepsilon\to 0$ the solution of the parabolic equation approaches a solution that possesses a discontinuity, as illustrated in Fig. 3. This is termed the vanishing viscosity solution of the parabolic equation.

Discontinuous Solutions

In order to consider the solution of Exner's model after the formation of the discontinuity, we must generalize the notion of a solution so that we can consider discontinuous functions as solutions as well, so-called weak or generalized solutions. To do so we must return to the integral form of Exner's equation.

Suppose that $x=s(t)$ defines a smooth curve in space–time along which the solution η has a discontinuity. On either side of the curve $x=s(t)$ we assume that the solution is continuous and differentiable. Looking at the integral form of Exner's model with the limits of integration defined such that $x_1<s(t)$ and $x_2>s(t)$ we have

$$\frac{d}{dt}\int_{x_1}^{s(t)}\eta dx + \frac{d}{dt}\int_{s(t)}^{x_2}\eta dx + \left[\frac{\alpha q_f}{h-\eta}\right]_{x_1}^{x_2} = 0 \tag{16}$$

where the range of integration has been split into two intervals on either side of s. With some manipulation and letting x_1 and $x_2\to s$, we arrive at the well-known Rankine–Hugoniot jump condition, which governs solutions of hyperbolic equations along discontinuities

$$\dot{s}[\eta] = [f(\eta)] \tag{17}$$

where $\dot{s}$=propagation speed of the discontinuity; and $[\cdot]=(\cdot)_L-(\cdot)_R$ represents the jump in the quantity across the discontinuity from the left, L, to the right, R. For Exner's model this is easily computed to be

$$\dot{s} = \frac{\alpha q_f}{(h-\eta_L)(h-\eta_R)} \tag{18}$$

where η_L and η_R=bed elevation to the left and right of the discontinuity, respectively. It is this jump condition that must be satisfied along discontinuities that may form in the bed.

Generalized solutions, however, are not necessarily unique and an additional criterion (in addition to the jump condition) known as the entropy condition must be satisfied at discontinuities. The term entropy condition originates from the condition in gas dynamics that requires that the entropy of a particle must increase across a discontinuity. This condition leads to unique, physically correct generalized solutions. In fact, it can be shown (see Kalashnikov 1959) that a generalized solution that satisfies an entropy condition corresponds to the vanishing viscosity solution of the related parabolic problem. The entropy condition relates the propagation speed of the discontinuity, $\dot{s}$, to the propagation speeds of the bed, c, ahead of and behind the discontinuity and is given by

$$c(\eta_L) > \dot{s} > c(\eta_R) \tag{19}$$

For convex flux functions this reduces to the condition

$$\eta_L > \eta_R \tag{20}$$

In the general theory of hyperbolic equations, discontinuities that satisfy both a jump condition and an entropy condition are termed shocks (as previously mentioned, in shallow water theory the term bore is used). In the present analysis, a discontinuity in the bed that satisfies both the jump condition given by Eq. (18) and the entropy condition given by Eq. (20) will be termed a sediment bore.

Example Problems

With the notion of a generalized solution defined, in this section, we consider the application of the theory developed in the previous section to some specific examples.

Exner's Dune Example

As a first example, we consider the generalized solution of Exner's dune problem. The initial condition is that of a wavy bed given by

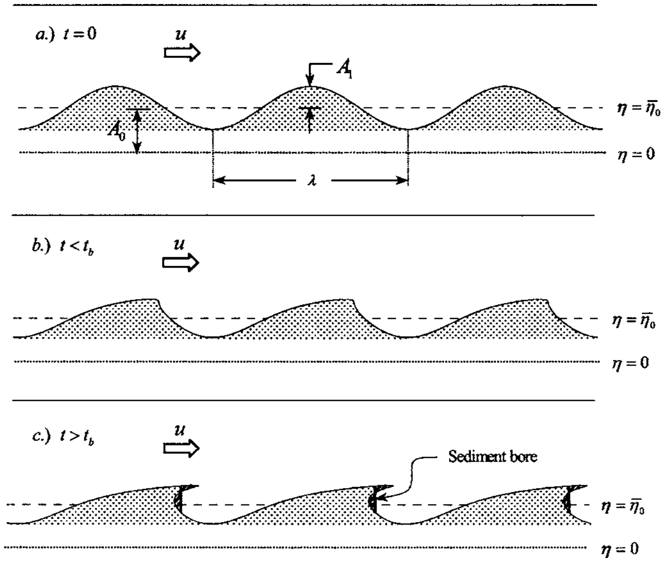

Fig. 4. Evolution of the bed for Exner's dune example at three different times: (a) $t=0$; (b) $t<t_b$; and (c) $t>t_b$

$$\eta(x,0)=A_0+A_1\cos\left(\frac{2\pi x}{\lambda}\right)\quad -\infty<x<\infty \tag{21}$$

where A_0, $A_1(A_0\geqslant A_1)$, and λ=positive constants as defined in Fig. 4(a). Following Exner (1925), as an example, we take the following numerical values for these parameters: $A_0=A_1=1$, $\lambda=20$, $h=3$, and $\alpha q_f=1$ (see, also, Graf 1971). The solution will be smooth up until the breaking time, t_b, which we can compute explicitly from the equation

$$t_b=-\frac{1}{\min\limits_{-\infty<\xi<\infty}[F'(\xi)]} \tag{22}$$

where $F(\xi)=c[\eta_0(\xi)]$; and $F'(\xi)$ denotes differentiation with respect to ξ. Using this equation with the given initial condition we compute the breaking time to be

$$t_b=\frac{5(5-\sqrt{7})^3}{4\pi(\sqrt{27}-4)^{1/2}}\approx 4.5685 \tag{23}$$

[It should be noted here that Exner's plot of the solution in his original work (Exner 1925), which is also reproduced by Graf (1971), is actually for the case $\lambda=10$, not $\lambda=20$, as reported in those works. For $\lambda=10$, Eq. (22) gives $t_b\approx 2.28$, which is consistent with what is shown in Exner's plots, i.e., the dune is shown to break between the plots for $t=2$ and $t=3$.]

For $t<t_b$ the solution is smooth and is given by Eq. (13). From the solution it can be observed (see Fig. 4) that the upstream portions of the mounds develop gentle slopes, while the downstream portions develop steep fronts, i.e., the wavy bed evolves into a train of dunes. For $t>t_b$ the solution will continue to be given by Eq. (13) in smooth regions, but the multivalued parts of the "solution" as given by this equation will be replaced by a discontinuity or sediment bore (the discontinuity obviously satisfies the entropy condition) propagating with speed $\dot{s}$ as given by Eq. (18).

The location of the sediment bore can be determined graphically by the so-called equal-area principle (see, for example, Whitham 1974, which also includes a quantitative means of determining the shock location), where it is placed at a position that separates the multivalued dune form, as given by the classical solution, into equal areas as shown by the dashed lines in Fig. 4(c). In this way, of course, sediment mass is still conserved. This can be seen by the fact that the area contained within the multivalued dune shape is equal to the area under the discontinuous solution. It should be noted that the generalized solution as shown in Fig. 4 closely resembles the form of dunes observed in nature with the exception, of course, that the downstream face of the dune would not be completely vertical due to some small diffusive effect that would be present in nature, which is not accounted for in the hyperbolic model. Finally, we remark that as the sediment bore continues to propagate downstream it experiences a decrease in height, and it can be shown that for periodic initial conditions the solutions tend to the mean value, $\bar{\eta}_0$, of the initial condition as $t\to\infty$ (see Lax 1973), that is, the bed eventually flattens out.

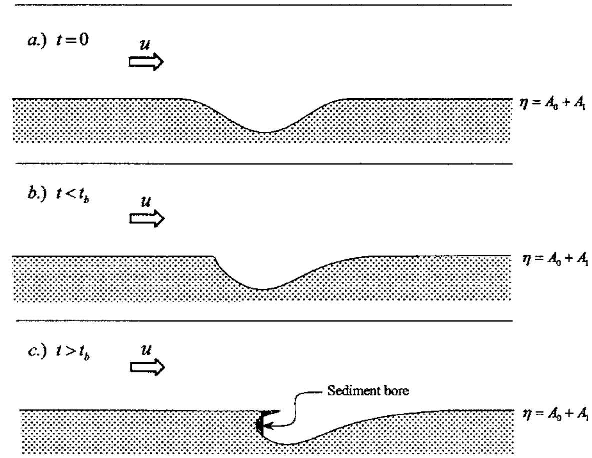

Fig. 5. Evolution of the bed for the dredged section at three different times: (a) $t=0$; (b) $t<t_b$; and (c) $t>t_b$

Dredged Section

Next we consider the evolution of an isolated dredged section given by the following initial condition:

$$\eta_0(x)=\begin{cases}A_0-A_1\cos(2\pi x/\lambda) & \text{for } -\lambda/2\leqslant x\leqslant\lambda/2\\ A_0+A_1 & \text{otherwise}\end{cases} \tag{24}$$

where the parameters are as previously defined. A similar problem was examined analytically by van de Kreeke et al. (2002) by considering a similar model to that of Exner's.

The solution to this problem is shown in Fig. 5 where it can be observed that the dredged section deforms in shape as it propagates downstream. A rarefaction wave can be observed in the downstream portion of the section while the upstream portion becomes gradually steeper, eventually developing a sediment bore at time t_b, which will propagate with a speed, $\dot{s}$, given by Eq. (18). From the entropy condition given by Eq. (19), it can be seen that as the sediment bore propagates further into the dredged section it will eventually "catch up" with the rarefaction wave [since $\dot{s}>c(\eta_R)$] after which time the height of the sediment bore will decay at $O(t^{-1/2})$ (see Lax 1973), i.e., the dredged section will fill with sediment.

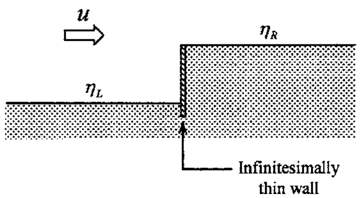

Fig. 6. Illustration of the Riemann problem for Exner's model showing two bed elevations η_L and η_R separated by an infinitesimally thin wall

Riemann Problem

Finally, we consider the so-called Riemann problem for Exner's equation, that is, Eq. (9) supplemented with an initial condition that is piecewise constant with a single jump discontinuity

$$\eta(x,0) = \begin{cases} \eta_L & x < 0 \\ \eta_R & x > 0 \end{cases} \tag{25}$$

In the present context, the discontinuity takes on the form of a simple step discontinuity in the bed. For example, consider the situation as illustrated in Fig. 6 where an infinitesimally thin wall at $x=0$ separates the bed into unique left, η_L, and right, η_R, states with the current flowing from the left to the right. Suppose then at time $t=0$ the wall is removed. This problem, for example, could represent the sudden failure of a submerged retaining wall or sediment trap. We seek the solution to this problem and consider the two possible cases below:

Case I: $\eta_L < \eta_R$
One possible generalized solution in this case would be the propagation of a sediment bore downstream with speed, $\dot{s}$, as given by Eq. (18). This solution would obviously satisfy the given jump condition, but it is clearly in violation of the entropy condition as given by Eq. (20). Intuitively, this makes sense, in that we would not expect a discontinuity in the bed that steps up from left to right in the direction of the current to be propagated downstream undisturbed but rather to be "washed out" or eroded away by the current. The physically correct solution, that is, the entropy-satisfying solution, is given by the rarefaction wave solution

$$\eta(x,t) = \begin{cases} \eta_L & x < c(\eta_L)t \\ h - \sqrt{\alpha t/x} & c(\eta_L)t \leq x \leq c(\eta_R)t \\ \eta_R & x > c(\eta_R)t \end{cases} \tag{26}$$

It can easily be verified that this rarefaction wave solution satisfies Eq. (9). An example of this solution for some time $t>0$ is shown in Fig. 7(a) where we see that the initial step in the bed is eroded away as it propagates downstream. Thus, it is seen how consideration of the entropy condition leads to physically correct solutions.

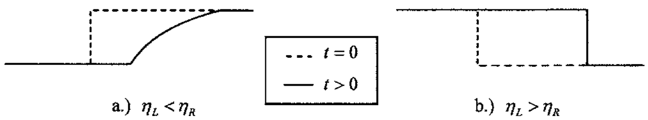

Fig. 7. Illustrations of the solution of the Riemann problem for Exner's model for (a) Case I: $\eta_L < \eta_R$; (b) Case II: $\eta_L > \eta_R$

Case II: $\eta_L > \eta_R$
In this case the generalized solution is given by

$$\eta(x,t) = \begin{cases} \eta_L & x < \dot{s}t \\ \eta_R & x > \dot{s}t \end{cases} \tag{27}$$

This scenario, which obviously satisfies the entropy condition, represents the steady propagation of a sediment bore downstream in the direction of the current, as shown in Fig. 7(a). Needham and Hey (1991) present an example of the generation and propagation of a sediment bore in an experimental flume.

Summary

In this paper, we have developed a simple theory for sediment bores by considering discontinuous solutions to a simplified bed evolution model originally due to Exner. We have extended solutions beyond the time of the breakdown of the classical solution, which will invariably occur if the initial condition has a negative slope. The solutions are extended by considering generalized solutions of the fundamental integral form of Exner's equation. Through the addition of a bed–slope-dependent sediment transport term, which acts in a viscous or diffusive manner, we have also considered the corresponding parabolic equation of Exner's model, the vanishing viscosity solution of which corresponds to the entropy-satisfying, generalized solution. We have demonstrated the application of these principles to three specific examples. Namely, we have extended Exner's solutions beyond the breaking time for problems involving the evolution of a dune and a dredged section, and we have considered the solution of the Riemann problem for Exner's model.

We conclude by noting that although the formation of sediment bores is intrinsically of theoretical interest, it is also an important point to consider in terms of developing numerical models for bed evolution. Many numerical methods for hyperbolic equations, such as finite-volume and discontinuous Galerkin methods, involve solving the Riemann problem. A theoretical understanding of this problem for Exner's model aids in the application and extension of such methods to the equations governing bed evolution. Additionally, many numerical methods will exhibit spurious oscillations in the presence of steep gradients or discontinuities. Working with the single equation of Exner's model provides a simplified setting in which to examine a numerical method, and the solutions developed in this paper could serve as rigorous "benchmark" test cases for bed evolution models. An example of the numerical solution of Exner's model is addressed by Kubatko et al. (2005).

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Notation

The following symbols are used in this paper:

A_0, A = positive constants;
c = propagation speed of the bed;

H = total height of the water column;
h = rigid lid elevation;
q_f = fluid flow discharge;
q_s = sediment flux;
$\dot{s}$ = shock propagation speed;
t = time;
t_b = breaking time;
u = depth-averaged velocity in the x-direction;
x, x_1, x_2 = Cartesian coordinates;
α = sediment flux constant of proportionality;
ε = diffusivity constant;
ζ = free surface elevation;
η = bed elevation; and
λ = positive constant.

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