Vertex algebra associated to abelian current Lie algebras

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Lie algebra, vertex algebra and related topics
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Definition of vertex algebras

A conformal vertex algebra is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

equipped with a linear map

$$\begin{aligned} Y(\cdot, x) : V &\to \mathrm{Hom}(V, V((x))) \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \ (\text{where } v_n \in \mathrm{End}\, V) \end{aligned}$$

and equipped with two distinguished vectors $\mathbf{1} \in V_{(0)}$, called vacuum vector, such that for $v \in V$, the following axioms hold:

1. $Y(\mathbf{1}, x) = \mathrm{id}_V$;
2. $Y(v, x)\mathbf{1} \in V[[x]]$, and $Y(v, x)\mathbf{1}|_{x=0} = v_{-1}\mathbf{1} = v$;

Definition of vertex algebras (continued)

and $\omega \in V_{(2)}$, called conformal vector, such that the following properties hold:

1. The Virasoro algebra relations
$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_V,$$
where $L(n) = \omega_{n+1}$, and $c_v \in \mathbb{C}$, called rank of V.
2. $Y(L(-1)v, x) = \frac{d}{dx}Y(v, x)$, for $v \in V$.
3. $L(0)v = nv$, for $v \in V_{(n)}$.

and for $u, v \in V$, the Jacobi identity (the main axiom) holds:

$$x_0^{-1}\delta(\frac{x_1 - x_2}{x_0})Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta(\frac{x_2 - x_1}{-x_0})Y(v, x_2)Y(u, x_1)$$
$$= x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})Y(Y(u, x_0)v, x_2),$$

where $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$.

Vertex operator algebras

Definition

A vertex operator algebra is a conformal vertex algebra

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

such that

$$\dim V_{(n)} < \infty \text{ for } n \in \mathbb{Z},$$

$$V_{(n)} = 0 \text{ for } n \text{ sufficiently negative}$$

(grading restriction property).

Strongly graded vertex algebras

Definition

Let A be an abelian group. A conformal vertex algebra

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

is said to be strongly graded with respect to A (or strongly A-graded) if it is equipped with a second grading by A,

$$V = \coprod_{\alpha \in A} V^{(\alpha)},$$

such that the grading restriction conditions hold:

1. $\dim V^{(\alpha)}_{(n)} < \infty$
2. $V^{(\alpha)}_{(n)} = 0$ for n sufficiently negative.

Examples: Vertex algebras associated with even lattices

Let L be an even lattice not necessarily positive definite. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Then we form a Heisenberg algebra

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n \in \mathbb{Z},\ n \neq 0} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c.$$

Let $(\widehat{L}, ^-)$ be a central extension of L by a finite cyclic group $\langle \kappa \mid \kappa^s = 1 \rangle$. Let $C\{L\}$ be a certain induced $\widehat{L}$-module isomorphic to $\mathbb{C}[L]$. Then

$$V_L = S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L\}$$

has a natural structure of conformal vertex algebra. For $\alpha \in L$, choose an $a \in \widehat{L}$ such that $\bar{a} = \alpha$. Define $\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$ and

$$V_L^{(\alpha)} = S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\iota(a).$$

Then V_L is equipped with a natural second grading given by L itself.

Strongly graded modules for strongly graded vertex algebras

Definition

Let $\tilde{A}$ be an abelian group containing A. A V-module $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ is said to be strongly graded with respect to $\tilde{A}$ (or strongly $\tilde{A}$-graded) if it is equipped with a second grading by $\tilde{A}$,

$$W = \coprod_{\beta\in\tilde{A}} W^{(\beta)}$$

such that the grading restriction conditions hold:

1. $\dim W_{(n)}^{(\beta)} < \infty$
2. $W_{(n+k)}^{(\beta)} = 0$ for $k \in \mathbb{Z}$ sufficiently negative.

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2. So far, the only source of strongly graded vertex algebras and their modules comes from V_L, where L is an even lattice.
3. We construct a new family of strongly graded vertex algebras along with a natural logarithmic module category.
4. We show some properties needed in Huang-Lepowsky-Zhang's logaritmic tensor category theory and expect that there is a natural logarithmic tensor category structure on the module category.

Current algebra of a finite dimensional Lie algebra has been studied in [CG1], [CG2] et al. Current Lie algebra is the standard parabolic subalgebra of an affine Lie algebra and its representation has broad applications.

Definition

Let $\mathbb{C}[t]$ be the ring of polynomials in an indeterminate t. The current algebra $\mathfrak{g}[t]$ of a Lie algebra $\mathfrak{g}$ is the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]$, where the Lie bracket is defined by

$$[x \otimes f, y \otimes g]_{\mathfrak{g}[t]} = [x, y]_{\mathfrak{g}} \otimes fg, \quad x, y \in \mathfrak{g}, \quad f, g \in \mathbb{C}[t].$$

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2. $\mathfrak{h}[t]$ has an invariant symmetric bilinear form induced from $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$:

$$\langle xt^m, yt^n \rangle_{\mathfrak{h}[t]} = \delta_{m,n} \langle x, y \rangle_{\mathfrak{h}}, \quad x, y \in \mathfrak{h}, \quad m, n \in \mathbb{N}.$$

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3. For every $\lambda \in \mathfrak{h}$, denote by $\mathbb{C}_\lambda$ the one-dimensional $\mathfrak{h}$-module with $h \in \mathfrak{h}$ acting as the scalar $\langle \lambda, h \rangle_{\mathfrak{h}}$. For every $a \in \mathbb{C}$ we define an $\mathfrak{h}[t]$-module $V(\lambda, a) = \mathbb{C}_\lambda$ as a vector space with action given by

$$(hf) \cdot v = f(a)\lambda(h)v, \quad h \in \mathfrak{h}, \ f \in \mathbb{C}[t], \ v \in \mathbb{C}_\lambda.$$

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1. Affinize $\mathfrak{h}[t]$ as follows:

$$\widehat{\mathfrak{h}[t]} = \mathfrak{h}[t] \otimes \mathbb{C}[s, s^{-1}] \oplus \mathbb{C}\mathbf{k},$$

equipped with the bracket relations

$$[xt^i \otimes s^m, yt^j \otimes s^n] = m\langle x, y\rangle_{\mathfrak{h}} \delta_{m+n,0}\delta_{i,j}\mathbf{k}$$

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2. Set

$$M(l) = \mathrm{Ind}_{\widehat{\mathfrak{h}[t]}_{\leq 0}}^{\widehat{\mathfrak{h}[t]}}(\mathbb{C}\mathbf{1}) = S(\widehat{\mathfrak{h}[t]}_{+}) \otimes \mathbb{C}\mathbf{1},$$

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3. $M(l)$ has a natural vertex algebra structure with operators given in the same way as the Heisenberg Vertex operator algebra.

Modules for $M(l)$

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1. We consider the induced module from the evaluation modules as follows:

$$W(\lambda, a, l) = \mathrm{Ind}_{\widehat{\mathfrak{h}[t]}_{\leq 0}}^{\widehat{\mathfrak{h}[t]}}(V(\lambda, a)) = S(\widehat{\mathfrak{h}[t]}_{+}) \otimes_{\mathbb{C}} V(\lambda, a),$$

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2. We are mostly interested in the case when $a = 0$.
3. We can also construct logarithmic modules for $M(l)$ in a similar way.

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2. Define

$$L(n) = \frac{1}{2l} \sum_{i=1}^{d} \sum_{j \in \mathbb{N}} \sum_{m \in \mathbb{Z}} {}^{\circ}_{\circ} (u^{(i)} t^j)(n-m)(u^{(i)} t^j)(m) {}^{\circ}_{\circ}$$

The operators $L(n)$ are well-defined since for each $w \in W(\lambda, 0, l)$, $L(n)w$ has only finitely many nonzero terms.

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3. $L(n)$ is also well-defined when $|a| < 1$.

Quasi-conformal structure on $M(l)$

Definition

A vertex algebra V is called quasi-conformal if it carries the operators $L(n)$ for $n \geq -1$ such that for $m, n \geq -1$

$$[L(m), L(n)] = (m-n)L(m+n)$$

and for $v \in V$,

$$[L(n), Y(v,x)] = \sum_{m \geq -1}^{n} \binom{n+1}{m+1} x^{n-m} Y(L(m)v, x).$$

Theorem

The vertex algebra $M(l)$ is quasi-conformal.

Strongly gradedness structure on $M(I)$ and $W(\lambda, a, I)$

1. We define an $\mathbb{N}$-grading for $M(l)$ by

$$\mathbb{N}\text{-wt } x_1 t^{i_1}(-n_1)\cdots x_k t^{i_k}(-n_k)\mathbf{1} = i_1 + \cdots + i_k.$$

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3. It is routine to check that under this second grading, $M(l)$ and $W(\lambda, a, l)$ satisfying strongly gradedness restrictions defined before.

Logarithmic intertwining operators

Definition

Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be logarithmic modules for a conformal (quasi-conformal) vertex algebra V. A logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3[\log x]\{x\},$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1} (\log x)^k$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$ {w_{(1)}}^{\mathcal{Y}}_{n+m;\,k} w_{(2)} = 0 \quad \text{for} \quad m \in \mathbb{N} \text{ sufficiently large;}$$

Logarithmic intertwining operators

Definition

the *Jacobi identity*:

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_3(v,x_1)\mathcal{Y}(w_{(1)},x_2)w_{(2)} \\ - \; & x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(w_{(1)},x_2)Y_2(v,x_1)w_{(2)} \\ = \; & x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y_1(v,x_0)w_{(1)},x_2)w_{(2)} \end{aligned}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$; the $L(-1)$*-derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)},x) = \frac{d}{dx}\mathcal{Y}(w_{(1)},x).$$

C_1-cofiniteness condition with respect to $\tilde{A}$

Definition

Let $C_1(W)$ be the subspace of W spanned by elements of the form $u_{-1}w$ for

$$u \in V_+ = \coprod_{n>0} V_{(n)}$$

and $w \in W$. The $\tilde{A}$-grading on W induces an $\tilde{A}$-grading on $W/C_1(W)$:

$$W/C_1(W) = \coprod_{\beta \in \tilde{A}} (W/C_1(W))^{(\beta)},$$

where

$$(W/C_1(W))^{(\beta)} = W^{(\beta)}/(C_1(W))^{(\beta)}$$

for $\beta \in \tilde{A}$. If $\dim\,(W/C_1(W))^{(\beta)} < \infty$ for $\beta \in \tilde{A}$, we say that W is C_1-cofinite with respect to $\tilde{A}$ or W satisfies the C_1-cofiniteness condition with respect to $\tilde{A}$.

Differential equations

Theorem

Let W_i for $i = 0, \dots, 3$ be strongly $\mathbb{N}$-graded generalized $M(l)$-modules satisfying C_1-cofiniteness condition with respect to $\mathbb{N}$. Then for any $w_i \in W_i$, there exist

$$a_k(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{-1}]$$

for $k = 1, \dots, m$ such that for any $M(l)$-modules W_4, and any logarithmic intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of types $\binom{W_0'}{W_1\ W_4}, \binom{W_4}{W_2\ W_3}$, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle$$

satisfies the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_1^m} + a_1(z_1, z_2)\frac{\partial^{m-1}\varphi}{\partial z_1^{m-1}} + \cdots + a_m(z_1, z_2)\varphi = 0,$$

Corollary

The $M(l)$-modules of the form $W(\lambda, a, l)$ is C_1-cofinite with respect to $\mathbb{N}$. Therefore, matrix elements of products and iterates of intertwining operators among triples of modules of the form $W(\lambda, a, l)$ satisfy the differential equations above.