THE ROGERS-RAMANUJAN IDENTITIES: FROM SUMS, HOPEFULLY TO PRODUCTS

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INTRODUCTION

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Rogers-Ramanujan 1

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... Using Jacobi triple product identity

SUMS TO PRODUCTS

SETUP

$$\mathcal{A} = \mathbb{C}[x_{-1}, x_{-2}, \ldots]$$

$$I_{\Lambda_0} = \text{Ideal generated by } \left\{ r_{-n} = \sum_{i=1}^{n-1} x_{-i}x_{-n+i} \,;\, n \geq 2 \right\}.$$

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Definition (actually, a Theorem of Cal-L-M): Principal Subspace

We call $W_{\Lambda_0} = \mathcal{A}/I_{\Lambda_0}$ a *principal subspace*.

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- In this paper, it comes up while calculating Hilbert-Poincaré series of arc space of a double point.
- Shows up in a lot of different problems — more later.

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First use the Jacobi triple product identity

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Could be explained via Euler-Poincaré principle applied to a resolution.

KOSZUL COMPLEX

$$\cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} \mathcal{A}\xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} \mathcal{A}\xi_{i_1} \xrightarrow{\partial_2} C_1 = \mathcal{A} \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \twoheadrightarrow 0$$

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  $\partial(r_{-i}) = 0$
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  $\partial(\xi_{-i,-j}) = r_{-i}\xi_{-j} - r_{-j}\xi_{-i}$
- $H_n = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$
  $H_0 = H_1 = 0$.

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$H_2 = \text{Ker}(\partial_2)/\text{Im}(\partial_3)$ measures the space of “non-trivial” relations.

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Euler-Poincaré principle

With χ being the “dimension” (actually, the (x, q)–character)

$$\chi(W_{\Lambda_0}; x, q) = \sum_{n \geq 1} (-1)^{n+1} \left(\chi(C_n; x, q) - \chi(H_n; x, q)\right).$$

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The Problem

Find the precise structure of H_ns and calculate $\chi(H_n; x, q)$.

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$$L_{-1}\mathrm{Ker}(\partial) \subset \mathrm{Ker}(\partial)$$

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$$\sigma \mathrm{Ker}(\partial) \subset \mathrm{Ker}(\partial)$$

$$\sigma \mathrm{Im}(\partial) \subset \mathrm{Im}(\partial)$$

Some obvious elements in the $\mathrm{Ker}(\partial_2)$:

$$\mu_{-4} = 2x_{-2}\xi_{-2} - x_{-1}\xi_{-3}$$
$$\partial(\mu_{-4}) = 2x_{-2} \cdot x_{-1}^2 - x_{-1} \cdot 2x_{-1}x_{-2} = 0$$

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$$\begin{aligned}
\mu_{-4} &= 2x_{-2}\xi_{-2} - x_{-1}\xi_{-3} \\
\mu_{-5} &= 4x_{-3}\xi_{-2} + x_{-2}\xi_{-3} - 2x_{-1}\xi_{-4} \\
\mu_{-6} &= 6x_{-4}\xi_{-2} + 3x_{-3}\xi_{-3} - 3x_{-1}\xi_{-5} \\
\mu_{-7} &= 8x_{-5}\xi_{-2} + 5x_{-4}\xi_{-3} + 2x_{-3}\xi_{-4} - x_{-2}\xi_{-5} - 4x_{-1}\xi_{-6} \\
\mu_{-8} &= 10x_{-6}\xi_{-2} + 7x_{-5}\xi_{-3} + 4x_{-4}\xi_{-4} + x_{-3}\xi_{-5} - 2x_{-2}\xi_{-6} - 5x_{-1}\xi_{-7}
\end{aligned}$$

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The second homology H_2 is generated by the elements $L_{-1}^s \cdot \mu_{-4}$ for $s \in \mathbb{N}$.

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Remark

The proof is very similar to the proof of presentation of W_{Λ_0} in Calinescu-Lepowsky-Milas '08.

Uses the "minimal counter-example" technique.

CONTEXT

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$$\langle a, b\rangle = \mathrm{Tr}(ab)$$
$$\mathfrak{n} = \mathbb{C}\, x_\alpha$$

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$
$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + \langle x, y\rangle n\delta_{m+n,0}c$$
$$[c, \widehat{\mathfrak{g}}] = 0$$
$$\widehat{\mathfrak{g}} \cong A_1^{(1)}$$

$$\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}\{x_\alpha, \alpha, x_{-\alpha}\}$$
$$\langle a, b\rangle = \mathrm{Tr}(ab)$$
$$\mathfrak{n} = \mathbb{C}\, x_\alpha$$

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$
$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + \langle x, y\rangle n\delta_{m+n,0}c$$
$$[c, \widehat{\mathfrak{g}}] = 0$$
$$\widehat{\mathfrak{g}} \cong A_1^{(1)}$$

$$\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}] \subset \widehat{\mathfrak{g}}$$
$$\bar{\mathfrak{n}}_- = \mathfrak{n} \otimes t^{-1}\mathbb{C}[t^{-1}] \subset \widehat{\mathfrak{g}}$$
$$[\bar{\mathfrak{n}}_-, \bar{\mathfrak{n}}_-] = 0.$$

$$L(\Lambda): \text{ Irreducible, integrable } \widehat{\mathfrak{g}} - \text{module}$$
$$\text{generated by highest wt. vector } v_\Lambda$$

$$W_\Lambda := \mathcal{U}(\bar{\mathfrak{n}}) \cdot v_\Lambda \cong \mathcal{U}(\bar{\mathfrak{n}}_-) \cdot v_\Lambda$$

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Theorem (Calinescu-Lepowsky-Milas '08)

$\mathrm{Ker}(f_{\Lambda_0}) = I_{\Lambda_0}$ and $\mathrm{Ker}(f_{\Lambda_1}) = I_{\Lambda_0} + \mathcal{A}x_{-1}$.

$$\longrightarrow E_2 = \langle v_{r_0 r_1 \cdot \Lambda_0} \rangle \longrightarrow E_1 = \langle v_{r_0 \cdot \Lambda_0} \rangle \longrightarrow E_0 = \langle v_{\Lambda_0} \rangle \longrightarrow L_{\Lambda_0}$$

$$\longrightarrow \bigoplus_{i,j \leq -2} \mathcal{A}\xi_{i,j} \longrightarrow \bigoplus_{i \leq -2} \mathcal{A}\xi_i \longrightarrow \mathcal{A} \longrightarrow W_{\Lambda_0}$$

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2. GARLAND-LEPOWSKY RESOLUTION

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$$v_{r_0 r_1 \cdot \Lambda_0} \mapsto 2x_\alpha(-2) v_{r_0 \cdot \Lambda_0} - x_\alpha(-1) L(-1) v_{r_0 \cdot \Lambda_0}$$

$$\longrightarrow E_2 = \langle v_{r_0 r_1 \cdot \Lambda_0} \rangle \longrightarrow E_1 = \langle v_{r_0 \cdot \Lambda_0} \rangle \longrightarrow E_0 = \langle v_{\Lambda_0} \rangle \longrightarrow L_{\Lambda_0}$$

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$$v_{r_0 r_1 \cdot \Lambda_0} \mapsto 2x_\alpha(-2) v_{r_0 \cdot \Lambda_0} - x_\alpha(-1) L(-1) v_{r_0 \cdot \Lambda_0}$$
$$\sim 2x_{-2}\xi_{-2} - x_{-1} L_{-1} \xi_{-2}$$

$$\longrightarrow E_2 = \langle v_{r_0 r_1 \cdot \Lambda_0} \rangle \longrightarrow E_1 = \langle v_{r_0 \cdot \Lambda_0} \rangle \longrightarrow E_0 = \langle v_{\Lambda_0} \rangle \longrightarrow L_{\Lambda_0}$$

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$$\begin{aligned} v_{r_0 r_1 \cdot \Lambda_0} &\mapsto 2x_\alpha(-2) v_{r_0 \cdot \Lambda_0} - x_\alpha(-1) L(-1) v_{r_0 \cdot \Lambda_0} \\ &\sim 2x_{-2}\xi_{-2} - x_{-1}L_{-1}\xi_{-2} = 2x_{-2}\xi_{-2} - x_{-1}\xi_{-3} \end{aligned}$$

$$\longrightarrow E_2 = \langle v_{r_0 r_1 \cdot \Lambda_0} \rangle \longrightarrow E_1 = \langle v_{r_0 \cdot \Lambda_0} \rangle \longrightarrow E_0 = \langle v_{\Lambda_0} \rangle \longrightarrow L_{\Lambda_0}$$

$$\longrightarrow \bigoplus_{i,j \leq -2} \mathcal{A}\xi_{i,j} \longrightarrow \bigoplus_{i \leq -2} \mathcal{A}\xi_i \longrightarrow \mathcal{A} \longrightarrow W_{\Lambda_0}$$

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$$v_{r_0 \Lambda_0} \sim \xi_{-2}$$

$$v_{r_0 r_1 \cdot \Lambda_0} \mapsto 2x_\alpha(-2) v_{r_0 \cdot \Lambda_0} - x_\alpha(-1) L(-1) v_{r_0 \cdot \Lambda_0}$$

$$\sim 2x_{-2}\xi_{-2} - x_{-1}L_{-1}\xi_{-2} = 2x_{-2}\xi_{-2} - x_{-1}\xi_{-3} = \mu_{-4}$$

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$$\mathrm{Kh}(T(n,\infty)) = \lim_{m\to\infty} q^{-(n-1)(m-1)+1}\mathrm{Kh}(T(n,m)).$$

This limit exists (Stošić).

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Conjecture (Gorsky-Oblomkov-Rasmussen '12)

$\mathrm{Kh}(T(n,\infty))$ is dual to the homology of the Koszul complex determined by the elements $r_{-2}, \ldots, r_{-n-1}$. (Note: the gradings are different than ours.)

Every Koszul complex is a dg-algebra:

$\xi_{-i_1,\ldots,-i_j} = \xi_{-i_1} \wedge \cdots \wedge \xi_{-i_j}.$

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$H = \bigoplus_{n \geq 0} H_n$ is a graded algebra.

Conjecture (Gorsky-Oblomkov-Rasmussen '12)

For $T(n,\infty)$, H is generated as by the elements $\mu_{-4},\ldots,\mu_{-n-2}$, with the defining relations being

$$\begin{aligned} x(z)^2 &= 0 \\ x(z)\mu(z) &= 0 \\ x''(z)\mu(z) - x'(z)\mu'(z) &= 0 \\ \mu(z)\mu'(z) &= 0, \end{aligned}$$

recall that $\mu(z) = 2x'(z)\xi(z) - x(z)\xi'(z)$.

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σ: unreduced $\longrightarrow$ reduced.

$$\begin{aligned} E_0 &\longrightarrow L(\Lambda_0) \\ R &= \mathcal{U}(\mathfrak{g}) x_{-1}^2 \mathbf{1} \subset E_0 \\ \bar{R} &= \text{ Coeffs of } r(x) = Y(r, x), r \in R \end{aligned}$$

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\bar{R}\mathbf{1} \otimes E_0 &\longrightarrow E_0 \\
u \otimes v &\longmapsto u_{-1} v \\
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Meurman-Primc '99, Primc '02

The kernel of the map above is generated, in some sense, by:

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\frac{d}{dx}\left(x_\theta(x)^2\right) \otimes x_\theta(x) - 2(x_\theta(z)^2) \otimes \frac{d}{dx}x_\theta(z)
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&L(-1)x_{-1}^2\mathbf{1} \otimes x_{-1}\mathbf{1} - 2x_{-1}^2\mathbf{1} \otimes L(-1)x_{-1}\mathbf{1}
\end{aligned}
$$

$$E_0 \longrightarrow L(\Lambda_0)$$
$$R = \mathcal{U}(\mathfrak{g})x_{-1}^2\mathbf{1} \subset E_0$$
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$$L(-1)x_{-1}^2\mathbf{1} \otimes x_{-1}\mathbf{1} - 2x_{-1}^2\mathbf{1} \otimes L(-1)x_{-1}\mathbf{1}$$
$$\sim L_{-1}\xi_{-2} \cdot x_{-1} - 2\xi_{-2} \cdot L_{-1}x_{-1} = \xi_{-3} \cdot x_{-1} - 2\xi_{-2} \cdot x_{-2} = \mu_{-4}.$$

Questions?