Explicit realization of affine vertex algebras and their applications

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Affine vertex algebras

Let $V^k(\mathfrak{g})$ be universal affine vertex algebra of level k associated to the affine Lie algebra $\hat{\mathfrak{g}}$.

$V^k(\mathfrak{g})$ is generated generated by the fields $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$, $x \in \mathfrak{g}$.

As a $\hat{\mathfrak{g}}$–module, $V^k(\mathfrak{g})$ can be realized as a generalized Verma module.

For every $k \in \mathbb{C}$, the irreducible $\hat{\mathfrak{g}}$–module $L_k(\mathfrak{g})$ carries the structure of a simple vertex algebra.

Affine Lie algebra $A_1^{(1)}$

Let now $\mathfrak{g} = sl_2(\mathbb{C})$

with generators e, f, h

and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

The corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is of type $A_1^{(1)}$.

The level $k = -2$ is called **critical level**.

For $x \in sl_2$ identify x with $x(-1)\mathbf{1}$. Let Θ be the automorphism of $V^k(sl_2)$ such that

$$\Theta(e) = f, \ \Theta(f) = e, \ \Theta(h) = -h.$$

Affine Lie algebra $A_1^{(1)}$ in principal graduation

Let $\widehat{sl_2}[\Theta]$ be the affine Lie algebra $\widehat{sl_2}$ in principal graduation [Lepowsky-Wilson]. $\widehat{sl_2}[\Theta]$ has basis:

$$\{K,\ h(m),\ x^+(n),\ x^-(p)\ | m, p \in \tfrac{1}{2} + \mathbb{Z},\ n \in \mathbb{Z}\}$$

with commutation relations:

$$\begin{aligned}
&[h(m), h(n)] = 2m\delta_{m+n,0}K \\
&[h(m), x^+(r)] = 2x^-(m+r) \\
&[h(m), x^-(n)] = 2x^+(m+n) \\
&[x^+(r), x^+(s)] = 2r\delta_{r+s,0}K \\
&[x^+(r), x^-(m)] = -2h(m+r) \\
&[x^-(m), x^-(n)] = -2m\delta_{m+n,0}K \\
&K \quad \text{in the center}
\end{aligned}$$

Proposition. (FLM)

The category of Θ–twisted $V^k(sl_2)$–modules coincides with the category of restricted modules for $\widehat{sl_2}[\Theta]$ of level k.

$N = 2$ superconformal algebra

$N = 2$ superconformal algebra (SCA) is the infinite-dimensional Lie superalgebra with basis $\mathcal{L}(n), \mathcal{H}(n), \mathcal{G}^{\pm}(r), C$, $n \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$ and (anti)commutation relations given by

$$
\begin{aligned}
&[\mathcal{L}(m), \mathcal{L}(n)] = (m-n)\mathcal{L}(m+n) + \tfrac{C}{12}(m^3 - m)\delta_{m+n,0},\\
&[\mathcal{H}(m), \mathcal{H}(n)] = \tfrac{C}{3} m\delta_{m+n,0}, \quad [\mathcal{L}(m), \mathcal{G}^{\pm}(r)] = (\tfrac{1}{2}m - r)\mathcal{G}^{\pm}(m+r),\\
&[\mathcal{L}(m), \mathcal{H}(n)] = -n\mathcal{H}(n+m), \quad [\mathcal{H}(m), \mathcal{G}^{\pm}(r)] = \pm\mathcal{G}^{\pm}(m+r),\\
&\{\mathcal{G}^{+}(r), \mathcal{G}^{-}(s)\} = 2\mathcal{L}(r+s) + (r-s)\mathcal{H}(r+s) + \tfrac{C}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0},\\
&[\mathcal{L}(m), C] = [\mathcal{H}(n), C] = [\mathcal{G}^{\pm}(r), C] = 0,\\
&\{\mathcal{G}^{+}(r), \mathcal{G}^{+}(s)\} = \{\mathcal{G}^{-}(r), \mathcal{G}^{-}(s)\} = 0
\end{aligned}
$$

for all $m, n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.
Let $V_c^{N=2}$ be the universal $N = 2$ superconformal vertex algebra.

$N = 2$ superconformal algebra

$N = 2$ superconformal algebra (SCA) admits the mirror map automorphism (terminology of K. Barron):

$$\kappa : \mathcal{G}^{\pm}(r) \mapsto \mathcal{G}^{\mp}(r), \ \mathcal{H}(m) \mapsto -\mathcal{H}(m), \ \mathcal{L}(m) \mapsto \mathcal{L}(m), \ C \mapsto C$$

which can be lifted to an automorphism of $V_c^{N=2}$.

Proposition. (K. Barron, ..)

The category of κ–twisted $V_c^{N=2}$–modules coincides with the category of restricted modules for the mirror twisted $N = 2$ superconformal algebra of central charge c.

Correspondence

When $k \neq -2$, the representation theory of the affine Lie algebra $A_1^{(1)}$ is related with the representation theory of the $N = 2$ superconformal algebra.

The correspondence is given by Kazama-Suzuki mappings.

We shall extend this correspondence to representations at the critical level by introducing a new infinite-dimensional Lie superalgebra $\mathcal{A}$.

Vertex superalgebras F and F_{-1}

The Clifford vertex superalgebra F is generated by fields $\Psi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \Psi^{\pm}(n + \frac{1}{2}) z^{-n-1}$,
whose components satisfy the (anti)commutation relations for the infinite dimensional Clifford algebra CL:

$$\{\Psi^{\pm}(r), \Psi^{\mp}(s)\} = \delta_{r+s,0}; \quad \{\Psi^{\pm}(r), \Psi^{\pm}(s)\} = 0 \qquad (r, s \in \tfrac{1}{2} + \mathbb{Z}).$$

Let $F_{-1} = M(1) \otimes \mathbb{C}[L]$ be the lattice vertex superalgebra associated to the lattice

$$L = \mathbb{Z}\beta, \quad \langle \beta, \beta \rangle = -1.$$

Vertex superalgebras F and F_{-1}

Let Θ_F be automorphism of order two of F lifted from the automorphism $\Psi^{\pm}(r) \mapsto \Psi^{\mp}(r)$ of the Clifford algebra.

F has two inequivalent irreducible Θ_F–twisted modules F^{T_i}.

Let $\Theta_{F_{-1}}$ be the automorphism of F_{-1}–lifted from the automorphism $\beta \mapsto -\beta$ of the lattice L.

F_{-1} has two inequivalent irreducible $\Theta_{F_{-1}}$–twisted modules $F_{-1}^{T_i}$ realized on

$$M_{\mathbb{Z}+\frac{1}{2}}(1) = \mathbb{C}[\beta(-\tfrac{1}{2}), \beta(-\tfrac{3}{2}), \dots].$$

$N = 2$ superconformal vertex algebra

Let $\mathfrak{g} = sl_2$. Consider the vertex superalgebra $V^k(\mathfrak{g}) \otimes F$. Define

$$\tau^+ = e(-1) \otimes \Psi^+(-\tfrac{1}{2}), \quad \tau^- = f(-1) \otimes \Psi^-(-\tfrac{1}{2}).$$

Then the vertex subalgebra of $V^k(\mathfrak{g}) \otimes F$

generated by τ^+ and τ^- carries the structure of a highest weight module for of the $N = 2$ SCA:

$\mathcal{G}^{\pm}(z) = \sqrt{\frac{2}{k+2}} Y(\tau^{\pm}, z) = \sum_{n \in \mathbb{Z}} \mathcal{G}^{\pm}(n + \frac{1}{2}) z^{-n-2}$

Kazama-Suzuki and "anti" Kazama-Suzuki mappings

Introduced by Fegin, Semikhatov and Tipunin (1997)

Assume that M is a (weak) $V^k(\mathfrak{g})$-module. Then $M \otimes F$ is a (weak) $V_c^{N=2}$–module with $c = 3k/(k+2)$.

Assume that N is a weak $V_c^{N=2}$–module. Then $N \otimes F_{-1}$ is a (weak) $V^k(sl_2)$–module.

This enables a classification of irreducible modules for simple vertex superalgebras associated to N=2 SCA (D.Adamović, IMRN (1998))

Kazama-Suzuki and "anti" Kazama-Suzuki mappings: twisted version

Let $c = 3k/(k+2)$.

- Assume that M^{tw} is a Θ–twisted $V^k(\mathfrak{g})$-module. Then $M^{tw} \otimes F^{T_i}$ is a κ–twisted $V_c^{N=2}$–module.

- Assume that N^{tw} is a κ–twisted $V_c^{N=2}$–module. Then $N^{tw} \otimes F_{-1}^{T_i}$ is a Θ–twisted $V^k(\mathfrak{g})$–module.

Lie superalgebra $\mathcal{A}$

$\mathcal{A}$ is infinite-dimensional Lie superalgebra with generators $S(n), T(n), G^{\pm}(r), C$, $n \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, which satisfy the following relations

$$
\begin{aligned}
& S(n), T(n), C \quad \text{are in the center of } \mathcal{A}, \\
& \{G^{+}(r), G^{-}(s)\} = 2S(r+s) + (r-s)T(r+s) + \tfrac{C}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0}, \\
& \{G^{+}(r), G^{+}(s)\} = \{G^{-}(r), G^{-}(s)\} = 0
\end{aligned}
$$

for all $n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

The (universal) vertex algebra $\mathcal{V}$

$\mathcal{V}$ is strongly generated by the fields

$$G^{\pm}(z) = Y(\tau^{\pm}, z) = \sum_{n \in \mathbb{Z}} G^{\pm}(n + \tfrac{1}{2}) z^{-n-2},$$
$$S(z) = Y(\nu, z) = \sum_{n \in \mathbb{Z}} S(n) z^{-n-2},$$
$$T(z) = Y(j, z) = \sum_{n \in \mathbb{Z}} T(n) z^{-n-1}.$$

The components of these fields satisfy the (anti)commutation relations for the Lie superalgebra $\mathcal{A}$.

Let $\Theta_{\mathcal{V}}$ be the automorphism of $\mathcal{V}$ lifted from the automorphism of order two of $\mathcal{A}$ such that

$$G^{\pm}(r) \mapsto G^{\mp}(r), \ \ T(r) \mapsto -T(r), \ \ S(r) \mapsto S(r), C \mapsto C.$$

Lie superalgebra $\mathcal{A}^{tw}$

$\mathcal{A}^{tw}$ has the basis

$$\mathcal{S}(n),\ \mathcal{T}(n+1/2),\ \mathcal{G}(r),\ C, \qquad n \in \mathbb{Z},\ r \in \tfrac{1}{2}\mathbb{Z}$$

and anti-commutation relations:

$$\{\mathcal{G}(r),\mathcal{G}(s)\} = (-1)^{2r+1}(2\delta^{\mathbb{Z}}_{r+s}\mathcal{S}(r+s) - \delta^{\frac{1}{2}+\mathbb{Z}}_{r+s}(r-s)\mathcal{T}_{r+s} + \tfrac{C}{3}\delta^{\mathbb{Z}}_{r+s}(r^2-\tfrac{1}{4})\,\delta_{r+s,0}),$$

$$\mathcal{S}(n), \mathcal{T}(n+1/2), C \quad \text{in the center},$$

with $\delta^S_m = 1$ if $m \in S$, $\delta^S_m = 0$ otherwise.

Proposition.

The category of $\Theta_{\mathcal{V}}$–twisted $\mathcal{V}$–modules coincides with the category of restricted modules for the Lie superalgebra $\mathcal{A}^{tw}$.

Theorem (A, CMP 2007)

Assume that U is an irreducible $\mathcal{V}$–module such that U admits the following $\mathbb{Z}$–gradation

$$U = \bigoplus_{j \in \mathbb{Z}} U^j, \quad \mathcal{V}^i . U^j \subset U^{i+j}.$$

Let F_{-1} be the vertex superalgebra associated to lattice $\mathbb{Z}\sqrt{-1}$. Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \textit{where} \ \ \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}$$

and for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible $A_1^{(1)}$–module at the critical level.

Weyl vertex algebra

The Weyl vertex algebra W is generated by the fields

$$a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}, \quad a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n)z^{-n},$$

whose components satisfy the commutation relations for infinite-dimensional Weyl algebra

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m,0}.$$

Wakimoto modules

Assume that $\chi(z) \in \mathbb{C}((z))$.

On the vertex algebra W exists the structure of the $A_1^{(1)}$–module at the critical level defined by

$$\begin{aligned} e(z) &= a(z), \\ h(z) &= -2 : a^*(z)a(z) : -\chi(z) \\ f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) - a^*(z)\chi(z). \end{aligned}$$

This module is called the Wakimoto module and it is denoted by $W_{-\chi(z)}$.

Theorem (D.A., CMP 2007, Contemp. Math. 2014)

The Wakimoto module $W_{-\chi}$ is irreducible if and only if $\chi(z)$ satisfies one of the following conditions:

(i) *There is $p \in \mathbf{Z}_{>0}$, $p \geq 1$ such that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_p \neq 0.$$

(ii) $\chi(z) = \sum_{n=0}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad$ *and* $\quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z})$.

(iii) *There is $\ell \in \mathbb{Z}_{\geq 0}$ such that*

$$\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

and $S_\ell(-\chi) \neq 0$, where $S_\ell(-\chi) = S_\ell(-\chi_{-1}, -\chi_{-2}, \dots)$ is a Schur polynomial.

The structure of Wakimto modules

Theorem

Assume that $\chi \in \mathbb{C}((z))$ *such that* W_χ *is reducible. Then*

(1) W_χ *is indecomposable.*

(2) *The maximal* sl_2*–integrable submodule* W_χ^{int} *is irreducible.*

(3) W_χ / W_χ^{int} *is irreducible.*

Theorem (D. Adamovic, N. Jing, K. Misra, 2014-2015)

Assume that U^{tw} is an irreducible, restricted $\mathcal{A}^{tw}$–module, and $F_{-1}^{T_i}$ twisted F_{-1}–module. Then $U^{tw} \otimes F_{-1}^{T_i}$ has the structure of irreducible $\widehat{sl_2}[\Theta]$–module at the critical level.

Construction of $\mathcal{A}^{tw}$–modules

Let F^{T_i} irreducible Θ_F–twisted F–module.

F^{T_i} is an irreducible module the the twisted Clifford algebra CL^{tw} with generators $\Phi(r)$, $r \in \frac{1}{2}\mathbb{Z}$, and relations

$$\{\Phi(r), \Phi(s)\} = -(-1)^{2r}\delta_{r+s,0}; \quad (r, s \in \frac{1}{2}\mathbb{Z})$$

Let $\chi \in \mathbb{C}((z^{1/2}))$. The $\mathcal{A}^{tw}$–module $F^{T_i}(\chi)$ is uniquely determined by

$$G(z) = \partial_z \Phi(z) + \chi(z)\Phi(z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} G(n) z^{-n-1}.$$

Theorem (D. Adamovic, N. Jing, K. Misra, 2014-2015)

Assume that $p \in \frac{1}{2}\mathbf{Z}_{>0}$ *and that*

$$\chi(z) = \sum_{k=-2p}^{\infty} \chi_{-\frac{k}{2}} z^{\frac{k}{2}-1}.$$

Then $F^{T_i}(\chi)$ *is irreducible* $\mathcal{A}^{tw}$*–module if and only if one of the following conditions hold:*

$$p > 0 \quad \textit{and } \chi_p \neq 0, \tag{1}$$

$$p = 0 \quad \textit{and } \chi_0 \in (\mathbb{C} \setminus \tfrac{1}{2}\mathbb{Z}) \cup \{\frac{1}{2}\}, \tag{2}$$

$$p = 0 \quad \textit{and } \chi_0 - \frac{1}{2} = \ell \in \tfrac{1}{2}\mathbf{Z}_{>0} \quad \textit{and } \det(A(\chi)) \neq 0 \tag{3}$$

$$A(\chi) = \begin{pmatrix} 2S(-1) & 2S(-2) & \cdots & & 2S(-2\ell) \\ \ell^2-(\ell-1)^2 & 2S(-1) & 2S(-2) & \cdots & 2S(-2\ell-1) \\ 0 & \ell^2-(\ell-2)^2 & 2S(-1) & \cdots & 2S(-2\ell-2) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \ell^2-(\ell-2\ell+1)^2 & 2S(-1) \end{pmatrix}$$

and

$$S(z) = \frac{1}{2}(\chi^{(1)}(z))^2 + \partial_z \chi^{(1)}(z)) = \sum_{n\in\mathbb{Z}} S(n) z^{-n-2}$$

(here $\chi^{(1)}$ is the integral part of χ).

N=4 superconformal vertex algebra $V_c^{N=4}$

$V_c^{N=4}$ is generated by the Virasoro field L, three primary fields of conformal weight 1, J^0, J^+ and J^- (even part) and four primary fields of conformal weight $\frac{3}{2}$, $G^{\pm}$ and $\overline{G}^{\pm}$ (odd part).
The remaining (non-vanishing) λ–brackets are

$$
\begin{array}{rl}
[J^0_\lambda, J^{\pm}] = \pm 2J^{\pm} & [J^0_\lambda J^0] = \dfrac{c}{3} I \\
[J^+_\lambda J^-] = J^0 + \frac{c}{6}\lambda & [J^0_\lambda G^{\pm}] = \pm G^{\pm} \\
[J^0_\lambda \overline{G}^{\pm}] = \pm \overline{G}^{\pm} & [J^+_\lambda G^-] = G^+ \\
[J^-_\lambda G^+] = G^- & [J^+_\lambda \overline{G}^-] = -\overline{G}^+ \\
[J^-_\lambda \overline{G}^+] = -\overline{G}^- & [G^{\pm}_\lambda \overline{G}^{\pm}] = (T + 2\lambda) J^{\pm} \\
[G^{\pm}_\lambda \overline{G}^{\mp}] = & L \pm \dfrac{1}{2} T J^0 \pm \lambda J^0 + \dfrac{c}{6}\lambda^2
\end{array}
$$

Let $L_c^{N=4}$ be its simple quotient.

N=4 superconformal vertex algebra $L_c^{N=4}$ with $c=-9$

We shall present some results from D.Adamović, arXiv:1407.1527. (to appear in Transformation Groups)

Theorem

(i) The simple affine vertex algebra $L_k(sl_2)$ with $k=-3/2$ is conformally embedded into $L_c^{N=4}$ with $c=-9$.
(ii)

$$L_c^{N=4} \cong (M \otimes F)^{int}$$

where $M \otimes F$ is a maximal sl_2–integrable submodule of the Weyl-Clifford vertex algebra $M \otimes F$.

$L_c^{N=4}$ with $c = -9$ as an $\widehat{sl_2}$–module

$L_c^{N=4}$ with $c = -9$ is completely reducible $\widehat{sl_2}$–module and the following decomposition holds:

$$L_c^{N=4} \cong \bigoplus_{m=0}^{\infty} (m+1) L_{A_1}(-(\frac{3}{2}+n)\Lambda_0 + n\Lambda_1).$$

$L_c^{N=4}$ is a completely reducible $sl_2 \times \widehat{sl_2}$–modules. sl_2 action is obtained using screening operators for Wakimoto realization of $\widehat{sl_2}$–modules at level $-3/2$.

The affine vertex algebra $L_k(sl_3)$ with $k = -3/2$.

Theorem

(i) The simple affine vertex algebra $L_k(sl_3)$ with $k = -3/2$ is realized as a subalgebra of $L_c^{N=4} \otimes F_{-1}$ with $c = -9$. In particular $L_k(sl_3)$ can be realized as subalgebra of

$$M \otimes F \otimes F_{-1}.$$

(ii) $L_c^{N=4} \otimes F_{-1}$ is a completely reducible $A_2^{(1)}$–module at level $k = -3/2$.

On representation theory of $L_c^{N=4}$ with $c = -9$

- $L_c^{N=4}$ has only one irreducible module in the category of strong modules. Every $\mathbf{Z}_{>0}$–graded $L_c^{N=4}$–module with finite-dimensional weight spaces (with respect to $L(0)$) is semisimple ("Rationality in the category of strong modules")
- $L_c^{N=4}$ has two irreducible module in the category $\mathcal{O}$. There are non-semisimple $L_c^{N=4}$–modules from the category $\mathcal{O}$.
- $L_c^{N=4}$ has infinitely many irreducible modules in the category of weight modules.
- $L_c^{N=4}$ admits logarithmic modules on which $L(0)$ does not act semi-simply.

Theorem (D.A, 2014)

Assume that U is an irreducible $L_c^{N=4}$–module with $c = -9$ such that $U = \bigoplus_{j \in \mathbb{Z}} U^j$ is $\mathbb{Z}$–graded (in a suitable sense).
Let F_{-1} be the vertex superalgebra associated to lattice $\mathbb{Z}\sqrt{-1}$. Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \textit{where} \;\; \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}$$

and for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible $A_2^{(1)}$–module at level $-3/2$.

Connection with C_2–cofinite vertex algebras appearing in LCFT

Drinfeld-Sokolov reduction maps:

$L_c^{N=4}$ to doublet vertex algebra $\mathcal{A}(p)$ and even part $(L_c^{N=4})^{even}$ to triplet vertex algebra $\mathcal{W}(p)$ with $p = 2$

(symplectic-fermion case)

Vacuum space of $L_k(sl_3)$ with $k = -3/2$ contains the vertex algebra $\mathcal{W}_{A_2}(p)$ with $p = 2$ (which is conjecturally C_2–cofinite).

Connection with C_2–cofinite vertex algebras appearing in LCFT:

Vacuum space of $L_k(sl_3)$ with $k = -3/2$ contains the vertex algebra $\mathcal{W}_{A_2}(p)$ with $p = 2$ (which is conjecturally C_2–cofinite).

Affine vertex algebra $L_k(sl_2)$ for $k + 2 = \frac{1}{p}$, $p \geq 2$ can be conformally embedded into the vertex algebra $\mathcal{V}^{(p)}$ generated by $L_k(sl_2)$ and 4 primary vectors $\tau^{\pm}_{(p)}, \overline{\tau}^{\pm}_{(p)}$.

$\mathcal{V}^{(p)} \cong L_c^{N=4}$ for $p = 2$.

Drinfeld-Sokolov reduction maps $\mathcal{V}^{(p)}$ to the doublet vertex algebra $\mathcal{A}(p)$ and even part $(\mathcal{V}^{(p)})^{even}$ to the triplet vertex algebra $\mathcal{W}(p)$. (C_2–cofiniteness and RT of these vertex algebras were obtain in a work of D.A and A. Milas)

The Vertex algebra $\mathcal{W}_{A_2}(p)$: Definition

We consider the lattice

$$\sqrt{p}A_2 = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2, \quad \langle \gamma_1, \gamma_1 \rangle = \langle \gamma_2, \gamma_2 \rangle = 2p, \ \langle \gamma_1, \gamma_2 \rangle = -p.$$

Let $M_{\gamma_1,\gamma_2}(1)$ be the s Heisenberg vertex subalgebra of $V_{\sqrt{p}A_2}$ generated by the Heisenberg fields $\gamma_1(z)$ and $\gamma_2(z)$.

$$\mathcal{W}_{A_2}(p) = \mathrm{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_1/p} \bigcap \mathrm{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_2/p}.$$

We also have its subalgebra:

$$\mathcal{W}^0_{A_2}(p) = \mathrm{Ker}_{M_{\gamma_1,\gamma_2}(1)} e_0^{-\gamma_1/p} \bigcap \mathrm{Ker}_{M_{\gamma_1,\gamma_2}(1)} e_0^{-\gamma_2/p}$$

$\mathcal{W}_{A_2}(p)$ and $\mathcal{W}^0_{A_2}(p)$ have vertex subalgebra isomorphic to the simple $\mathcal{W}(2,3)$–algebra with central charge $c_p = 2 - 24\frac{(p-1)^2}{p}$.

The Vertex algebra $\mathcal{W}_{A_2}(p)$: Conjecture

(i) $\mathcal{W}_{A_2}(p)$ is a C_2–cofinite vertex algebra for $p \geq 2$ and that it is a completely reducible $\mathcal{W}(2,3) \times sl_3$–module.

(ii) $\mathcal{W}_{A_2}(p)$ is strongly generated by $\mathcal{W}(2,3)$ generators and by $sl_3 . e^{-\gamma_1 - \gamma_2}$, so by 8 primary fields for the $\mathcal{W}(2,3)$–algebra.

Note that $\mathcal{W}_{A_2}(p)$ is a generalization of the triplet vertex algebra $\mathcal{W}(p)$ and $\mathcal{W}^0_{A_2}(p)$ is a generalization of the singlet vertex subalgebra of $\mathcal{W}(p)$.

Relation with parafermionic vertex algebras for $p = 2$

(i) Let $K(sl_3, k)$ be the parafermion vertex subalgebra of $L_k(sl_3)$ (C. Dong talk).

(iii) For $k = -3/2$ we have

$$K(sl_3, k) = \mathcal{W}^0_{A_2}(p).$$

General conjectures

Conjecture

Assume that $L_k(\mathfrak{g})$ is a simple affine vertex algebra of affine type and $k \in (\mathbb{Q} \setminus \mathbb{Z}_{\geq 0})$ is admissible. Then

- *The vacuum space*

$$\Omega(L_k(\mathfrak{g})) = \{v \in L_k(\mathfrak{g}) \mid h(n).v = 0 \quad h \in \mathfrak{h}, \ n \geq 1\}$$

  is extension of certain C_2–cofinite, irrational vertex algebra.
- *$L_k(\mathfrak{g})$ admits logarithmic representations.*

(1) $\Omega(L_k(sl_2)) \cong W(2)$ for $k = -1/2$.

(2) $\Omega(L_k(sl_2)) \cong A(3)$ for $k = -4/3$, where $A(3)$ is SCE extension of triplet vertex algebra $W(3)$.

(3) $\Omega(L_k(sp_{2n}))$ for $k = -1/2$ is $\mathbb{Z}_2$–orbifold of symplectic fermions.

Realization of simple W–algebras

Let $F_{-p/2}$ denotes the generalized lattice vertex algebra associated to the lattice $\mathbb{Z}(\frac{p}{2}\varphi)$ such that

$$\langle \varphi, \varphi \rangle = -\frac{2}{p}.$$

Let $\mathcal{R}^{(p)}$ by the subalgebra of $\mathcal{V}^{(p)} \otimes F_{-p/2}$ generated by $x = x(-1)\mathbf{1} \otimes 1$, $x \in \{e, f, h\}$, $1 \otimes \varphi(-1)\mathbf{1}$ and

$$e_{\alpha_1,p} := \frac{1}{\sqrt{2}} \tau^{+}_{(p)} \otimes e^{\frac{p}{2}\varphi} \tag{4}$$

$$f_{\alpha_1,p} := \frac{1}{\sqrt{2}} \overline{\tau}^{-}_{(p)} \otimes e^{-\frac{p}{2}\varphi} \tag{5}$$

$$e_{\alpha_2,p} := \frac{1}{\sqrt{2}} \overline{\tau}^{+}_{(p)} \otimes e^{-\frac{p}{2}\varphi} \tag{6}$$

$$f_{\alpha_2,p} := \frac{1}{\sqrt{2}} \tau^{-}_{(p)} \otimes e^{\frac{p}{2}\varphi} \tag{7}$$

Realization of simple W–algebras

$\mathcal{R}^{(2)} \cong L_{A_2}(-\frac{3}{2}\Lambda_0)$.

$\mathcal{R}^{(3)} \cong \mathcal{W}_k(sl_4, f_\theta)$ with $k = -8/3$.

(Conjecture) $\mathcal{R}^{(p)}$ and $\mathcal{V}^{(p)}$ have finitely many irreducible modules in the category $\mathcal{O}$.

$\mathcal{R}^{(p)}$ and $\mathcal{V}^{(p)}$ have infinitely many irreducible modules outside of the category $\mathcal{O}$ and admit logarithmic modules.

Thank you