Representation Theory, Q-systems, and Generalizations: A Preliminary Report

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Conference on Lie Algebras, Vertex Operator Algebras, and Related Topics
A Conference in Honor of J. Lepowsky and R. Wilson

August 18, 2015

Overview

Use the **homogeneous realization** of the **basic representation** of $\widehat{sl_2\mathbb{C}}$ to compute certain **tau functions** for the Toda lattice (Kasman '96).

Bergvelt obeserved that these tau functions satisfy the A_∞ $\mathbf{Q}-$ **system** (see Kedem and Di Francesco papers as well as Kirillov and Reshetikhin '90).

We are working to **generalize** Bergvelt's idea by using the **homogeneous realization** of the basic representation of $\widehat{sl_3\mathbb{C}}$ to obtain new (more complicated) functions.

We are currently working to understand what sort of relations are satisfied by these new tau functions.

The $\widehat{sl_2\mathbb{C}}$ Case (Bergvelt)

We take the **homogeneous realization** of the basic representation of $\widehat{sl_2\mathbb{C}}$.

By a theorem of Frenkel and Kac, '80, this representation is isomorphic to

$$\oplus_{k\in\mathbb{Z}} T^k v_{\Lambda_0} \otimes \mathbb{C}[t_1, t_2, t_3, \cdots]$$

where $T = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$ and v_{Λ_0} is the vacuum vector.

Since this is an **integrable** representation, we can consider the action of the loop group element,

$$g = \begin{bmatrix} 1 & 0 \\ C(z) & 1 \end{bmatrix}$$ on the vacuum vector, v_{Λ_0}, where

$$C(z) = \sum_{i=0}^{\infty} \frac{c_i}{z^{i+1}}$$ where $c_i \in \mathbb{C}$.

Since the basic representation is isomorphic to

$\oplus_{k\in\mathbb{Z}} T^k v_{\Lambda_0} \otimes \mathbb{C}[t_1, t_2, t_3, \cdots]$,

$$g \cdot v_{\Lambda_0} = \sum_{k\in\mathbb{Z}} \tau_k(t_1, t_2, t_3, \cdots) T^k v_{\Lambda_0} \text{ for some } \tau_k \in \mathbb{C}[t_1, t_2, t_3, \cdots].$$

We take these $\tau_{\mathbf{k}}$ to be the definition of our tau functions.

Using the fact that

$$g = \exp \begin{bmatrix} 0 & 0 \\ C(z) & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ C(z) & 0 \end{bmatrix} = \mathrm{Res}_w (C(w) \sum_{i \in \mathbb{Z}} \begin{bmatrix} 0 & 0 \\ z^i & 0 \end{bmatrix} w^{-i-1}),$$

we can calculate the action of g, by calculating the action of the current,

$$\sum_{i \in \mathbb{Z}} \begin{bmatrix} 0 & 0 \\ z^i & 0 \end{bmatrix} w^{-i-1},$$

on

$$\oplus_{k \in \mathbb{Z}} T^k v_{\Lambda_0} \otimes \mathbb{C}[t_1, t_2, t_3, \cdots]$$

We then find that, for $k \geq 0$,

$$\tau_k(t) = \det \begin{bmatrix} c_0^{\mathbf{t}} & c_1^{\mathbf{t}} & \cdots & c_{k-1}^{\mathbf{t}} \\ c_1^{\mathbf{t}} & c_2^{\mathbf{t}} & \cdots & c_k^{\mathbf{t}} \\ \vdots & \vdots & \cdots & \vdots \\ c_{k-1}^{\mathbf{t}} & c_k^{\mathbf{t}} & \cdots & c_{2(k-1)}^{\mathbf{t}} \end{bmatrix}$$

where $c_i^{\mathbf{t}} = \mathrm{Res}_w(w^i C(w) \exp(\sum_{j>0} w^j t_j))$.

Also notice that τ_k is the determinant of a **Hankel matrix**.

The Desnanot-Jacobi Identity and Difference Relations

Since we are only concerned with our difference relations and not concerned with dependence on the t_is, we may take

$$\tau_k = \det \begin{bmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & \cdots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2(k-1)} \end{bmatrix}$$

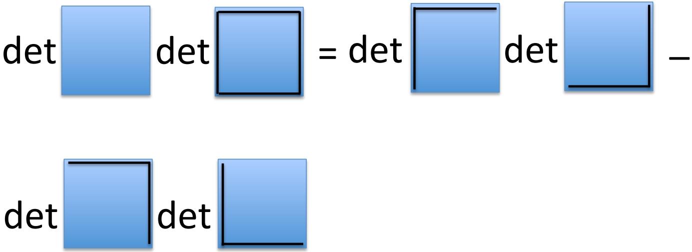
Since this is a **Hankel matrix**, applying the **Desnanot − Jacobi Identity** is particularly nice.

The Desnanot-Jacobi Identity

Given a $k \times k$ matrix, M, let M_i^j denote the matrix obtained by deleting the ith row and jth column of M. For $1 \leq i_1 < i_2 \leq k$ and $1 \leq j_1 < j_2 \leq k$, $M_{i_1,i_2}^{j_1,j_2}$ denotes the matrix obtained from M by deleting the i_1, i_2 rows and the j_1, j_2 rows.

We then have the "**Desnanot – Jacobi Identity**":

$$\det M \det M_{1,k}^{1,k} = \det M_1^1 \det M_k^k - \det M_1^k \det M_k^1$$


det det = det det –
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In order to get our A_∞ Q-system, we first need to expand our definition of tau-functions to allow "shifted tau functions"...

Expanding our definition of τ functions to include "**shifted** τ-**functions**" amounts to working in the space,

$$\oplus_{j,k\in\mathbb{Z}} Q^j T^k v_{\Lambda_0} \otimes \mathbb{C}[t_1, t_2, t_3, \cdots], \text{ where } Q = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

These new tau functions are then the coefficients of the $Q^a T^k v_{\Lambda_0}$s in $gQ^a v_{\Lambda_0}$s.

They are given by $(-1)^{ak}\det \begin{bmatrix} c_a & c_{a+1} & \cdots & c_{a+k-1} \\ c_{a+1} & c_{a+2} & \cdots & c_{a+k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{a+k-1} & c_{a+k} & \cdots & c_{a+2(k-1)} \end{bmatrix}$.

It is convenient to write

$$\tau_k^n = (-1)^{(n-k+1)k} \det \begin{bmatrix} c_{n-k+1} & c_{n-k+2} & \cdots & c_n \\ c_{n-k+2} & c_{n-k+3} & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_n & c_{n+1} & \cdots & c_{n+k-1} \end{bmatrix}$$

Applying the Desnanot-Jacobi Identity, we have:

$(\tau_k^n)^2 + \tau_{k+1}^n \tau_{k-1}^n = \tau_k^{n+1} \tau_k^{n-1}$ for all $k \geq 0$ and $n \in \mathbb{Z}$,

which are precisely the equations which define an $\mathbf{A}_\infty$ $\mathbf{Q}-$**system** (see Kedem Di Francesco papers, Kirillov-Reshetikhin '90)

Additionally, we have orthogonal polynomials:

$$p_k(z) = \det \begin{bmatrix} c_0 & \cdots & c_{k-1} & 1 \\ c_1 & \cdots & c_k & z \\ \vdots & \dots & \vdots & \vdots \\ c_k & \cdots & c_{2k-1} & z^k \end{bmatrix},$$

$c(p_m(z)p_n(z)) = 0$ if $m \neq n$, where $c(f(z)) = \text{Res}_z(C(z)f(z))$.

The orthogonality of these polynomials is implied by **Hirota Equations** (If time permits, I will briefly mention these later. See Kac-Raina '87), satisfied by our $\widehat{sl_2\mathbb{C}}$ τ-functions.

We'd like to find an analogous system of polynomials for our $\widehat{sl_3\mathbb{C}}$ case.

Generalizing the Above to the $\widehat{sl_3\mathbb{C}}$ Case

Take the **homogeneous realization** of the basic representation of $\widehat{\mathbf{sl_3}\mathbb{C}}$, which is isomorphic (Frenkel, Kac, '80) to

$$\oplus_{k,\ell\in\mathbb{Z}} T_1^k T_2^\ell v_{\Lambda_0} \otimes \mathbb{C}[t_1, t_2, t_3, \cdots]$$

where $T_1 = \begin{bmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

We now consider the action of a group element

$$g = \begin{bmatrix} 1 & 0 & 0 \\ C(z) & 1 & 0 \\ D(z) & E(z) & 1 \end{bmatrix}$$ on the vacuum vector, where

$C(z) = \sum_{i=0}^{\infty} \frac{c_i}{z^{i+1}}$, $D(z) = \sum_{i=0}^{\infty} \frac{d_i}{z^{i+1}}$, and $E(z) = \sum_{i=0}^{\infty} \frac{e_i}{z^{i+1}}$ where

$c_i, d_i, e_i \in \mathbb{C}$.

As before, we have

$$\mathbf{g} \cdot \mathbf{v}_{\mathbf{\Lambda_0}} = \sum_{\mathbf{k},\ell \in \mathbb{Z}} \tau_{\mathbf{k},\ell}(\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, \cdots) \mathbf{T}_1^{\mathbf{k}} \mathbf{T}_2^{\ell} \mathbf{v}_{\mathbf{\Lambda_0}}$$

for some $\tau_{\mathbf{k},\ell} \in \mathbb{C}[t_1, t_2, t_3, \cdots]$ and **we take these $\tau_{k,\ell}$s to be the definition of our new tau functions**.

We again ignore dependence on the t_is and focus instead on the discrete evolution.

These new functions are, in general, much more complicated than before...

A Few Examples

$$\tau_{k,0} = \det \begin{bmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & \ldots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2(k-1)} \end{bmatrix},$$

$$\tau_{0,k} = \det \begin{bmatrix} e_0 & e_1 & \cdots & e_{k-1} \\ e_1 & e_2 & \cdots & e_k \\ \vdots & \vdots & \ldots & \vdots \\ e_{k-1} & e_k & \cdots & e_{2(k-1)} \end{bmatrix}, \tau_{1,1} = -d_0,$$

$$\tau_{1,2} = -\det \left[\begin{array}{c|c} e_0 & d_0 \\ e_1 & d_1 \end{array}\right], \tau_{2,1} = \det \left[\begin{array}{c|cc} 1 & 0 & c_0 \\ \hline 0 & c_0 & c_1 \\ \hline e_0 & d_0 & d_1 \end{array}\right],$$

$$\tau_{2,2} = -\det \left[\begin{array}{c|cc} 1 & 0 & c_0 \\ \hline e_0 & d_0 & d_1 \\ e_1 & d_1 & d_2 \end{array}\right], \tau_{1,3} = -\det \left[\begin{array}{cc|c} e_0 & e_1 & d_0 \\ e_1 & e_2 & d_1 \\ e_2 & e_3 & d_2 \end{array}\right]$$

$$\tau_{3,2} = \det \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & c_0 \\ 1 & 0 & 0 & c_0 & c_1 \\ \hline 0 & 0 & c_0 & c_1 & c_2 \\ \hline e_0 & e_1 & d_0 & d_1 & d_2 \\ e_1 & e_2 & d_1 & d_2 & d_3 \end{array}\right],$$

$$\tau_{4,2} = \det \left[\begin{array}{ccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & c_0 \\ 0 & 1 & 0 & 0 & 0 & c_0 & c_1 \\ 1 & 0 & 0 & 0 & c_0 & c_1 & c_2 \\ \hline 0 & 0 & 0 & c_0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 & c_4 \\ \hline e_0 & e_1 & e_2 & d_0 & d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 & d_1 & d_2 & d_3 & d_4 \end{array}\right]$$

We're working to understand what sort of relations are satisfied by these new tau functions.

It's not at all apparent how one would use the Desnanot-Jacobi Identity to find these relations, so we need another approach.

We'll need to use **Hirota Equations**, which are implied by **Plücker Relations**. We are currently working to use these Hirota Equations to try to find new and interesting relations satisfied by our $\widehat{sl_3\mathbb{C}}$ τ-functions.

Plücker Relations in the Finite Dimensional Case:

We have an action of $\mathbf{G} := \mathbf{GL_k}(\mathbb{C})$ on the finite wedge space, $\mathbf{\Lambda^n}\mathbb{C}^\mathbf{k}$, where $n \leq k$:

$\mathbf{g} \cdot (\mathbf{w_1} \wedge \cdots \wedge \mathbf{w_n}) = \mathbf{gw_1} \wedge \cdots \wedge \mathbf{gw_n}$ for all $\mathbf{g} \in \mathbf{G}$ and $\mathbf{w} = \mathbf{w_1} \wedge \cdots \mathbf{w_n} \in \mathbf{\Lambda^n}\mathbb{C}^\mathbf{k}$

We define an operator, $\mathbf{S} : \mathbf{\Lambda^n}\mathbb{C}^\mathbf{k} \otimes \mathbf{\Lambda^n}\mathbb{C}^\mathbf{k} \to \mathbf{\Lambda^{n+1}}\mathbb{C}^\mathbf{k} \otimes \mathbf{\Lambda^{n-1}}\mathbb{C}$ by $\mathbf{S}(\mathbf{v} \otimes \mathbf{w}) = \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{e_i} \wedge \mathbf{v} \otimes \mathbf{e_i}\urcorner \mathbf{w}$, where the $\mathbf{e_i}$ are the standard basis vectors of $\mathbb{C}^\mathbf{k}$, and $e_i\wedge$ and $e_i\urcorner$ are the wedging and contracting operators, respectively.

$\mathbf{S}$ commutes with the action of $\mathbf{G}$ and
$\mathbf{S}(\mathbf{e_1} \wedge \cdots \wedge \mathbf{e_n} \otimes \mathbf{e_1} \wedge \cdots \wedge \mathbf{e_n}) = \mathbf{0}$, so
$\mathbf{S}(\mathbf{g} \cdot (\mathbf{e_1} \wedge \cdots \wedge \mathbf{e_n}) \otimes \mathbf{g} \cdot (\mathbf{e_1} \wedge \cdots \wedge \mathbf{e_n})) = \mathbf{0}$ for all $\mathbf{g} \in \mathbf{G}$,

This gives us relations, called "Plücker relations", for elements in the orbit, $\mathbf{G} \cdot \mathbf{e_1} \wedge \cdots \wedge \mathbf{e_n}$.

An Infinite Dimensional Analogue of the Above

If we consider the basic representation of $\widehat{sl_2\mathbb{C}}$ on the two-component fermionic Fock space, we have an infinite dimensional analogue of the previous slide.

We can define an operator S, that commutes with the action of $\widehat{SL_2\mathbb{C}}$ and is such that $S(v_{\Lambda_0} \otimes v_{\Lambda_0}) = 0$.

In particular, this S commutes with the action of our group element, $g = \begin{bmatrix} 1 & 0 \\ C(z) & 1 \end{bmatrix}$, so we get new **Plücker Relations** relations from $S(g \cdot v_{\Lambda_0} \otimes g \cdot v_{\Lambda_0}) = 0$.

The **Hirota Equations** are then obtained from these **Plücker Relations** by defining a bilinear product on the two component Fermionic Fock space and using the fact that the bilinear product between $S(g \cdot v_{\Lambda_0} \otimes g \cdot v_{\Lambda_0}) = 0$ and anything else is 0.

We are currently in the process of writing out **Hirota Equations** for the $\widehat{\mathbf{sl_3}\mathbb{C}}$ case, and hope that these will give us new and interesting relations satisfied by our tau functions.

Since some of our new tau functions are determinants of Hankel matrices, certain subsets of our collection of tau functions give us $\mathbf{A}_\infty$ **Q-systems** as before.

We'd like to find some **unifying set of relations** between our tau functions, and so expect to get some sort of "**generalized" Q-system**.

Since Q-**systems** appear in many places in representation theory and in combinatorics, once we understand what our new **"generalized" Q-system** looks like, it would be exciting to then find other situations in which it appears.

Thank you.

Happy birthday to Professor Lepowsky and Professor Wilson!

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For Q-systems, see Di Francesco and Kedem papers.

Thanks to Maarten Bergvelt and Rinat Kedem for their helpful comments and suggestions.